Commun. math. Phys. 60, 37—72 (1978)

Communications in Mathematical Physics

Soliton Mass and Surface Tension in the $(\lambda|\phi|^4)_2$ Quantum Field Model

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Abstract. The spectrum of the mass operator on the soliton sectors of the anisotropic $(\lambda|\phi|^4)_2$—and the $(\lambda\phi^4)_2$—quantum field models in the two phase region is analyzed. It is proven that, for small enough $\lambda>0$, the mass gap $m_s(\lambda)$ on the soliton sector is positive, and $m_s(\lambda)=0(\lambda^{-1})$. This involves estimating $m_s(\lambda)$ from below by a quantity $\tau(\lambda)$ analogous to the surface tension in the statistical mechanics of two dimensional, classical spin systems and then estimating $\tau(\lambda)$ by methods of Euclidean field theory. In principle, our methods apply to any two dimensional quantum field model with a spontaneously broken, internal symmetry group.

1. Introduction: Main Subject, Models, Main Results

1.1

During the past few years the quantization of nonlinear waves (solitary solutions of nonlinear,classical field equations)has attracted a lot of interest and has been studied from various—more and less rigorous—points of view; see [1–6] and references given there, and [7–10] for a mathematically rigorous analysis. From these efforts emerged the (heuristic) picture that the homotopy classes of *finite energy* solutions to some classical, nonlinear field equation are, for small enough $\hbar$ ($\propto$ Planck's constant), in a one-one correspondence with non-trivial, charged *superselection (soliton) sectors* of the relativistic quantum field theory formally determined by the same nonlinear field equation. It is felt that this picture might be a key to understanding some of the conservation laws and some of the (hadronic) extended particles observed in elementary particle physics.

So far, however, many workers in the field have concentrated on the analysis of quantum field models (or quantum spin systems [11]) in two space-time

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** A Sloan Foundation Fellow; Research supported in part by the U.S. National Science Foundation under Grant No. MPS 75-11864. Present address: I.H.É.S., F-91440 Bures-sur-Yvette, France

*** Supported in part by the National Science Foundation under Grant No. PHY 76-17191

0010-3616/78/0060/0037/$07.20

dimensions. The reason behind this (somewhat surprising) enthusiasm for two dimensions is that in two space-time dimensions there are plenty of simple, superrenormalizable quantum field models with soliton sectors, e.g. the $(\lambda\phi^4)_2$-, [10] the pseudoscalar Yukawa_2 model and the quantum sine-Gordon equation [12, 13], whereas in higher dimensions soliton sectors appear to occur only in gauge theories (with matter fields; see [5, 14, 7]) or in non-renormalizable field theories with chiral symmetries [15]. This does *not* mean that standard, renormalizable field theories in higher dimensions do not have *bound states* in the vacuum sector corresponding to non-trivial solitary solutions of the classical field equation in the homotopy class of the (constant) vacuum solutions ("non-topological solitions") [16]. Such bound states are of considerable interest, but they do not concern us in the present paper. We leave this topic with the remark that non-topological solitons can sometimes be thought of as bound states of two or more confined, topological "would be" solitons, and that perturbations of the dynamics that lift the degeneracy of the physical vacuum and make the soliton sectors disappear generally give rise to new bound states ("non-topological solitons") in the vacuum sector [7]; see also [17, 18].

1.2

By now the *general mechanism* behind the phenomenon of nontrivial super selection sectors with topological charge is rather well understood: It is intimately connected with the existence of several, inequivalent ("orthogonal") physical vacua, i.e. phase transitions, at least in two dimensions [7, 8]. (In gauge theories in three or more dimensions it appears to be connected with the existence of non-unitary gauge transformations which permute different, but *physically equivalent* vacua among each other.) Phase transitions—generally, but not always [8], accompanied by symmetry breaking—in two dimensional quantum field theories, in turn, give rise to the existence of several nontrivial, local Poincaré cocycles [1] from which the soliton sectors can be reconstructed and which yield Poincaré covariance of the soliton sectors [7, 8, 10]. This last point of view was inspired by the deep analysis of super selection sectors due to [19] and the study [7] of special two dimensional models with phase transitions [17, 34] such as the $(\lambda\phi^4)_2$ model. It was realized in [19] (see also §6 of [7]) that a general theory of Poincaré covariant superselection sectors in arbitrary dimension could be developed in terms of Poincaré cocycles. A similar point of view has been advocated in [21], where the main accent is placed on the concept of local cohomology, but Poincaré covariance is unfortunately not emphasized. (The concept of local cohomology has previously been pioneered in a somewhat different framework in [22, 23].)

Let us finally comment on the difference in the point of view adopted in the more heuristic literature on quantum solitons [1–6] and the one adopted in [24, 7–10]: In the more heuristic literature various approximation schemes and algorithms, in particular semi-classical methods, for the calculation of the mass spectrum on vacuum—and soliton sectors and of some special scattering amplitudes have been developed and have provided a wealth of formulas and some

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1 These cocycles are localized objects (local observables) which describe, physically speaking, the operation of transfering some charge from one space-time region to another; see e.g. [21, 10]

rather detailed insight. In particular, in the case of the quantum sine-Gordon equation the semiclassical methods of Dashen et al. [4] appear to give exact results. (There are many indications for this belief to be correct, but *no* rigorous proof, yet.) What is, however, *missing* in this part of the literature is a construction of the *states* that constitute the soliton sectors, of *quasi-local fields* with non-vanishing matrix elements between the physical vacuum and the one soliton states, and (exactly because of this circumstance) of a general, *multi-soliton scattering theory*, except perhaps in the quantum sine-Gordon equation (where, apart from time delays and charge transfer, there appears to be no scattering [25]).

In contrast, in [7–10] soliton sectors have been constructed rigorously, quasi-local soliton fields with non-vanishing matrix elements between the physical vacuum and the one soliton states (if they exist as discrete particles) have been given, at least for some nontrivial, two dimensional models, and as a consequence a Haag-Ruelle multi-soliton scattering theory has been obtained. The drawback of this more constructive and rigorous approach is that it is very difficult to extract from it *explicit* information on the mass spectrum and the scattering matrix. In this paper we propose to do a first step in this direction.

1.3

We feel it is necessary to add a few references to early work in the history of the quantum soliton which seem to have escaped the attention of many workers in the field. Apart from recommending [26, 19] to the reader's attention we wish to mention that early work concerning non-trivial superselection sectors in models (notably the two dimensional, massless scalar free field) has been done in [12, 11, 24, 27], and Refs. given there. The reader may consult [28] for an account of the early history.

1.4

Next we introduce the two dimensional models studied in this paper and summarize our main results which concern the mass gap on their soliton sectors. These results were announced in [29]. Here we give the details and present the proofs.

Space-time points in $\mathbb{R}^2$ are denoted by $x=(\boldsymbol{x},t)$; $\boldsymbol{x}$ is the space—and t the time coordinate. Partial derivatives with respect to $\boldsymbol{x}$, t are denoted $\partial_{\boldsymbol{x}}$, ∂_t, respectively.

Consider the *classical Hamilton density* of the well known ϕ^4-theory

$$\mathscr{H}(\pi,\phi)=\mathscr{H}_0(\pi,\phi)+\mathscr{H}_I(\phi) \tag{1.1}$$

with

$$\mathscr{H}_0(\pi,\phi)=1/2\{\pi(x)^2+(\partial_x\phi(x))^2\}, \tag{1.2}$$

and

$$\mathscr{H}_I(\phi)=\phi(x)^4-1/4\phi(x)^2+1/64, \tag{1.3}$$

where ϕ is a real, scalar field and π the momentum canonically conjugate to it. The constant term in $\mathscr{H}_I(\phi)$ is so chosen that $\mathscr{H}(\pi,\phi)$ is *non-negative*. The Hamilton equations of motion derived from (1.1)–(1.3) give the following classical field equation

$$\Box\phi(x) = -4\phi(x)^3 + 1/2\phi(x). \tag{1.4}$$

A complete existence theory for solutions to (1.4) is available (see [52] and Refs. given there). The solutions ϕ_0 of (1.4) on which the Hamilton functional

$$H(\pi, \phi) = \int_{-\infty}^{+\infty} d\boldsymbol{x} \mathscr{H}(\pi(\boldsymbol{x}, 0), \phi(\boldsymbol{x}, 0)) \geqq 0 \tag{1.5}$$

takes a *finite value*,

$$E(\phi_0) = \int_{-\infty}^{+\infty} d\boldsymbol{x} \mathscr{H}(\partial_t\phi_0(\boldsymbol{x}, 0), \phi(\boldsymbol{x}, 0)) < \infty, \tag{1.6}$$

called *finite energy solutions*, fall into *four different homotopy classes* (Hilbert sectors [52]) represented by the stationary solutions

$$\phi_+ = 8^{-1/2}, \phi_- = -8^{-1/2} \tag{1.7}$$

and

$$\phi_s = 8^{-1/2} \tanh\left(\frac{\boldsymbol{x}}{2}\right)$$

$$\phi_{\bar{s}} = -8^{-1/2} \tanh\left(\frac{\boldsymbol{x}}{2}\right). \tag{1.8}$$

Heuristic, *canonical quantization* of the super-renormalizable model, defined by (1.1)–(1.5), consists in replacing products of $\phi(x)$ and $\pi(x)$ in (1.1)–(1.5) by *normal (Wick) ordered* products of operator valued distributions $\phi_\hbar(x)$ and $\pi_\hbar(x)$ acting on some Hilbert space $\mathscr{H}$ (to be constructed!) and satisfying the canonical commutation relations

$$[\phi_\hbar(\boldsymbol{x}, t), \pi_\hbar(\boldsymbol{y}, t)] = i\hbar\delta(\boldsymbol{x} - \boldsymbol{y}), \tag{1.9}$$

and proving selfadjointness of the Hamiltonian on $\mathscr{H}$, "obtained" from (1.5) in this way. This provides then an existence theory for q-number solutions of the field Equation (1.4). It is customary to introduce new quantum fields

$$\phi(x) = \hbar^{-1/2}\phi_\hbar(x), \pi(x) = \hbar^{-1/2}\pi_\hbar(x) \tag{1.10}$$

satisfying the (normalized) canonical commutation relations

$$[\phi(\boldsymbol{x}, t), \pi(\boldsymbol{y}, t)] = i\delta(\boldsymbol{x} - \boldsymbol{y}), \tag{1.11}$$

in terms of which the *formal* quantum Hamiltonian is given by

$$H = \hbar^{-1} :H(\pi_\hbar, \phi_\hbar): \equiv H_0 + H_I, \tag{1.12}$$

where

$$H_0 = 1/2 \int_{-\infty}^{+\infty} d\boldsymbol{x}\{:\pi(\boldsymbol{x}, 0)^2: + :\partial_{\boldsymbol{x}}\phi(\boldsymbol{x}, 0)^2: + :\phi(\boldsymbol{x}, 0)^2:\},$$

$$H_I = \int_{-\infty}^{+\infty} d\boldsymbol{x}\left\{\hbar:\phi(\boldsymbol{x}, 0)^4: - 3/4:\phi(\boldsymbol{x}, 0)^2: + \frac{1}{64\hbar}\right\}. \tag{1.13}$$

The double colons indicate Wick ordering with respect to the neutral, scalar free field of mass 1. In constructive field theory it is customary to denote $\hbar$ by λ. Clearly we may replace the coefficient $-3/4$ of the quadratic term in H_I by a general coefficient $-\sigma/2$, with $\sigma > 1$ [2], adjusting the constant in H_I in such a way that the classical Hamiltonian is again non-negative. This is no gain in generality and would only complicate our formulas.

According to the heuristic picture described in 1.1 and (1.7), (1.8) we expect *four inequivalent superselection sectors* of the relativistic quantum field model heuristically described above, when $\lambda \equiv \hbar$ is small enough, namely

two vacuum sectors $\mathscr{H}_+$ and $\mathscr{H}_-$ corresponding to ϕ_+, ϕ_-, resp.; see [17, 9],
one soliton sector $\mathscr{H}_s$ corresponding to ϕ_s; and
one "anti-soliton" sector $\mathscr{H}_{\bar{s}}$ corresponding to $\phi_{\bar{s}}$; see [7, 10].

The symmetry $\phi \to -\phi$, $\pi \to -\pi$ of the Hamiltonian H—see (1.12), (1.13)—is *spontaneously broken* on all sectors, but (quite obviously) the substitution $\phi \to -\phi$, $\pi \to -\pi$ takes $\mathscr{H}_+$ to $\mathscr{H}_-$ and $\mathscr{H}_s$ to $\mathscr{H}_{\bar{s}}$ (and consersely), so that the physics on $(\mathscr{H}_+, \mathscr{H}_s)$ is the same as the physics on $(\mathscr{H}_-, \mathscr{H}_{\bar{s}})$. (For this reason it is claimed in some references that the ϕ^4-model has only *two* sectors, one vacuum and one soliton sector, a reasonable point of view. One knows however, from the two dimensional Ising model which has a similar structure that in certain calculations all four sectors should be retained.) It has been shown [7] that space reflection takes $\mathscr{H}_s$ to $\mathscr{H}_{\bar{s}}$ and that all vectors in $\mathscr{H}_s$ (resp. $\mathscr{H}_{\bar{s}}$) are eigenvectors of the topological charge

$$Q = \int dx \partial_x \phi(x, 0) \tag{1.14}$$

with eigenvalue q (resp. $-q$), where

$$\lambda^{1/2} q \approx \phi_s(x = +\infty) - \phi_s(x = -\infty) = 2 \cdot 8^{-1/2}. \tag{1.15}$$

A well known, heuristic approach to calculating the physics on $\mathscr{H}_\pm$ is standard perturbation theory about mean field theory (see e.g. [30]) (only in the sine-Gordon model there are more powerful methods [4]). Perturbation theory will *miss* multisoliton thresholds in the vacuum sector.

A useful approach to calculating soliton effects (see e.g. [3, 5, 6, 16, 18]) is to start from (1.11)–(1.13) and to express the fields π and ϕ in terms of some soliton coordinate—and momentum operators and fluctuation fields in such a way that the canonical commutation relations are formally preserved. One ends up with *non-polynomial* Hamiltonians whose renormalizability is far from obvious. Conceptually this approach is *problematic*, as it ignores the crucial significance of boundary conditions (at infinity, [17, 9]). This will become obvious in our proofs. (But see [31] for a mathematically rigorous implementation of such ideas in a slightly more restrictive context.)

We follow a different route, summarized in 1.5–1.8, which permits us to apply rigorous methods. But we end up with similar conclusions, at least with regard to the properties of the *mass spectrum*, which, in our case, are *theorems*.

We now summarize some rigorous information on the mass spectrum on $\mathscr{H}_\pm$, $\mathscr{H}_s$, $\mathscr{H}_{\bar{s}}$.

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[2] This guarantees that, for small $\lambda > 0$, there are at least two phases

Glimm et al. [17] have shown that the mass gap $m(\lambda)$ on $\mathscr{H}_\pm$ is, for small $\lambda>0$, *strictly positive* with

$$m(\lambda)=0(1) \tag{1.16}$$

$[m(\lambda)\to 1$, as $\lambda\downarrow 0]$.

The *main result* of this paper is

Theorem A. *For small enough $\lambda>0$, the mass gap $m_s(\lambda)$ on the soliton sectors $\mathscr{H}_s$, $\mathscr{H}_{\bar{s}}$ is strictly positive, and*

$$m_s(\lambda)=0(\lambda^{-1}). \tag{1.17}$$

This result has been announced in [29]. In this paper we give precise statements and proofs.

As to the existence of *one particle states* in the vacuum, and the soliton sectors with mass $m(\lambda)$, resp. $m_s(\lambda)$, this problem has not been rigorously settled, yet. However, we emphasize that the necessary methods to solve it appear to have been developed [32–34], and there cannot be any doubt that $m(\lambda)$ and $m_s(\lambda)$ are the masses of stable particles. (To supply detailed proofs is presumably quite hard, though.) If this is correct it follows from [17,7,9] that there are quasi-local fields with non-vanishing matrix elements between the physical vacuum and these one-particle states and a Haag-Ruelle *multi-soliton scattering theory* exists [19, 7, 10].

We wish to point out that the situation described here for the ϕ^4-model is typical of all *two dimensional models* with two inequivalent vacuum sectors (i.e. a phase transition) such as the anisotropic $|\phi|_2^4$-model or the pseudoscalar Yukawa$_2$-model with the *proviso* that the *three* sectors $\mathscr{H}_+$, $\mathscr{H}_-$, $\mathscr{H}_s$ are physically inequivalent if $\phi\to-\phi$, $\pi\to-\pi$ is *not* a symmetry of the Hamilton function, as is the case in a class of $P(\phi)_2$*-models with* P positive and "almost even" [8]; $\mathscr{H}_{\bar{s}}$ is still the mirror image of $\mathscr{H}_s$. (But in this case the only rigorous construction of $\mathscr{H}_s$ and $\mathscr{H}_{\bar{s}}$ proposed so far [7] looks very indirect and artificial.)

In the quantum sine-Gordon equation and a class of related models, however, we encounter infinitely many soliton sectors labelled by charges that are integer multiples of some elementary charge [7, 35]. (This is no surprise, as the sine-Gordon model is equivalent to the massive Thirring model describing a charged Dirac two spinor field, provided $\hbar$ is small enough (and the total charge vanishes) [36].) Our methods apply to this model, too.

1.5

Before we can describe our main result in more detail we must recall the construction of the relativistic quantum field theory heuristically defined in (1.11)–(1.13). At the present time this construction is always done in two steps:

Step 1 [17]. Construction of the vacuum sectors $\mathscr{H}_+$ and $\mathscr{H}_-$ ("quantization in the vacuum representation"; see also [34]).

Step 2 [7, 8]. Construction of the soliton sectors $\mathscr{H}_s$ and $\mathscr{H}_{\bar{s}}$; ("quantization in the soliton representation").

As a preliminary to Step 2 one needs:

Step 1' [38, 41]. Construction of a net of local von Neumann algebras satisfying the Haag-Kastler axioms [26] (which is not an automatic consequence of Step 1 in the form [17, 34]).

1.6

Quantization in the vacuum representation:

Description of Step 1

At the present time, the universal approach of constructive field theorists to constructing the vacuum sectors $\mathscr{H}_\pm$ comes from the Euclidean description of relativistic quantum field theory, in particular Euclidean field theory [34, 42, 43]. This approach has not only been very successful in constructive quantum field theory, but it has also led to the discovery of the instanton [44] (a phenomenon which is more difficult to extract from the Hamiltonian formalism).

For reasons of technical simplicity described in Step 2 we henceforth consider the anisotropic $|\boldsymbol{\phi}|_2^4$-model; see e.g. [9, 45]. This is the model describing a pair $\boldsymbol{\phi} = (\phi_1, \phi_2)$ of real, scalar fields with classical Hamilton density

$$\mathscr{H}(\boldsymbol{\pi}, \boldsymbol{\phi}) = \mathscr{H}_0(\boldsymbol{\pi}, \boldsymbol{\phi}) + \mathscr{H}_I(\boldsymbol{\phi}),$$

$$\mathscr{H}_0(\boldsymbol{\pi}, \boldsymbol{\phi}) = 1/2\{|\boldsymbol{\pi}(x)|^2 + |\partial_x \boldsymbol{\phi}(x)|^2 + |\boldsymbol{\phi}(x)|^2\}$$

$$\mathscr{H}_I(\boldsymbol{\phi}) = |\boldsymbol{\phi}(x)|^4 - 3/4\phi_1(x)^2 - 1/4\phi_2(x)^2 + 1/64, \tag{1.18}$$

with $\boldsymbol{\pi} = (\pi_1, \pi_2)$ canonically conjugate to $\boldsymbol{\phi}$.

With $\mathscr{H}$ one associates the classical Euclidean action

$$S(\boldsymbol{\phi}) = S_0(\boldsymbol{\phi}) + S_I(\boldsymbol{\phi}),$$

with

$$S_0(\boldsymbol{\phi}) = 1/2 \int \{|\nabla \boldsymbol{\phi}(x)|^2 + |\boldsymbol{\phi}(x)|^2\} d^2x \tag{1.19}$$

and

$$S_I(\boldsymbol{\phi}) = \int \{|\boldsymbol{\phi}(x)|^4 - 3/4\phi_1(x)^2 - 1/4\phi_2(x)^2 + 1/64\} d^2x.$$

The integrals extend over all of Euclidean spacetime; $\boldsymbol{\phi} = (\phi_1, \phi_2)$ is the classical Euclidean field.

To construct the Euclidean Green's or Schwinger functions ($\equiv$ Wightman functions at the Euclidean points) one wants to interpret $\boldsymbol{\phi}$ as a pair of *real random fields* over $\mathbb{R}^2$ and one replaces the classical action by

$$1/\lambda : S(\lambda^{1/2}\boldsymbol{\phi}) : \quad (\lambda \equiv \hbar), \tag{1.20}$$

where the double colons indicate normal ordering of random fields [42] (sometimes called Ito-ordering) with respect to the free (Gaussian) Euclidean field of mass 1.

The Euclidean Green's functions (EGF's) are then given by the *Euclidean Gell'Mann-Low formula*

$$\mathscr{S}_n(x_1, i_1, \ldots, x_n, i_n) \equiv \left\langle \prod_{j=1}^{n} \phi_{i_j}(x_j) \right\rangle_S$$

$$= {}'' [\int e^{-1/\lambda : S(\lambda^{1/2}\boldsymbol{\phi}):} \prod_x \mathscr{D}\boldsymbol{\phi}(x)]^{-1}$$

$$\times \int \prod_{j=1}^{n} \phi_{i_j}(x_j) e^{-1/\lambda : S(\lambda^{1/2}\boldsymbol{\phi}):} \prod_x \mathscr{D}\boldsymbol{\phi}(x)''. \tag{1.21}$$

For the model discussed here this heuristic formula has been given a rigorous sense in [34, 17] (see also [42, 9]) in such a way that $\{\mathscr{S}_n(x_1, i_1, \ldots, x_n, i_n)\}_{n=0}^{\infty}$ satisfy all the axioms of Osterwalder and Schrader [43] and hence are the Wightman functions restricted to the Euclidean points of a unique relativistic quantum field theory.

The rigorous construction of the EGF's starts with first setting $S_I = 0$, i.e. $S = S_0$.In this case the expectation $\langle - \rangle_{S_0}$ is simply the one of the Gaussian process with mean 0 and covariance $(-\Delta + 1)^{-1}$ indexed by the Sobolev space $\mathscr{H}_{-1}$. It is given by a Gaussian measure $d\mu_0(\boldsymbol{\phi})$ on $\mathscr{S}' \equiv \mathscr{S}'_{\text{real}}(\mathbb{R}^2)^{\times 2}$ of mean 0 and covariance $(-\Delta + 1)^{-1}$.

Let $L \times T$ denote the rectangle $(-L/2, L/2) \times (-T/2, T/2)$, and define

$$S_I(L \times T) = \int_{L \times T} d^2 x \lambda^{-1} :S_I(\lambda^{1/2} \boldsymbol{\phi}):. \tag{1.22}$$

Furthermore

$$\boldsymbol{\phi}_\pm = (\pm (8\lambda)^{-1/2}, 0), \tag{1.23}$$

and

$$\delta \tilde{S}_\pm(L \times T) = \int_{L \times T} d^2 x \{\boldsymbol{\phi}_\pm \cdot \boldsymbol{\phi}(x) - 1/2 |\boldsymbol{\phi}_\pm|^2\}. \tag{1.24}$$

Consider the measure

$$d\mu_{\lambda, \pm}^{L \times T}(\boldsymbol{\phi}) = \tilde{Z}_\pm(L \times T)^{-1} e^{-S_I(L \times T) - \delta \tilde{S}_\pm(L \times T)} d\mu_0(\boldsymbol{\phi} - \boldsymbol{\phi}_\pm), \tag{1.25}$$

where $\tilde{Z}_\pm(L \times T)$ is a normalization factor chosen so that $d\mu_{\lambda, \pm}^{L \times T}$ is a probability measure on $\mathscr{S}'$. Note that $d\mu_0(\boldsymbol{\phi} - \boldsymbol{\phi}_\pm)$ is the Gaussian measure with mean $\boldsymbol{\phi}_\pm$ and covariance $(-\Delta + 1)^{-1}$.

We now define space-time cutoff EGF's

$$\left\langle \prod_{j=1}^n \phi_{i_j}(x_j) \right\rangle_{\lambda, \pm} (L \times T) = \int_{\mathscr{S}'} \prod_{j=1}^n \phi_{i_j}(x_j) d\mu_{\lambda, \pm}^{L \times T}(\boldsymbol{\phi}).$$

Such integrals are discussed in [17, 34, 42] and Refs. given there and exist in the distributional sense. We now state a basic theorem due to Glimm et al. [17].

Theorem 1. *For λ small enough, the limits*

$$\left\langle \prod_{j=1}^n \phi_{i_j}(x_j) \right\rangle_{\lambda, \pm} \equiv \lim_{L, T \to \infty} \left\langle \prod_{j=1}^n \phi_{i_j}(x_j) \right\rangle_{\lambda, \pm} (L \times T)$$

exist (independent of order) and satisfy all Osterwalder-Schrader axioms including a strictly positive mass gap $m(\lambda)$ with $m(\lambda) \to 1$, as $\lambda \downarrow 0$.

Definition. Let $\mathcal{O}$ be an open set in $\mathbb{R}^2$. Define $\Sigma_{\mathcal{O}}$ to be the *smallest* σ-algebra on $\mathscr{S}'$ with the property that all random variables generated by

$$\{\boldsymbol{\phi}(f) : f_j \in \mathscr{H}_{-1}, \operatorname{supp} f_j \subset \mathcal{O}, j = 1, 2\}$$

are $\Sigma_{\mathcal{O}}$-measurable. Heuristically, a $\Sigma_{\mathcal{O}}$-measurable function F on $\mathscr{S}'$ has the property that $F(\boldsymbol{\phi}) = F(\boldsymbol{\phi}')$, for all $\boldsymbol{\phi}' \in \mathscr{S}'$ coinciding on $\mathcal{O}$ with $\boldsymbol{\phi}$.

Let χ be a C^∞ function on $\mathbb{R}^2$ with the properties that $0 \leqq \chi(x) \leqq 1$, $\chi(x)=1$, for all $x \in L \times T$, $\chi(x)=0$, for all x in the complement of $(L+1)\times(T+1)$.

Lemma 2. *For all* $\mathcal{O} \subset L \times T$

$$d\mu_0(\phi - \phi_\pm)\upharpoonright \Sigma_{\mathcal{O}} = d\mu_0(\phi - \phi_\pm \chi)\upharpoonright \Sigma_{\mathcal{O}}.$$

Proof. The functions spanned by

$$\{e^{i\phi(f)} : f_j \in \mathcal{S}, \operatorname{supp} f_j \subset \mathcal{O}, j=1,2\}$$

are dense in $L^1(\mathcal{S}', \Sigma_{\mathcal{O}}, d\mu_0)$; $\mathcal{S} \equiv \mathcal{S}_{\text{real}}(\mathbb{R}^2)$. Therefore it suffices to show

$$\int d\mu_0(\phi - \phi_\pm) e^{i\phi(f)} = \int d\mu_0(\phi - \phi_\pm \chi) e^{i\phi(f)},$$

for f's with support in $\mathcal{O}$. This follows by substituting $\phi := \phi' + \phi_\pm$, resp. $\phi := \phi' + \phi_\pm \cdot \chi$ on the l.h.s., the r.h.s., resp. □

Corollary 3. *For* $\mathcal{O} \subset L \times T$ *and* F *any* $\Sigma_{\mathcal{O}}$*-measurable function on* $\mathcal{S}'$

$$\langle F \rangle_{\lambda,\pm}(L \times T) = Z'_\pm(L \times T)^{-1} \int_{\mathcal{S}'} F(\phi) e^{-S_I(L\times T) - \delta\tilde{S}_\pm(L\times T)} \cdot e^{\phi_\pm \phi((-\Delta+1)\chi) - 1/2|\phi_\pm|^2(\chi,(-\Delta+1)\chi)} d\mu_0(\phi),$$

where $Z'_\pm(L \times T)$ is a normalization factor.

Proof. This is an obvious consequence of Lemma 2 and the equation

$$\frac{d\mu_0(\phi - g)}{d\mu_0(\phi)} = e^{\phi((-\Delta+1)g) - 1/2(g,(-\Delta+1)g)}. \quad \text{Q.E.D.}$$

Let $\chi_L(x)$ be a C^∞ function on $\mathbb{R}$ with $0 \leqq \chi_L(x) \leqq 1$,

and

$$\chi_L(x) = \begin{cases} 1, & \text{on } [-L/2, L/2] \\ 0, & \text{on } \left(-\infty, -\dfrac{L+1}{2}\right] \cup \left[\dfrac{L+1}{2}, \infty\right). \end{cases}$$

We define

$$\delta S_{\pm\pm}(L \times T) = \int_{-T/2}^{T/2} dt \left(\int_{-\frac{L+1}{2}}^{-L/2} + \int_{L/2}^{L+1/2} \right) dx \{\phi_\pm \cdot \phi(x) - 1/2|\phi_\pm|^2 \chi_L(x)\} \cdot (-\partial_x^2 + 1)\chi_L(x)$$

$$\delta S_{-+}(L \times T) = \int_{-T/2}^{T/2} dt \left[\int_{-\frac{L+1}{2}}^{-L/2} dx \{\phi_- \cdot \phi(x) - 1/2|\phi_-|^2 \chi_L(x)\} \cdot (-\partial_x^2 + 1)\chi_L(x) + \int_{\frac{L}{2}}^{\frac{L+1}{2}} dx \{\phi_+ \cdot \phi(x) - 1/2|\phi_+|^2 \chi_L(x)\}(-\partial_x^2 + 1)\chi_L(x) \right], \quad (1.26)$$

and

$$Z_{\pm +}(L\times T)=\int_{\mathscr{S}'} e^{-S_I(L\times T)-\delta S_{\pm +}(L\times T)}d\mu_0(\phi). \tag{1.27}$$

Corollary 4.

$$\langle F\rangle_{\lambda,\pm}=\lim_{L\to\infty}\langle F\rangle_{\lambda,\pm}(L), \tag{1.28}$$

where

$$\langle G\rangle_{\lambda,\pm}(L)\equiv\lim_{T\to\infty}\langle G\rangle_{\lambda,\pm}(L\times T) \tag{1.29}$$

$$=\lim_{T\to\infty}Z_{\pm\pm}(L\times T)^{-1}\int_{\mathscr{S}'} Ge^{-S_I(L\times T)-\delta S_{\pm\pm}(L\times T)}d\mu_0(\phi), \tag{1.30}$$

for all $\Sigma_{(-\frac{L}{2},\frac{L}{2})\times\mathbb{R}}$-measurable functions G on $\mathscr{S}'$.

Proof. Equation (1.28) follows from (1.29) and Theorem 1. If we reexpress the r.h.s. of (1.29) using Corollary 3 and then use a simple "transfer matrix" argument (see e.g. [42]) to control the limit $T\to\infty$ we obtain (1.30). Q.E.D.

Next we recall the connection between the *Euclidean* field theory *formalism* summarized above and the *Hamiltonian formalism* [37].

Let $\mathscr{F}$ be the usual, symmetric Fock space of the free, neutral, scalar fields ϕ_1, ϕ_2 with mass 1, and let H_0 denote the free Hamiltonian. We define

$$H_I(L)=\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\{\lambda:(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2:(x,0)-3/4:\phi_1^2:(x,0)-1/4:\phi_2^2:(x,0)+(64\lambda)^{-1}\} \tag{1.31}$$

and the double colons indicate Wick ordering with respect to the free field of mass 1;

$$\delta H_{\pm\pm}(L)=\left(\int_{-\frac{L+1}{2}}^{-\frac{L}{2}}+\int_{\frac{L}{2}}^{\frac{L+1}{2}}\right)d\boldsymbol{x}\{\boldsymbol{\phi}_\pm\cdot\boldsymbol{\phi}(\boldsymbol{x},0)-1/2|\boldsymbol{\phi}_\pm|^2\chi_L(\boldsymbol{x})\}\cdot(-\partial_{\boldsymbol{x}}^2+1)\chi_L(\boldsymbol{x}) \tag{1.32}$$

$$\delta H_{-+}(L)=\int_{-\frac{L+1}{2}}^{-\frac{L}{2}} d\boldsymbol{x}\{\boldsymbol{\phi}_-\cdot\boldsymbol{\phi}(\boldsymbol{x},0)-1/2|\boldsymbol{\phi}_-|^2\chi_L(\boldsymbol{x})\}\cdot(-\partial_{\boldsymbol{x}}^2+1)\chi_L(\boldsymbol{x})$$
$$+\int_{\frac{L}{2}}^{\frac{L+1}{2}} d\boldsymbol{x}\{\boldsymbol{\phi}_+\cdot\boldsymbol{\phi})\boldsymbol{x},0)-1/2|\boldsymbol{\phi}_+|^2\chi_L(x)\}\cdot(-\partial_{\boldsymbol{x}}^2+1)\chi_L(\boldsymbol{x}). \tag{1.33}$$

Then the operators

$$\tilde{H}_{\pm\pm}(L)=H_0+H_I(L)+\delta H_{\pm\pm}(L),$$

and

$$\tilde{H}_{-+}(L)=H_0+H_I(L)+\delta H_{-+}(L) \tag{1.34}$$

are selfadjoint on $\mathscr{D}(H_0)\cap\mathscr{D}(H_I(L))$ [37], and $\exp[-t\tilde{H}_{\pm +}(L)]$ is the transition function of a Markov process on the spectrum of the abelian von Neumann algebra generated by all bounded functions of the time 0-fields $\phi_1(\cdot,0)$, $\phi_2(\cdot,0)$. The *Feynman-Kac formula*—see [42, 46]—tells us that the expectation $\langle - \rangle_{\lambda,\pm}(L)$ is precisely given by the *path space measure* of the process with transition function $\exp[-t\tilde{H}_{\pm\pm}(L)]$, and

$$Z_{++}(L\times T)=Z_{--}(L\times T)=(\Omega_0, \exp[-T\tilde{H}_{\pm\pm}(L)]\Omega_0) \tag{1.35}$$

$$Z_{-+}(L\times T)=(\Omega_0, \exp[-T\tilde{H}_{-+}(L)]\Omega_0), \tag{1.36}$$

where $\Omega_0 \in \mathscr{F}$ is the bare vacuum, $(H_0\Omega_0=0)$.

It is known—see [38, 42], and Refs. given there—that $\tilde{H}_{\pm\pm}(L)$ has a unique groundstate $\Omega_{\pm}(L)\in\mathscr{F}$ corresponding to the simple eigenvalue

$$E_{++}(L)=E_{--}(L)=\inf\operatorname{spec}(\tilde{H}_{\pm\pm}(L))>-\infty.$$

Defining $H_{\pm\pm}(L)=\tilde{H}_{\pm\pm}(L)-E_{\pm\pm}(L)$ we have $H_{\pm\pm}(L)\geqq 0$. We set

$$E_{-+}(L)=\inf\operatorname{spec}(\tilde{H}_{-+}(L)),$$

and

$$H_{-+}(L)=\tilde{H}_{-+}(L)-E_{++}(L), \tag{1.37}$$

whence $\inf\operatorname{spec}(H_{-+}(L))=E_{-+}(L)-E_{++}(L)$.

Our *main result*, Theorem A, (1.17) of Section 1.4, can now be reformulated as follows.

Theorem A′. *The mass gap $m_s(\lambda)$ on the soliton sectors of the anisotropic $|\phi|_2^4$-model described above satisfies*

$$m_s(\lambda)\geqq\tau(\lambda)\equiv\varlimsup_{L\to\infty}\tau_L(\lambda), \tag{1.38}$$

where

$$\tau_L(\lambda)\equiv E_{-+}(L)-E_{++}(L) \tag{1.39}$$

$$=-\lim_{T\to\infty}\frac{1}{T}\log\frac{Z_{-+}(L\times T)}{Z_{++}(L\times T)}, \tag{1.40}$$

and

$$\frac{Z_{-+}(L\times T)}{Z_{++}(L\times T)}\leqq 0(L)e^{-0(\lambda^{-1})T}, \tag{1.41}$$

for $L\gg 1$.

Remarks. Of course we still owe to the reader a review of the construction of the soliton sectors and a technically convenient definition of $m_s(\lambda)$; see Sections 1.8 and 2. Apart from that the anatomy of our main result should now be clear from the form of Theorem A′. The proof of (1.38)–(1.39) is given in Section 2. Equation (1.40) is standard; see [39, 47]. Equation (1.39) can be viewed as the definition of a *surface tension* in a system of fields confined to the strip $\left(-\frac{L}{2},\frac{L}{2}\right)\times\mathbb{R}$ with "$-$boundary conditions" at $x=-\frac{L}{2}$, and "$+$boundary conditions" at $x=+\frac{L}{2}$; $\tau(\lambda)$ is then the surface tension of the corresponding infinite system.

Estimate (1.41) and (1.38)–(1.40) give

$$m_s(\lambda) \geqq 0(\lambda^{-1}).$$

The *proof* of (1.41) employs methods that are somewhat similar to the ones used in the discussion of the surface tension in Ising models [48]. This analogy has been suggested in [8].

1.7

Construction of local algebras of bounded operators:

Step 1'

First we recall an extended version of the Feynman-Kac formula. We define the interacting quantum field with space cutoff by

$$\phi_i^{(L,\pm)}(\boldsymbol{x},t) = e^{itH_{\pm}\pm(L)}\phi_i(\boldsymbol{x},0)e^{-itH_{\pm}\pm(L)},$$

where $\phi_i(\boldsymbol{x},0)$ is the free field at time 0.

Theorem 5 (see [42, 40, 43]). *The moments* $\left\langle \prod_{j=1}^{n} \phi_{i_j}(x_j) \right\rangle_{\lambda,\pm}(L)$ *are the EGF's of the spatially cutoff Wightman distributions*

$$\left(\Omega_{\pm}(L), \prod_{j=1}^{n} \phi_{i_j}^{(L,\pm)}(\boldsymbol{x}_j, t_j)\Omega_{\pm}(L)\right),$$

i.e. for $t_1 < t_2 < \ldots < t_n$,

$$\left\langle \prod_{j=1}^{n} \phi_{i_j}(\boldsymbol{x}_j, t_j) \right\rangle_{\lambda,\pm}(L)$$

$$= \left(\Omega_{\pm}(L), \prod_{j=1}^{n} \phi_{i_j}(\boldsymbol{x}_j, 0)e^{-(t_{j+1}-t_j)H_{\pm}\pm(L)}\phi_{i_n}(\boldsymbol{x}_n, 0)\Omega\pm(L)\right). \tag{1.42}$$

Remark. From Theorem 5, Corollary 4 and a well known theorem concerning convergence of boundary values [in $\mathscr{S}'(\mathbb{R}^{2n})$] of a convergent sequence of holomorphic functions of several complex variables (satisfying some uniform bounds [40, 47]) we conclude that the limit

$$\mathscr{W}_{n,\pm}(i_1, x_1, \ldots, i_n, x_n) = \lim_{L\to\infty}\left(\Omega_{\pm}(L), \prod_{j=1}^{n} \phi_{i_j}^{(L,\pm)}(x_j)\Omega_{\pm}(L)\right) \tag{1.43}$$

exist in $\mathscr{S}'(\mathbb{R}^{2n})$ and, using Theorem 1, Corollary 4 and the Osterwalder-Schrader reconstruction theorem [43], we see that the moments

$$\left\{\left\langle \prod_{j=1}^{n} \phi_{i_j}(x_j) \right\rangle_{\lambda,\pm}\right\}_{n=0}^{\infty} \quad \text{are the EGF's of} \quad \{\mathscr{W}_{n,\pm}\}_{n=0}^{\infty}$$

and $\{\mathscr{W}_{n,\pm}\}_{n=0}^{\infty}$ satisfy all Wightman axioms including a positive mass gap $m(\lambda)$, [17]. The Hilbert spaces $\mathscr{H}_{\pm}$ obtained by Wightman reconstruction are called *vacuum sectors.*

Let $\phi(x,t)$ be the relativistic quantum field, and $H_\pm$ the Hamiltonian reconstructed from the $\{\mathscr{W}_{n,\pm}\}_{n=0}^\infty$. It was shown in [40, 47] that for $f \in \mathscr{S}_{\text{real}}(\mathbb{R}^2)^{\times 2}$

$$\pm\phi(f) \leqq |f|(H_\pm + 1), \tag{1.44}$$

in the sense of quadratic forms on $\mathscr{H}_\pm$. Here $|\cdot|$ is some norm continuous on $\mathscr{S}(\mathbb{R}^2)^{\times 2}$. In [40] it was shown that this estimate implies essential selfadjointness of $\phi(f)$ on any core for $H_\pm$. This permits us to define local von Neumann algebras:

Let $\mathcal{O}$ be a bounded, open set in $\mathbb{R}^2$ (typically a double cone). We define $\mathscr{A}_\pm(\mathcal{O})$ to be the von Neumann algebra generated by

$$\{e^{i\phi(f)} : f_j \in \mathscr{S}, \operatorname{supp} f_j \subset \mathcal{O}, j = 1, 2\}$$

on the Hilbert space $\mathscr{H}_\pm$.

Theorem 6 [41]. *The Wightman axioms for* $\{\mathscr{W}_{n,\pm}\}_{n=0}^\infty$ and the estimate (1.44) *guarantee that the algebras* $\{\mathscr{A}_\pm(\mathcal{O})\}$ *form a net of local algebras satisfying all the axioms of Haag and Kastler.*

We let $\mathscr{A}_\pm$ be the norm closure of $\bigcup \mathscr{A}_\pm(\mathcal{O})$. Furthermore, $U_\pm : \xi = (\Lambda, a) \in \mathscr{P}_+^\uparrow \mapsto U_\pm(\xi)$ denotes the unitary representation of the Poincaré group on $\mathscr{H}_\pm$. For $A \in \mathscr{A}_\pm$ we define

$$\tau_\xi(A) = U_\pm(\xi) A U_\pm(\xi)^* \tag{1.45}$$

Theorem 6 asserts that the group $\{\tau_\xi : \xi \in \mathscr{P}_+^\uparrow\}$ is a representation of $\mathscr{P}_+^\uparrow$ by **-automorphisms of* $\mathscr{A}_\pm$.

Next, we recall a basic theorem due to Glimm and Jaffe. We let $\mathscr{A}_{\mathscr{F}}(\mathcal{O})$ denote the local von Neumann algebra generated by all bounded functions of the free, scalar field of mass 1 smeared out with test functions supported in $\mathcal{O}$, in the Fock representation.

Theorem 7 [39]. *For all bounded, open double cones in* $\mathbb{R}^2$ *the algebras* $\mathscr{A}_\pm(\mathcal{O})$ *and* $\mathscr{A}_{\mathscr{F}}(\mathcal{O})$ *are isomorphic (and unitarily equivalent).*

This theorem permits us to identify the algebras $\mathscr{A}_+(\mathcal{O})$, $\mathscr{A}_-(\mathcal{O})$ and $\mathscr{A}_{\mathscr{F}}(\mathcal{O})$ and hence $\mathscr{A}_\pm$ and $\mathscr{A}_{\mathscr{F}}$ [the norm closure of $\cup \mathscr{A}_{\mathscr{F}}(\mathcal{O})$], and we omit the subscripts hence forth.

From estimate (1.44) and (1.43) follows (by a simple argument [40]).

Corollary 8. *For all* $A \in \mathscr{A}$ *the limit*

$$\omega_\pm(A) \equiv (\Omega_\pm, A\Omega_\pm) = \lim_{L \to \infty} (\Omega_\pm(L), A\Omega_\pm(L))$$

exists.

Remark. We omit reference to the specific representations $\pi_{\mathscr{H}_\pm}$, $\pi_{\mathscr{F}}$ of $\mathscr{A}$ on $\mathscr{H}_\pm$, resp. $\mathscr{F}$, so that A denotes both, the element of the abstract C^* algebra $\mathscr{A}$, and its representative on $\mathscr{H}_\pm$ or $\mathscr{F}$.

We may now proceed to the construction of the soliton sectors.

1.8

Quantization in the soliton representation:

Step 2

Consider the formal equations

$$\begin{aligned}\sigma(\phi_1(x,0)) &= \cos\theta(x)\phi_1(x,0) + \sin\theta(x)\phi_2(x,0)\\ \sigma(\phi_2(x,0)) &= -\sin\theta(x)\phi_1(x,0) + \cos\theta(x)\phi_2(x,0)\end{aligned} \tag{1.46}$$

+ identical equations with (ϕ_1, ϕ_2) replaced by (π_1, π_2), the momenta canonically conjugate to (ϕ_1, ϕ_2) (at time $t=0$).

The function $\theta(x)$ is C^∞, $\partial_x\theta(x)$ has compact support, and

$$\lim_{x\to-\infty}\theta(x) = \pi\,, \qquad \lim_{x\to+\infty}\theta(x) = 0\,. \tag{1.46'}$$

It has been proven in [7] that equations (1.46)–(1.46′) uniquely determine a *-automorphism σ of the C^*-algebra $\mathscr{A}$ with the property that

$$\sigma(\mathscr{A}(\mathscr{O})) \subseteq \mathscr{A}(\mathscr{O})\,, \tag{1.47}$$

for all double cones $\mathscr{O} \subset \mathbb{R}^2$.

Moreover, in the Fock representation, σ is unitarily implemented by the operator $\exp iL(\theta)$, where

$$L(\theta) = \int dx\theta(x)[\phi_1(x,0)\pi_2(x,0) - \phi_2(x,0)\pi_1(x,0)]\,. \tag{1.48}$$

(See Lemma 2 of [7].)

Consider now the states $\omega_\pm \circ \sigma$ defined by

$$\omega_\pm \circ \sigma(A) = \omega_\pm(\sigma(A)), \quad \text{for all} \quad A \in \mathscr{A}\,.$$

According to the Gel'fand-Naimark-Segal construction there exist Hilbert spaces $\mathscr{H}_s$, $\mathscr{H}_{\bar{s}}$, representations of $\mathscr{A}$ on $\mathscr{H}_s$ and $\mathscr{H}_{\bar{s}}$ and cyclic vectors $\Omega_s \in \mathscr{H}_s$ and $\Omega_{\bar{s}} \in \mathscr{H}_{\bar{s}}$ such that

$$\begin{aligned}\omega_+ \circ \sigma(A) &= (\Omega_s, A\Omega_s)\\ \omega_- \circ \sigma(A) &= (\Omega_{\bar{s}}, A\Omega_{\bar{s}})\,.\end{aligned} \tag{1.49}$$

One of the main results of [7, 8] says that $\mathscr{H}_s$ and $\mathscr{H}_{\bar{s}}$ can be interpreted as the soliton sectors of this model. This is due to

Theorem 9 [7, 8]. (1) *There exists a continuous, unitary representation U_s of $\mathscr{P}_+^\uparrow$ on $\mathscr{H}_s$ such that*

$$U_s(\xi)AU_s(\xi)^*\Psi = \tau_\xi(A)\Psi\,,$$

for all $A \in \mathscr{A}$, $\Psi \in \mathscr{H}_s$.

The generators (H_s, P_s) of the space-time translations $\{U_s(1,a);\, a \in \mathbb{R}^2\}$ satisfy the relativistic spectrum condition, i.e. $\operatorname{spec}(H_s, P_s) \subset \bar{V}_+$. *Moreover,* $\operatorname{spec}(H_s, P_s)$ *is purely continuous, i.e. $\mathscr{H}_s$ does not contain any vacuum state.*

(2) *There exists a selfadjoint charge operator* Q, *formally given by*

$$Q = \int dx \partial_x \phi_1(x,t) ,$$

with

$$Q\Psi = 0, \quad \textit{for all} \quad \Psi \in \mathcal{H}_\pm ,$$

$$Q\Psi = q\Psi, \quad \textit{for all} \quad \Psi \in \mathcal{H}_s ,$$

where $q = (\Omega_+, \phi_1(x)\Omega_+) - (\Omega_-, \phi_1(x)\Omega_-) \simeq 2(8\lambda)^{-1/2}$.

Of course Theorem 9 is also true for $\mathcal{H}_{\bar{s}}$, but in (2) q must be replaced by $-q$. In fact, it has been shown in [7] that *space reflection* takes $\mathcal{H}_s$ to $\mathcal{H}_{\bar{s}}$, and establishes an isomorphism between the physics on $\mathcal{H}_s$ and the one on $\mathcal{H}_{\bar{s}}$.

If $\mathcal{O}$ is a double cone $\mathcal{O}(\xi)$ is the region obtained by applying the Poincaré transformation ξ to $\mathcal{O}$. Furthermore $\mathcal{O}_\xi$ is the smallest double cone with base at $t=0$ containing $\mathcal{O}$ and $\mathcal{O}(\xi)$. We let $\mathrm{supp}\,\sigma$ denote the double cone with base $\mathrm{supp}(\partial_x \theta)$.

The proof of Theorem 9, (1) is based on the following

Theorem 10. *To each* $\xi \in \mathcal{P}_+^\uparrow$ *there exists a unitary element* $\Gamma(\xi)$ *of* $\mathcal{A}((\mathrm{supp}\,\sigma)_\xi)$ *such that*

$$\sigma(\tau_\xi(A)) = \Gamma(\xi)\tau_\xi(\sigma(A))\Gamma(\xi)^* ,$$

and

$$(A\Omega_s, U_s(\xi)B\Omega_s) = (\sigma(A)\Omega_+, \Gamma(\xi)\tau_\xi(\sigma(B))\Omega_+) , \tag{1.50}$$

for all A, B *in* $\mathcal{A}$.

Moreover, $\Gamma(\xi)$ *is strongly continuous in* ξ *in every locally normal representation of* $\mathcal{A}$.

Theorem 10 has been proven in [7, 8] for the models studied in this paper and extended in [10], where it is used as one of the central elements of a general theory of Poincaré-covariant superselection sectors. The operators $\Gamma(\xi)$ satisfy the cocycle identity

$$\Gamma(\xi_1 \cdot \xi_2) = \Gamma(\xi_1)\tau_{\xi_1}(\Gamma(\xi_2)) \tag{1.51}$$

and are therefore called local Poincaré cocycles. They were introduced in [7, 8] for purely technical reasons, but it was already realized in [19] that they play a central role in the theory of Poincaré-covariant superselection sectors; see also §6 of [7], [21, 10]. (In particular, the existence of the local Poincaré cocycles $\Gamma(\xi)$ implies that $\pi \circ \sigma$ is a Poincaré-covariant representation of $\mathcal{A}$, whenever π is one [19]).

Moreover $\sigma(A) = \lim_{a \to \infty} \Gamma((1,a))A\Gamma((1,a))^*$, when a tends to ∞ in a space-like direction, [21, 10], i.e. σ can be reconstructed from Γ.)

In the proof of our main result we need an explicit construction of the special cocycles

$$\Gamma(t) = \Gamma(1, a = (0,t)) .$$

It is based in part on the following lemma due to Glimm and Jaffe [38].

Lemma 11 [38]. *(Finite propagation speed.) Let $\mathcal{O}$ be a bounded, open double cone and T some positive number. Suppose the base of $\mathcal{O}_{(1,(\mathbf{0},T))}$ (at time $t=0$) is contained in $[-L/2, L/2]$. Then, for all $t\in(-T,T)$, all $A\in\mathscr{A}(\mathcal{O})$ and all $\Psi\in\mathscr{F}$,*

$$\tau_t(A)\Psi = e^{itH_{\pm\pm}(L)} A e^{-itH_{\pm\pm}(L)}\Psi$$
$$(= e^{itH_{-+}(L)} A e^{-itH_{-+}(L)}\Psi) .$$

Remark. In [38] a different space cutoff was used in the proof of this result. However the proof can easily be extended to the cutoff and boundary conditions used here.

Theorem 12 [7]. *Let $T\equiv T(L)$ be the largest positive number such that the base of the double cone $(\operatorname{supp}\sigma)_{\xi=(1,(\mathbf{0},T))}$ is contained in $[-L/2, L/2]$. Then, for all $t\in(-T,T)$,*

$$\Gamma(t) = e^{iL(\theta)} e^{itH_{-+}(L)} e^{-iL(\theta)} e^{-itH_{++}(L)} ,$$

where $L(\theta)$ has been defined in (1.48).

Remark. In [7] (which appeared before [17]) the choice of boundary conditions in the spatially cutoff Hamiltonians used to construct $\Gamma(t)$ was not the one made here. However, the proof of Theorem 12 is identical to the one of Lemma 3 of [7], once one notes that

$$e^{iL(\theta)} e^{is\delta H_{-+}(L)} e^{-iL(\theta)} = e^{is\delta H_{++}(L)} , \quad \text{for all} \quad s ,$$

if $\operatorname{supp}\partial_x\theta\subset(-L/2, L/2)$; see pp. 284–287 in [7].

This completes our summary of the construction of the vacuum, and the soliton sectors in the anisotropic $|\phi|_2^4$-model. We remark that all these results can be proven for the usual ϕ_2^4-or the pseudoscalar Yukawa$_2$ model, but in these models the construction of the *-automorphism σ is somewhat complicated [10]. For this reason we exemplify our techniques in the context of the simpler $|\phi|_2^4$-model; but see [10].

2. Estimating the Soliton Mass in Terms of the Surface Tension

2.1

In this section we provide the proof of Theorem A', (1.38)–(1.40) (see Section 1.6), i.e. we show that the mass gap $m_s(\lambda)$ is bounded below by the surface tension $\tau(\lambda)$. Estimating $\tau(\lambda)$ is deferred to Section 3.

In a remark following Theorem 9 (Section 1.8) we have noted that the physics on $\mathscr{H}_s$ is isomorphic to the one on $\mathscr{H}_{\bar{s}}$. Therefore we may henceforth concentrate on analyzing the spectrum of the energy-momentum operator (H_s, P_s) on $\mathscr{H}_s$.

According to Theorem 9, (1) $\operatorname{spec}(H_s, P_s)$ is contained in $\bar{V}_+$, is Poincaré-invariant, and purely continuous. Therefore, the mass gap $m_s = m_s(\lambda)$ on $\mathscr{H}_s$ is given by

$$m_s = \inf \operatorname{spec} H_s \geqq 0 . \qquad (2.1)$$

In the introduction to Section 1.8 [see (1.49)] we have noted that

$$\{A\Omega_s : A \in \mathscr{A}\}$$

is dense in $\mathscr{H}_s$; (this follows from the construction of $\mathscr{H}_s$!).

Since $\mathscr{A}$ is the norm closure of $\mathring{\mathscr{A}} \equiv \bigcup \mathscr{A}(\mathscr{O})$

$$\{A\Omega_s : A \in \mathring{\mathscr{A}}\}$$

is dense, too.

Since H_s is selfadjoint on $\mathscr{H}_s$, it has a spectral decomposition

$$H_s = \int_{m_s}^{\infty} \lambda dE_s(\lambda) ,$$

and $\{E_s(\cdot)\}$ are the spectral projections. Thus, given any $\varepsilon_1 > 0$, there exists some $A \in \mathring{\mathscr{A}}$ such that

$$m_s \geqq -\lim_{t\to\infty} 1/t \log(A\Omega_s, e^{-tH_s} A\Omega_s) - \varepsilon_1 . \tag{2.2}$$

Since $1/t \log \|A\|^2$ tends to 0, as $t \to \infty$, for $0 < \|A\| < \infty$, we may suppose that

$$\|A\| = 1 . \tag{2.3}$$

Since $A \in \mathring{\mathscr{A}}$ there exists a bounded, open double cone $\mathscr{O}$ such that

$$A \in \mathscr{A}(\mathscr{O}) . \tag{2.4}$$

From the spectral decomposition of e^{-tH_s} we see that

$$-1/t \log(A\Omega_s, e^{-tH_s} A\Omega_s)$$

is monotone decreasing in t. Hence, given any $\varepsilon_2 > 0$ there exists $\tau = \tau(\varepsilon_2) < \infty$ such that

$$m_s \geqq -1/\tau \log(A\Omega_s, e^{-\tau H_s} A\Omega_s) - \varepsilon_1 - \varepsilon_2 . \tag{2.5}$$

The idea is now to approximate the r.h.s. of (2.5) by the corresponding expressions with space cutoff.

2.2

We first consider

$$F(t) \equiv (A\Omega_s, e^{itH_s} A\Omega_s) . \tag{2.6}$$

By Theorem 10, (1.50)

$$F(t) = (\sigma(A)\Omega_+, \Gamma(t)\tau_t(\sigma(A))\Omega_+) , \tag{2.7}$$

where $t \equiv (1, (\mathbf{0}, t)) \in \mathscr{P}_+^\uparrow$.

By Theorem 10, $\Gamma(t) \in \mathscr{A}((\operatorname{supp}\sigma)_t)$, i.e. $\Gamma(t)$ is a strictly local observable. Moreover $\sigma(A)$ and $\tau_t(\sigma(A))$ are elements of the local algebras $\mathscr{A}(\mathscr{O})$, $\mathscr{A}(\mathscr{O}(t))$, a consequence of (2.4) and the local action of σ; see (1.47).

We may therefore apply Corollary 8 and get

$$(\sigma(A)\Omega_+, \Gamma(t)\tau_t(\sigma(A))\Omega_+) = \lim_{L\to\infty} (\sigma(A)\Omega_+(L), \Gamma(t)\tau_t(\sigma(A))\Omega_+(L)) \,. \tag{2.8}$$

Next, suppose that L is so big that the bases of $\mathcal{O}_T$ and $(\operatorname{supp}\sigma)_T$ at time $t=0$ are contained in $[-L/2, L/2]$. Then, by (1.47) and Lemma 11,

$$\tau_t(\sigma(A))\Omega_+(L) = e^{itH_{++}(L)}\sigma(A)\Omega_+(L) \,, \tag{2.9}$$

for all $|t|<T$, [where we have used $e^{-itH_{++}(L)}\Omega_+(L)=\Omega_+(L)$]. Moreover, by Theorem 12,

$$\Gamma(t) = e^{iL(\theta)}e^{itH_{-+}(L)}e^{-iL(\theta)}e^{-itH_{++}(L)} \,, \tag{2.10}$$

for all $|t|<T$.

Equations (2.9)–(2.10) give

$$(\sigma(A)\Omega_+(L), \Gamma(t)\tau_t(\sigma(A))\Omega_+(L)) = (\sigma(A)\Omega_+(L), e^{iL(\theta)}e^{itH_{-+}(L)}e^{-iL(\theta)}\sigma(A)\Omega_+(L)) \,. \tag{2.11}$$

We set

$$F_L(t) \equiv (e^{-iL(\theta)}\sigma(A)\Omega_+(L), e^{itH_{-+}(L)}e^{-iL(\theta)}\sigma(A)\Omega_+(L)) \,. \tag{2.12}$$

Summarizing (2.6)–(2.11) we have:

For all $|t|<\infty$

$$F(t) = \lim_{L\to\infty} F_L(t) \,. \tag{2.13}$$

We now anticipate the main result of Section 3: For λ sufficiently small and all sufficiently large L

$$H_{-+}(L) \geqq 0(\lambda^{-1}) \,, \tag{2.14}$$

[in particular, $H_{-+}(L)$ is positive].

Thus, for λ small enough and all sufficiently large L, the function $F_L(t)$ is the boundary value of a function $F_L(z)$ analytic in z and uniformly bounded on the half plane $\operatorname{Im} z>0$. By Theorem 9, (1) the same is true of $F(z)$.

Hence (2.13) and the identity principle for analytic functions imply

$$F(i\tau) = \lim_{L\to\infty} F_L(i\tau) \,, \quad \text{for all} \quad \tau>0 \,, \tag{2.15}$$

or, in view of (2.6) and (2.12),

$$(A\Omega_s, e^{-\tau H_s}A\Omega_s) = \lim_{L\to\infty} (e^{-iL(\theta)}\sigma(A)\Omega_+(L), e^{-\tau H_{-+}(L)}e^{-iL(\theta)}\sigma(A)\Omega_+(L)) \,. \tag{2.16}$$

2.3

We now combine inequality (2.5) with Equation (2.16). This gives

Theorem 13. *Given any $\varepsilon>0$, there exists some $L<\infty$ such that*

$$m_s(\lambda) \geqq E_{-+}(L) - E_{++}(L) - \varepsilon .$$

Proof. From (2.5) and (2.16) we learn that, given any $\varepsilon_3>0$, there exists some $L<\infty$ such that, for τ large enough,

$$\begin{aligned} m_s(\lambda) &\geqq -1/\tau \log(e^{-iL(\theta)}\sigma(A)\Omega_+(L), e^{-\tau H_{-+}(L)}e^{-iL(\theta)}\sigma(A)\Omega_+(L)) \\ &\quad -\varepsilon_1-\varepsilon_2-\varepsilon_3 \\ &\geqq -1/\tau \log\{\|e^{-iL(\theta)}\sigma(A)\|^2\|e^{-\tau H_{-+}(L)}\|\} \\ &\quad -\varepsilon_1-\varepsilon_2-\varepsilon_3 . \end{aligned}$$

But $e^{-iL(\theta)}$ is unitary on $\mathscr{F}$, and $\|\sigma(A)\| = \|A\| = 1$, by (2.3). Moreover

$$\begin{aligned} \|e^{-\tau H_{-+}(L)}\| &\leqq e^{-\tau(\inf \operatorname{spec} H_{-+}(L))} \\ &= e^{-\tau(E_{-+}(L)-E_{++}(L))} , \end{aligned}$$

by (1.37), Section 1.6. Taking the logarithm of this inequality and setting $\varepsilon=\varepsilon_1+\varepsilon_2+\varepsilon_3$ completes the proof. Q.E.D.

Since $\varepsilon>0$ is arbitrarily small for L large enough, we have

$$m_s(\lambda) \geqq \overline{\lim_{L\to\infty}}\,(E_{-+}(L)-E_{++}(L)) . \tag{2.17}$$

This is Theorem A′, (1.38)–(1.39). It is well known (see e.g. [39, 42, 47]) that

$$E_{\pm+}(L) = -\lim_{T\to\infty} 1/T \log(\Omega_0, e^{-tH_{\pm+}(L)}\Omega_0) . \tag{2.18}$$

This and the Feynman-Kac formula (1.35)–(1.36) imply

$$\begin{aligned} m_s(\lambda) \geqq \tau(\lambda) &\equiv \overline{\lim_{L\to\infty}}\,(E_{-+}(L)-E_{++}(L)) \\ &= \overline{\lim_{L\to\infty}}\left(-\lim_{T\to\infty}\frac{1}{T}\log\frac{Z_{-+}(L\times T)}{Z_{++}(L\times T)}\right) \end{aligned} \tag{2.19}$$

which completes the proof of Theorem A′, (1.38)–(1.40). Without proof we quote

Theorem 14 [10].

$$m_s(\lambda) = -\lim_{T\to\infty}\frac{1}{T}\log(\Omega_s, e^{-TH_s}\Omega_s) ,$$

i.e. we can set $A=1$ in (2.2).

Remark. A proof of Theorem 14 has been sketched in [9]. For the purposes of this paper Theorem 14 is irrelevant. It is however significant for scattering theory: It tells us that the *soliton fields* constructed in [7] couple the vacuum $\Omega_+\in\mathscr{H}_+$ to the lowest excited state in $\mathscr{H}_s$, i.e. the one soliton state. This is an important input for Haag-Ruelle theory. (The proof of Theorem 14 requires a more subtle version of the Feynman-Kac formula and can therefore not be given here; but see [10].)

In the proof of our main result (see Theorem A, 1.4 and Theorem A′, 1.6) we are left with estimating $E_{-+}(L)-E_{++}(L)$, uniformly in L, or, in other words, with proving (2.14). This is the issue of the next section.

3. Estimating $E_{-+}(L)-E_{++}(L)$

3.1

In this section we have to prove inequality (2.14), i.e. for $\lambda>0$ sufficiently small, there exists some $L_0<\infty$ such that *uniformly* in $L\geqq L_0$

$$H_{-+}(L)\geqq 0(\lambda^{-1}) \tag{3.1}$$

By definition, inf spec $H_{-+}(L)=E_{-+}(L)-E_{++}(L)$, [see (1.37)] so that (3.1) is equivalent to

$$E_{-+}(L)-E_{++}(L)\geqq 0(\lambda^{-1})\,. \tag{3.2}$$

By (2.18) and the Feynman-Kac formula (1.35)–(1.36)—see also (2.19)—

$$E_{-+}(L)-E_{++}(L)=-\lim_{T\to\infty}\frac{1}{T}\log\frac{Z_{-+}(L\times T)}{Z_{++}(L\times T)}\,. \tag{3.3}$$

Let $d\mu_0^{(P,T)}(\phi)$ denote the Gaussian measure on $\mathscr{S}'$ with mean 0 and covariance $(-\Delta^{(P,T)}+1)^{-1}$, where $\Delta^{(P,T)}$ is the two dimensional Laplacean with periodic boundary conditions at $t=\pm T/2$.

We define

$$Z^P_{\pm +}(L\times T)=\int_{\mathscr{S}'} e^{-S_I(L\times T)-\delta S_{\pm +}(L\times T)}d\mu_0^{(P,T)}(\phi)\,, \tag{3.4}$$

where $S_I(L\times T)$ and $\delta S_{\pm +}(L\times T)$ are the actions defined in Section 1.6, (1.22), and (1.26), resp.

The following lemma is by now well known,(see e.g. [42, 49, 8]).

Lemma 15.

$$E_{-+}(L)-E_{++}(L)=-\lim_{T\to\infty}\frac{1}{T}\log\frac{Z^P_{-+}(L\times T)}{Z^P_{++}(L\times T)}\,. \tag{3.5}$$

Remark. This lemma follows from the standard equation

$$\begin{aligned} E_{\pm +}(L)&=-\lim_{T\to\infty}\frac{1}{T}\log(\Omega_0, e^{-TH_{\pm +}(L)}\Omega_0)\\ &=-\lim_{T\to\infty}\frac{1}{T}\log Z_{\pm +}(L\times T)\,, \end{aligned} \tag{3.6}$$

and the independence of the r.h.s. in (3.6) of boundary conditions at $t=\pm\frac{T}{2}$, [42, 49, 8]. In the following periodic boundary conditions at $t=\pm\frac{T}{2}$ are technically somewhat more convenient.

In order to prove (3.1)–(3.2) we now must estimate

$$Z^P_{-+}(L\times T)/Z^P_{++}(L\times T)$$

for $T \gg L$, and uniformly in $L > L_0$.

This is done by means of the *Peierls argument* in the form due to [20] and the chess board estimates of [50]. We follow [8] in presentation.

3.2. Generalities about the Peierls Argument

Let $1/2 \leqq \ell \leqq 3/2$ and $1/2 \leqq \ell \leqq 3/2$ be such that $n_{\boldsymbol{x}} \equiv \ell^{-1}L$ is an odd and $n_t \equiv \ell^{-1}T$ is an even integer; (it is assumed that $L \geqq 1/2$, $T \geqq 1$).

We cover $(L+2\ell)\times T$ with a grid of disjoint rectangles

$$\{\Delta_j : j=(j_1,j_2), j_1=0,\ldots,n_{\boldsymbol{x}}+1, j_2=1,\ldots,n_t\}$$

with sides, parallel to the coordinate axes, of length ℓ, ℓ, respectively.

Let $\chi_\pm$ be the characteristic functions of $[0,\infty)$, $(-\infty,0]$, and

$$\chi_\pm(j) \equiv \chi_\pm\left(\int_{\Delta_j} d^2x\phi_1(x)\right). \tag{3.7}$$

Clearly

$$\chi_-(j)+\chi_+(j)=1. \tag{3.8}$$

Let $\mathscr{C}$ be the family of all functions (called *configurations*) c defined on

$$\bar{\Lambda} \equiv \{j : j=(j_1,j_2), j_1=0,\ldots,n_{\boldsymbol{x}}+1, j_2=1,\ldots,n_t\}, \tag{3.9}$$

with

$$c((0,j_2))=-, c((n_{\boldsymbol{x}}+1,j_2)=+, c(j)\in\{-,+\}, \tag{3.10}$$

for all

$$j\in\Lambda \equiv \{j : j=(j_1,j_2), j_1=1,\ldots,n_{\boldsymbol{x}}, j_2=1,\ldots,n_t\}. \tag{3.11}$$

If we insert the l.h.s. of (3.8) into the r.h.s. of (3.4), for all $j\in\Lambda$, and expand we obtain

$$Z^P_{-+}(L\times T)=\sum_{c\in\mathscr{C}} Z^P_{-+}(c;L\times T) \tag{3.12}$$

where

$$Z^P_{-+}(c,L\times T)=\int_{\mathscr{S}'} e^{-S_I(L\times T)-\delta S_{-+}(L\times T)}\prod_{j\in\Lambda}\chi_{c(j)}(j)d\mu_0^{(P,T)}(\phi). \tag{3.13}$$

We define a *contour* γ to be a *connected* line consisting of sides of the rectangles $\{\Delta_j : j\in\Lambda\}$ decomposing $(L+2\ell)\times T$ into two *disjoint* connected regions B_1 and B_2 with the properties that

$$\left\{x : x=(\boldsymbol{x},t), \boldsymbol{x}=-\frac{L}{2}-\ell, -\frac{T}{2}<t<\frac{T}{2}\right\}\subset B_1$$

$$\left\{x : x=(\boldsymbol{x},t), \boldsymbol{x}=+\frac{L}{2}+\ell, -\frac{T}{2}<t<\frac{T}{2}\right\}\subset B_2.$$

We let $N(\gamma)$ be the collection of *all nearest neighbor pairs* of sites (j^1,j^2) such that $\Delta_{j^1}\in B_1$, $\Delta_{j^2}\in B_2$, (i.e. Δ_{j^1} and Δ_{j^2} have a common face contained in γ).

Given a configuration $c\in\mathscr{C}$ there exists a unique contour $\gamma=\gamma(c)$ *minimizing the area of* B_1, such that if $(j^1,j^2)\in N(\gamma)$ then $c(j^1)=-$ and $c(j^2)=+$ [recall that $c((0,j_2))=-$, $c((n_x+1,j_2))=+$].

We define

$$\begin{aligned}\hat{\chi}_\pm(j &\equiv \chi_\pm(j), \quad \text{for } all \quad \Delta_j\subset L\times T\\ \hat{\chi}_\pm(j)&\equiv 1, \qquad \text{for } all \quad \Delta_j\not\subset L\times T, (\Delta_j\subset(L+2\ell)\times T)\,.\end{aligned}\tag{3.14}$$

From (3.12)–(3.14) and the above definitions we obtain

$$\begin{aligned}\sum_{c\in\mathscr{C}} Z^P_{-+}(c;L\times T)&=\sum_\gamma \sum_{\{c:\,\gamma(c)=\gamma\}} Z^P_{-+}(c;L\times T)\\ &<\sum_\gamma Z^P_{-+}(\gamma;L\times T)\,,\end{aligned}\tag{3.15}$$

where

$$\begin{aligned}Z^P_{-+}(\gamma;L\times T)\equiv &\int_{\mathscr{S}'} e^{-S_I(L\times T)-\delta S_{-+}(L\times T)}\\ &\cdot\prod_{(j^1,j^2)\in N(\gamma)} \hat{\chi}_-(j^1)\hat{\chi}_+(j^2)d\mu_0^{(P,T)}(\phi)\,.\end{aligned}\tag{3.16}$$

Our next task is to estimate $Z^P_{-+}(\gamma;L\times T)$ and prove it has an upper bound contaning a convergence factor $\exp[-0(\lambda^{-1})|\gamma|]$, where $|\gamma|$ is the legnth of the contour γ. For the expert such estimates follow by inspection. In the following we develop some tools used for proving this upper bound.

3.3. *Estimates on* $Z^P_{-+}(\gamma;L\times T)$ *and* $Z^P_{++}(L\times T)$

Clearly

$$\sum_{m=-m_0-1}^{m_0}\left|\gamma\cap\left\{\boldsymbol{x}=\frac{2m+1}{2}\ell\right\}\right|\leqq|\gamma|\,,$$

so that, for some $\bar{m}$, $-m_0-1\leqq\bar{m}\leqq m_0$

$$\left|\gamma\cap\left\{\boldsymbol{x}=\frac{2\bar{m}+1}{2}\ell\right\}\right|\leqq|\gamma|/(2m_0+2)\,.$$

We set $\ell_0\equiv\frac{2\bar{m}+1}{2}\ell$ and define

$$\gamma'_-=\gamma\cap\{\boldsymbol{x}<\ell_0\},\gamma'_+=\gamma\cap\{\boldsymbol{x}>\ell_0\}\,,$$

so that

$$\gamma'_-\cup\gamma'_+\cup(\gamma\cap\{\boldsymbol{x}=\ell_0\})=\gamma\,.$$

Clearly we have

$$|\gamma'_-|+|\gamma'_+|\geqq|\gamma|\left(1-\frac{1}{2m_0+2}\right).\tag{3.17}$$

Let γ_-, γ_+ be the translates of γ'_-, resp. γ'_+ by the space-like vector $(-\ell_0, 0)$, and let $\Theta\gamma_\pm$ be the reflection of $\gamma_\pm$ at $\{\boldsymbol{x}=0\}$. Let $\Sigma_\pm$ be the σ-algebra on $\mathcal{S}'$ corresponding to the open sets $\{x : x=(\boldsymbol{x}, t),\ \boldsymbol{x} \gtrless 0\}$; see Section 1.6 (following Theorem 1) for a definition. If F is a $\Sigma_\pm$-measurable function on $\mathcal{S}'$, ΘF denotes its reflection at $\boldsymbol{x}=0$ which is the $\Sigma_\mp$-measurable function defined by

$$\Theta F(\phi) = F(\phi_\Theta), \quad \text{where, for all} \quad \phi \in \mathcal{S}',$$
$$\phi_\Theta(f) = \phi(f_\Theta) \quad \text{with} \quad f_\Theta(\boldsymbol{x}, t) = f(-\boldsymbol{x}, t),$$

for all $f \in \mathcal{S}^{\times 2}$.

Next we define

$$\begin{aligned} &Z^P_{\pm\pm}(\gamma_\pm \cup \Theta\gamma_\pm; (L \pm 2\ell_0) \times T) \\ &= \int_{\mathcal{S}'} e^{-S_I((L\pm 2\ell_0)\times T) - \delta S_{\pm\pm}((L\pm 2\ell_0)\times T}} \\ &\cdot \prod_{(j^1,j^2)\in N(\gamma_\pm)} \hat{\chi}_\mp(j^1)\hat{\chi}_\pm(j^2) \\ &\cdot \prod_{(j^1,j^2)\in N(\Theta\gamma_\pm)} \hat{\chi}_\mp(j^1)\hat{\chi}_\pm(j^2) d\mu_0(\phi) \end{aligned} \tag{3.18}$$

and $Z^P_{++}(\gamma_- \cup \Theta\gamma_-; (L+2\ell_0)\times T)$ is obtained from $Z^P_{--}(\gamma_- \cup \Theta\gamma_-; (L+2\ell_0)\times T)$ by replacing $\delta S_{--}((L+2\ell_0)\times T)$ by $\delta S_{++}((L+2\ell_0)\times T)$ and $\hat{\chi}_\pm$ by $\hat{\chi}_\mp$ on the r.h.s. of (3.18).

We note that

$$\begin{aligned} &\prod_{(j^1,j^2)\in N(\Theta\gamma_\pm)} \hat{\chi}_\pm(j^1)\hat{\chi}_\mp(j^2) \\ &= \Theta\left(\prod_{(j^1,j^2)\in N(\gamma_\pm)} \hat{\chi}_\mp(j^1)\hat{\chi}_\pm(j^2)\right). \end{aligned} \tag{3.19}$$

Lemma 16.

$$\begin{aligned} Z^P_{-+}(\gamma; L\times T) \leqq\, & Z^P_{++}(\gamma_- \cup \Theta\gamma_-; (L+2\ell_0)\times T)^{1/2} \\ & \cdot Z^P_{++}(\gamma_+ \cup \Theta\gamma_+; (L-2\ell_0)\times T)^{1/2}. \end{aligned}$$

Proof. From Osterwalder-Schrader positivity [43] for $d\mu_0$ follows the Schwarz inequality

$$\begin{aligned} \int_{\mathcal{S}'} FG d\mu_0^{(P,T)}(\phi) \leqq & \left(\int_{\mathcal{S}'} \overline{\Theta F} \cdot F d\mu_0^{(P,T)}(\phi)\right)^{1/2} \\ & \cdot \left(\int_{\mathcal{S}'} \overline{\Theta G} \cdot G d\mu_0^{(P,T)}(\phi)\right)^{1/2}, \end{aligned} \tag{3.20}$$

whenever F is Σ_-- and G is Σ_+-measurable.

We now recall definition (3.16) of $Z^P_{-+}(\gamma; L\times T)$, from which follows

$$Z^P_{-+}(\gamma; L\times T) \leqq Z^P_{-+}(\gamma'_- \cup \gamma'_+; L\times T). \tag{3.21}$$

Since $d\mu_0^{(P,T)}$ is translation invariant, we may shift the integrand in the integral for $Z^P_{-+}(\gamma'_- \cup \gamma'_+; L\times T)$ by $(-\ell_0, 0)$. Then we apply the Schwarz inequality (3.20). This gives

$$Z^P_{-+}(\gamma'_- \cup \gamma'_+; L\times T)$$
$$\leqq Z^P_{--}(\gamma_- \cup \Theta\gamma_-; (L+2\ell_0)\times T)^{1/2} Z^P_{++}(\gamma_+ \cup \Theta\gamma_+; (L-2\ell_0)\times T)^{1/2}.$$

By the $\phi \to -\phi$ symmetry of $d\mu_0^{(P,T)}$ and S_I we have

$$Z^P_{--}(\gamma_- \cup \Theta\gamma_-; (L+2\ell_0)\times T) = Z^P_{++}(\gamma_- \cup \Theta\gamma_-; (L+2\ell_0)\times T). \quad \text{Q.E.D.}$$

Remark. Inequalities such as Lemma 16 are by now a routine. For previous applications of these ideas, see [50, 41]; also [47, 51]. We are left with estimating $Z^P_{++}(\gamma; (L\pm 2\ell_0)\times T)$, with $\gamma \equiv \gamma_\pm \cup \Theta\gamma_\pm$. We set

$$L\pm 2\ell_0 = L_\pm, \phi = \phi_+ + \tilde{\phi}, \tag{3.22}$$

where $\phi_+ = ((8\lambda)^{-1/2}, 0)$, and

$$\tilde{S}_I(L_\pm \times T) = \int_{L_\pm \times T} d^2x[\lambda:(\tilde{\phi}\cdot\tilde{\phi})^2:(x) + \sqrt{2\lambda}:\tilde{\phi}_1^3:(x)], \tag{3.23}$$

where Wick ordering is done with respect to bare mass 1. We note that

$$\begin{aligned}&\lambda:(\tilde{\phi}\cdot\tilde{\phi})^2:(x) + \sqrt{2\lambda}:\tilde{\phi}_1^3:(x)\\ &= \lambda:(\phi\cdot\phi)^2:(x) - \tfrac{3}{4}:\phi_1^2:(x) - \tfrac{1}{4}:\phi_2^2:(x)\\ &+ (64\lambda)^{-1} + \phi_+\cdot\phi(x) - \tfrac{1}{2}|\phi_+|^2.\end{aligned} \tag{3.24}$$

We denote $\hat{\chi}_\pm\left(\int_{\Delta_j} d^2x[\tilde{\phi}_1 + (8\lambda)^{-\frac{1}{2}}]\right)$ by $\tilde{\chi}_\pm(j)$. Applying now Lemma 2, Section 1.6, we obtain, using (3.24)

$$Z^P_{++}(L\times T) = \int_{\mathscr{S}'} e^{-\tilde{S}_I(L\times T)} d\mu_0^{(P,T)}(\phi), \tag{3.25}$$

and

$$\begin{aligned} Z^P_{++}(\gamma; L_\pm \times T) = \int_{\mathscr{S}'} & e^{-\tilde{S}_I(L_\pm\times T)}\\ & \cdot \prod_{(j^1,j^2)\in N(\gamma_\mp)} \tilde{\chi}_\pm(j^1)\tilde{\chi}_\mp(j^2)\\ & \cdot \prod_{(j^1,j^2)\in N(\Theta\gamma_\mp)} \tilde{\chi}_\pm(j^1)\tilde{\chi}_\mp(j^2) d\mu_0^{(P,T)}(\tilde{\phi}).\end{aligned} \tag{3.26}$$

We define the vacuum energy densities of our model by

$$\alpha_T(\lambda) = \lim_{L\to\infty} \frac{1}{L\cdot T} \log Z^P_{++}(L\times T) \tag{3.27}$$

and

$$\alpha_\infty(\lambda) = \lim_{T\to\infty} \alpha_T(\lambda). \tag{3.28}$$

From a technical point of view the following are the main estimates of this paper.

Theorem 17. *For λ so small that Theorem 1 applies (i.e. the expansion of [17] converges)*

(1) $Z^P_{++}(\gamma; L_\pm \times T) \leqq e^{\alpha_T(\lambda)L_\pm \cdot T} e^{-0(\lambda^{-1})|\gamma|}$

(2) $\lim_{T\to\infty} \frac{1}{T} \log Z^P_{++}(L\times T) \geqq L\alpha_\infty(\lambda) - \bar{\beta}(\lambda)$,

with $\bar{\beta}(\lambda) \leqq o(1)$, uniformly in $L \geqq L_0$, for some sufficiently large L_0.

We defer an outline of the proof of Theorem 17 to Section 3.4. Estimates similar to the ones asserted in Theorem 17 are also used in [17, 20, 8].

Corollary 18. *Under the hypotheses of Theorem 17*

$$E_{-+}(L) - E_{++}(L) \geqq 0(\lambda^{-1}),$$

uniformly in $L \geqq L_0$.

Proof. We choose a scale in which $\ell = \ell = 1$, L is an odd and T an even integer. (This just serves to simplify our notations).

$$\begin{aligned} &\sum_\gamma Z^P_{-+}(\gamma; L\times T) \\ &= \sum_n \sum_{\{\gamma:|\gamma|=n\}} Z^P_{-+}(\gamma; L\times T) \\ &\leqq \sum_n \sum_{\{\gamma:|\gamma|=n\}} Z^P_{++}(\gamma_- \cup \Theta\gamma_-; (L+2\ell_0)\times T)^{1/2} \\ &\quad \cdot Z^P_{++}(\gamma_+ \cup \Theta\gamma_+; (L-2\ell_0)\times T)^{1/2} \qquad (3.29) \\ &\leqq e^{\alpha_T(\lambda)L\cdot T} \sum_n \#\{\gamma:|\gamma|=n\} e^{-[|\gamma_-|+|\gamma_+|]0(\lambda^{-1})}, \qquad (3.30) \end{aligned}$$

where $\ell_0 = \ell_0(\gamma)$ is chosen as explained above, see (3.17), such that

$$|\gamma_-| + |\gamma_+| \geqq \left(1 - \frac{1}{2m_0+2}\right)|\gamma| \geqq (1-\varepsilon)|\gamma|,$$

for arbitrary $\varepsilon > 0$ and $m_0 \geqq 1/2\cdot\varepsilon^{-1}$; see (3.17). Furthermore $\#\{\gamma:|\gamma|=n\}$ is the total number of contours γ [such as defined above, (3.13)–(3.14)] of length $|\gamma| = n$. Inequality (3.29) is Lemma 16 and inequality (3.30) follows immediately from Theorem 17, (1). Clearly

$$\#\{\gamma:|\gamma|=n\} = 0,$$

unless $n \geqq T$, since each γ has at least length T, as a consequence of our definition of contours. For $n \geqq T$ a standard argument (a very rough estimate; see e.g. [20]) gives

$$\#\{\gamma:|\gamma|=n\} \leqq L\cdot 3^n. \qquad (3.31)$$

Combining (3.30) and (3.31) we obtain

$$\sum_\gamma Z^P_{-+}(\gamma; L\times T) \leqq e^{\alpha_T(\lambda)L\cdot T} L \sum_{n=T}^\infty 3^n e^{-0(\lambda^{-1})n} \leqq e^{\alpha_T(\lambda)L\cdot T} L e^{-0(\lambda^{-1})T},$$

provided $\lambda > 0$ is sufficiently small. Hence

$$-\lim_{T\to\infty} \frac{1}{T} \log Z^{P}_{-+}(\gamma; L\times T) \geqq -\alpha_{\infty}(\lambda)L + 0(\lambda^{-1}). \tag{3.32}$$

By Theorem 17, (2)

$$\lim_{T\to\infty} \frac{1}{T} \log Z^{P}_{++}(L\times T) \geqq \alpha_{\infty}(\lambda)L - \bar{\beta}(\lambda), \tag{3.33}$$

with

$$\bar{\beta}(\lambda) \leqq o(1), \quad \text{for} \quad L \geqq L_0. \tag{3.34}$$

Adding (3.32) and (3.33) and applying (3.34) completes the proof of Corollary 18. We are left with proving Theorem 17.

3.4. *Proof of Theorem 17*, (1)

Without loss of generality we may assume that $t^{-1}T = 4m$, $m \in \mathbb{Z}_+$, and $\ell^{-1}L$ is an odd positive integer. Here t and ℓ are the lengths of the sides parallel to the time, resp. the space axis of the rectangles $\{\Delta_j\}_{j\in\Lambda}$. To simplify notations we again use a scale such that $t = \ell = 1$.

Let $\mathbb{Z}^{(T)} = \mathbb{Z} \cap \left[-\frac{T}{2}, \frac{T}{2}\right]$; let $\{\Delta_j\}_{j\in\mathbb{Z}\times\mathbb{Z}^{(T)}}$ be a covering of the strip $\left\{x : x = (x, s),\ -\frac{T}{2} \leqq s \leqq \frac{T}{2}\right\}$ by unit squares.

Let $\Delta = \Delta_{j^1} \cup \Delta_{j^2}$, where j^1 and j^2 are nearest neighbor sites in $\mathbb{Z}\times\mathbb{Z}^{(T)}$.

We cover the strip $\left\{x : x = (\boldsymbol{x}, s),\ -\frac{T}{2} \leqq s \leqq \frac{T}{2}\right\}$ with a union of *disjoint* translates of Δ. Since $\frac{T}{2}$ is even $\left\{x : x = (\boldsymbol{x}, s), s \in \left[0, \pm\frac{T}{2}\right]\right\}$ contains an *integer* number of rows consisting of disjoint translates of Δ, and this number is *even* if $j^1 - j^2$ points in the $\boldsymbol{x}$-direction. We set $\varepsilon_1 = 2$, $\varepsilon_2 = 1$ if $j^1 - j^2$ points in the x-direction and $\varepsilon_1 = 1$, $\varepsilon_2 = 2$ if $j^1 - j^2$ points in the t-direction. For $\beta = (\beta_1, \beta_2) \in \mathbb{Z}^2$, we define $\varepsilon\beta = (\varepsilon_1\beta_1, \varepsilon_2\beta_2)$. Let F be some Σ_Δ-measurable function on $\mathscr{S}'$; (see Section 1.6, following Theorem 1).

We define $F^{[\beta]}$ as follows; (see [50]): If β_1 and β_2 are both even $F^{[\beta]}$ is the translate of F to the rectangle $\Delta_{\varepsilon\beta}$ which is a translate of the rectangle Δ whose left lower corner is located at $\varepsilon\beta$. If β_1 (resp. β_2) is odd and β_2 (resp. β_1) is even we reflect F at the line $t = 0$, (resp. $\boldsymbol{x} = 0$) and translate to $\Delta_{\varepsilon\beta}$; (reflections at the t-axis were defined in 3.3; $\Theta : F \to \Theta F$, where ΘF is $\Sigma_{\Theta(\Delta)}$-measurable if F is Σ_Δ-measurable. In the same way reflections at the $\boldsymbol{x}$-axis are defined). If both, β_1 and β_2, are odd we reflect in both lines and translate from $-\Delta$ to $\Delta_{\varepsilon\beta}$.

Next we define a "pressure" associated with F:

$$p_T(F) = \lim_{n\to\infty} (2nT)^{-1} \log\left\{\int_{\mathscr{S}'} \prod_{\varepsilon\beta\in\mathbb{Z}^{(2n)}\times\mathbb{Z}^{(T)}} F^{[\beta]} d\mu_0^{(P,T)}(\phi)\right\} \tag{3.35}$$

and

$$\not{p}_\infty(F) = \lim_{T \to \infty} \not{p}_T(F). \tag{3.36}$$

The limits in (3.35) and (3.36) were shown to exist in [50]; (see also [8]).

We now consider some *special examples*:

$$(1) \quad \Delta = \Delta_{(0,0)} \cup \Delta_{(1,0)}, \qquad F_1 = \chi_+((0,0))\chi_-((1,0))e^{-S_I(\Delta_{(0,0)} \cup \Delta_{(1,0)})}, \tag{3.37}$$

with S_I as in (1.22), Section 1.6.

We set

$$\not{p}_T(F_1) = \not{p}_{T,1}(\lambda). \tag{3.38}$$

(2) A similar example is:

$$\Delta = \Delta_{(0,0)} \cup \Delta_{(0,1)},$$

$$F_1' = \chi_+((0,0))\chi_-((0,1))e^{-S_I(\Delta_{(0,0)} \cup \Delta_{(0,1)})}, \tag{3.39}$$

and we set

$$\not{p}_T(F_1') = \not{p}_{T,1}'(\lambda). \tag{3.40}$$

It is quite easy to show that

$$\not{p}_{\infty,1}(\lambda) = \not{p}_{\infty,1}'(\lambda), \tag{3.41}$$

but we shall not need this.

(3) Next we consider

$$\Delta = \Delta_{(0,0)} \cup \Delta_{(1,0)},$$

$$F_2 = e^{+\Phi_+ \cdot \Phi(\Delta_{(0,0)})} e^{-1/2|\Phi_+|^2 |\Delta_{(0,0)}|} e^{-S_I(\Delta_{(1,0)})} \chi_-((1,0)) \tag{3.42}$$

we note that $|\Delta_{(0,0)}| = 1$ (in the length scale we use) and that the effect of the first two exponentials on the r.h.s. of (3.42) is similar to the one of $\chi_+((0,0))$.

We set

$$\not{p}_T(F_2) = \not{p}_{T,2}(\lambda) = \not{p}_T(\Theta(F_2)). \tag{3.43}$$

The following portraits of $\not{p}_{T,1}(\lambda)$ [resp. $\not{p}_{T,2}(\lambda)$] and $\not{p}_{T,1}'(\lambda)$ are self-explanatory:

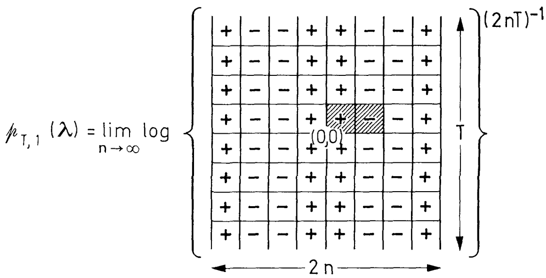

Fig. 1

This also serves as a portrait of $p_{T,2}(\lambda)$.

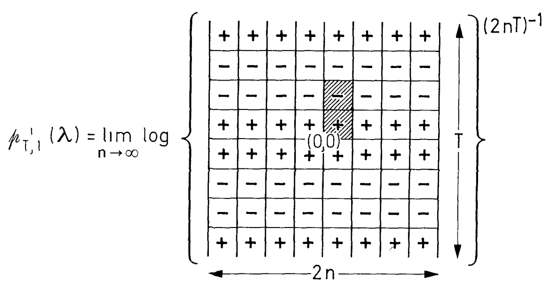

Fig. 2

Let γ be some curve consisting of sides of unit squares in $\{\Delta_j\}_{j\in\mathbb{Z}\times\mathbb{Z}^{(T)}}$ and suppose that $\gamma \subset \left[-\frac{L_\pm}{2}, \frac{L_\pm}{2}\right] \times \left(-\frac{T}{2}, \frac{T}{2}\right)$.

Let $N(\gamma)$ be the collection of all nearest neighbor pairs of sites (j^1, j^2), with j^1, j^2 in $\mathbb{Z}\times\mathbb{Z}^{(T)}$ such that Δ_{j^1} and Δ^{j^2} have one common side in γ.

Let $N_d(\gamma)$ be a *maximal collection* of *disjoint* nearest neighbor pairs of sites $(j^1, j^2) \in N(\gamma)$; $[(j^1, j^2)$ and $(\tilde{j}^1, \tilde{j}^2)$ are disjoint iff $j^1 \neq \tilde{j}^1$ and $j^2 \neq \tilde{j}^2]$.

Let $N_{(d),1}(\gamma)$ be all (j^1, j^2) in $N_{(d)}(\gamma)$ such that both, Δ_{j^1} and Δ_{j^2}, are contained in $L_\pm \times T$, and $j^2 - j^1 = (\pm 1, 0)$.

Let $N'_{(d),1}(\gamma)$ be all (j^1, j^2) in $N_{(d)}(\gamma)$ such that both, Δ_{j^1} and Δ_{j^2} are contained in $L_\pm \times T$, and $j^2 - j^1 = (0, \pm 1)$.

Finally let $N_{(d),2}(\gamma)$ be all (j^1, j^2) in $N_{(d)}(\gamma)$ with

$$\Delta_{j^1} \not\subset L_\pm \times T, \quad \Delta_{j^2} \subset L_\pm \times T, \quad j^2 - j^1 = (\pm 1, 0).$$

Clearly $N_{(d)}(\gamma) = N_{(d),1}(\gamma) \cup N'_{(d),1}(\gamma) \cup N_{(d),2}(\gamma)$.

Let $|N_{d,1}(\gamma)|$ denote the total number of nearest neighbor pairs in $N_{d,1}(\gamma)$, etc., and let $|\gamma|$ be the length of γ. Obviously

$$|N_{d,1}(\gamma)| + |N'_{d,1}(\gamma)| + |N_{d,2}(\gamma)| \geqq \frac{|\gamma|}{4}.$$

Let

$$\tilde{\chi}_{\pm}(j) \equiv \hat{\chi}_{\pm}(j) = \hat{\chi}_{\pm}\left(\int_{\Delta_j} d^2x [\tilde{\phi}_1(x) + (8\lambda)^{-1/2}] \right), \tag{3.44}$$

with $\hat{\chi}_{\pm}(j) = \chi_{\pm}(j)$, when $\Delta_j \subset L_{\pm} \times T$, and $\hat{\chi}_{\pm}(j) = 1$, when $\Delta_j \not\subset L_{\pm} \times T$. In order to prove Theorem 17, (1) we must estimate

$$Z^P_{++}(\gamma; L_{\pm} \times T)$$

$$= \int_{\mathcal{S}'} e^{-\tilde{S}_I(L_{\pm} \times T)} \prod_{(j^1, j^2) \in N(\gamma)} \tilde{\chi}_+(j^1) \tilde{\chi}_-(j^2) d\mu_0^{(P,T)}(\phi), \tag{3.45}$$

$$\leqq \int_{\mathcal{S}'} e^{-\tilde{S}_I(L_{\pm} \times T)} \prod_{(j^1, j^2) \in N_{d,1}(\gamma) \cup N'_{d,1}(\gamma) \cup N_{d,2}(\gamma)} \tilde{\chi}_+(j^1) \tilde{\chi}_-(j^2) d\mu_0^{(P,T)}(\phi). \tag{3.46}$$

Equation (3.45) is Equation (3.26), Section 3.3. Inequality (3.46) follows from the facts that

$$0 \leqq \tilde{\chi}_{\pm}(j) \leqq 1, \quad \text{for all } j,$$

and

$$|N_d(\gamma)| \leqq |N(\gamma)| = |\gamma|.$$

The following is a self-explanatory portrait of $Z^P_{++}(\gamma; L_{\pm} \times T)$.

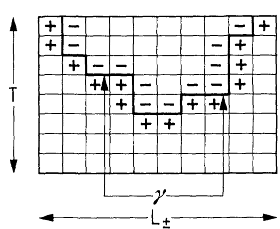

Fig. 3

Figure 3 is supposed to explain the notions introduced above. We recommend as an exercise to the reader to determine $N(\gamma)$, $N_d(\gamma)$, $N_{d,1}(\gamma)$, etc. for the situation sketched in Figure 3 and examine its relation with (3.46)

Proposition 19.

$$|Z^P_{++}(\gamma; L_{\pm} \times T)| \leqq e^{2 \not{p}_{T,1}(\lambda) |N_{d,1}(\gamma)|}$$

$$\cdot e^{2 \not{p}'_{T,1}(\lambda) |N'_{d,1}(\gamma)| + 2 \not{p}_{T,2}(\lambda) |N_{d,2}(\gamma)|}$$

$$\cdot e^{\alpha_T(\lambda)[L_{\pm} \cdot T - 2\{|N_{d,1}(\gamma)| + |N'_{d,1}(\gamma)|\} - |N_{d,2}(\gamma)|]}.$$

Proof. Similar estimates have been used at various places; see [20, 50, 8]. We first bound $Z^P_{++}(\gamma; L_{\pm} \times T)$ by the r.h.s. of (3.46). We then apply Theorem 2.2 of [50] ("chessboard estimate"). We then use the fact that the "pressures" $p_{T,1}(\lambda)$, $p'_{T,1}(\lambda)$, $p_{T,2}(\lambda)$, and $\alpha_T(\lambda)$ are invariant under the substitution $\tilde{\phi}=\phi-\phi_+$, (because we have imposed periodic boundary conditions at $t=\pm T/2$, so that shifting the field does not change the pressure). If we compare the result so obtained with (3.38), (3.40), and (3.43) we get Proposition 19. (There is some difference in notation between this paper and [50], but this should not cause any confusion. Since the proof of Theorem 2.2 of [50] is non-trivial and somewhat lengthy it is not repeated here.) The reader may also consult [8], especially Section 7 and Lemma 4.5.

In order to complete the proof of Theorem 17, (1) it now suffices to estimate

$$p_{T,1}(\lambda)-\alpha_T(\lambda)\,, \qquad p'_{T,1}(\lambda)-\alpha_T(\lambda)$$

and

$$p_{T,2}(\lambda)-1/2\alpha_T(\lambda)\,.$$

This is the content of

Proposition 20. *There exist positive constants λ_0, c_1 (independent of λ_0!); $T_0=T_0(\lambda_0,c_1)<\infty$ such that for all $0<\lambda<\lambda_0$, $T>T_0$*

(1) $\alpha_T(\lambda)\geqq 0(\lambda e^{-T})$

(2) (i) $p_{T,1}(\lambda)\leqq\alpha_T(\lambda)-c_1\lambda^{-1}$
(ii) $p'_{T,1}(\lambda)\leqq\alpha_T(\lambda)-c_1\lambda^{-1}$
(iii) $p_{T,2}(\lambda)\leqq 1/2\alpha_T(\lambda)-c_1\lambda^{-1}$.

Proof. The proof of (1) is simple: We recall (3.27) and (3.25), i.e.

$$\alpha_T(\lambda)=\lim_{L\to\infty}\frac{1}{L\cdot T}\log\left\{\int_{\mathcal{S}'} e^{-\tilde{S}_I(L\times T)}d\mu_0^{(P,T)}(\tilde{\phi})\right\}$$

$$\geqq \lim_{L\to\infty}\frac{1}{L\cdot T}\log\left\{e^{-\int_{\mathcal{S}'}\tilde{S}_I(L\times T)d\mu_0^{(P,T)}(\tilde{\phi})}\right\}=0(\lambda e^{-T})\,;$$

the inequality is Jensen's inequality; moreover

$$\frac{1}{L\cdot T}\int_{\mathcal{S}'}\tilde{S}_I(L\times T)d\mu_0^{(P,T)}(\tilde{\phi})=\lambda\int_{\mathcal{S}'}:(\tilde{\phi}\cdot\tilde{\phi})^2:(1\times 1)d\mu_0^{(P,T)}(\tilde{\phi})\leqq c_2\lambda e^{-T}\,, \tag{3.47}$$

for some finite constant c_2. This inequality is obtained by matching the Wick ordering of $:(\tilde{\phi}\cdot\tilde{\phi}):$ to $d\mu_0^{(P,T)}$ in (3.47) and using that the mass in the covariance of the Gaussian measure is $=1$; (see e.g. the Appendix of [50]).

Estimates (2), (i) and (ii) follow from the defining equations (3.35), (3.38), (3.40), (see also Figs. 1, 2) and inequalities (7.18)–(7.28) of [8] by making the following choices, (we adopt notations from [8], Section 7, (7.20)):

$$0\leqq\chi_1(i)\leqq F_1(i)\equiv e^{-2J\mu}e^{\mu[\phi_1(\square)-\phi_1(\square')]}\,,$$

with

$$J=\mu=\varepsilon(8\lambda)^{-1/2}$$
$$0\leqq\chi_2(i)\leqq F_2(i)\equiv e^{\frac{\sigma J^2}{2}(1-J^{-2}\phi_1(\square)^2)}, \tag{3.48}$$

with

$$\sigma=\tfrac{3}{2}, \quad J=\varepsilon(8\lambda)^{-1/2}, \text{ etc. }; \tag{3.49}$$

in (3.48)–(3.49) $\varepsilon=1/3$ so that

$$\frac{\sigma J^2}{2}=(96\lambda)^{-1}<(64\lambda)^{-1}.$$

In order to obtain estimates (2), (*i*) and (*ii*) one now applies Lemma 7.4, Section 7 of [8] with $\alpha_T(\lambda,1)\geqq\mathcal{O}(\lambda e^{-T})$, and (see (7.25) of [8]),

$$\hat{\alpha}_T(\lambda,1)\leqq-\frac{1}{64\lambda}+c_3 \tag{3.50}$$

for some finite constant c_3 independent of λ.

Note that α and $\hat{\alpha}$ such as defined in this paper differ from α and $\hat{\alpha}$ as defined in [8], here denoted $\alpha_{[8]}$, $\hat{\alpha}_{[8]}$, by $1/64\lambda$, i.e.

$$\alpha_{[8]}=\alpha+1/64\lambda, \quad \hat{\alpha}_{[8]}=\hat{\alpha}+1/64\lambda.$$

Inequality (3.50) is then seen to be Lemma 7.4, (1) of [8].

In order to prove estimate (2), (*iii*) one uses the following [see (3.42)]:

We set $\square=\Delta_{(0,0)}$, $\square'=\Delta_{(1,0)}$

$$\begin{aligned}&e^{\phi_+\cdot\phi(\square)-|\phi_+|^2/2}e^{-S_I(\square')}\chi_-((1,0))\\&\leqq e^{-J|\phi_+|-1/2|\phi_+|^2}e^{|\phi_+|(\phi_1(\square)-\phi_1(\square')}e^{-S_I(\square')}\\&\quad+e^{|\phi_+|\phi_1(\square)-1/2|\phi_+|^2/2}e^{\frac{J^2}{2}(1-J^{-2}\phi_1(\square'))}e^{-S_I(\square')}\end{aligned} \tag{3.51}$$

see (7.20), Section 7 of [8].

One inserts the r.h.s. of (3.51) into the r.h.s. of Equation (3.35) and expands. Then one applies the chess board estimate in the form of (the field theoretic version, e.g. Lemma 7.3 of [8], or) Lemma 4.5, (4.33) of [8] to the resulting terms. The expressions so obtained are bounded by using (7.23) and Lemma 7.4 of [8]. This gives the desired result: Estimate (2), (iii) of Proposition 20. Q.E.D.

Clearly Propositions 19 and 20 give Theorem 17, (1).

3.5. *Proof of Theorem 17*, (2)

We set

$$\alpha_L(\lambda)\equiv\lim_{T\to\infty}\frac{1}{L\cdot T}\log Z^P_{++}(L\times T). \tag{3.52}$$

It is well known that the limit exists and that

$$\alpha_\infty(\lambda) = \lim_{L\to\infty} \alpha_L(\lambda) = \lim_{T\to\infty} \alpha_T(\lambda), \tag{3.53}$$

where $\alpha_T(\lambda)$ and $\alpha_\infty(\lambda)$ have been introduced in (3.27)–(3.28).

If we compare (3.52)–(3.53) with Theorem 17, (2) we see that Theorem 17, (2) is equivalent to

$$L|\alpha_L(\lambda) - \alpha_\infty(\lambda)| \leqq o(1). \tag{3.54}$$

Our strategy to prove (3.54) is as follows:

First we show that $\alpha_L(\lambda)$ and $\alpha_\infty(\lambda)$ are (continuously) differentiable in λ for $\lambda \in (0, \lambda_0)$ (with λ_0 so small that for $0 < \lambda < \lambda_0$ the expansion [17] of Glimm et al. converges). Then we have

$$L|\alpha_L(\lambda) - \alpha_\infty(\lambda)| \leqq L \int_0^\lambda d\lambda' \left| \frac{\partial}{\partial\lambda'} \alpha_L(\lambda') - \frac{\partial}{\partial\lambda'} \alpha_\infty(\lambda') \right|.$$

We then show that

$$L \left| \frac{\partial}{\partial\lambda'} \alpha_L(\lambda') - \frac{\partial}{\partial\lambda'} \alpha_\infty(\lambda') \right| \leqq 0((\lambda')^{-1/2}),$$

and this will complete the proof of (3.54).

The convergence of the expansion [17] implies—by standard arguments—that $\frac{\partial}{\partial\lambda}\alpha_\infty(\lambda)$ exists, and

$$\frac{\partial}{\partial\lambda}\alpha_\infty(\lambda) = \frac{1}{L} \int_{L\times 1} dx \langle U'_\lambda(x) \rangle_{\lambda,+}, \tag{3.55}$$

where

$$U'_\lambda(x) = :(\tilde{\phi} \cdot \tilde{\phi})^2:(x) + \frac{1}{\sqrt{2\lambda}} :\tilde{\phi}_1^3:(x). \tag{3.56}$$

Differentiability of $\alpha_L(\lambda)$ in λ is more obvious. One easily shows (using e.g. the existence of a spatially cutoff Hamiltonian with a unique groundstate) that

$$\frac{\partial}{\partial\lambda}\alpha_L(\lambda) = \frac{1}{L} \int_{L\times 1} dx \langle U'_\lambda(x) \rangle_{\lambda,+}(L),$$

with

$$\langle - \rangle_{\lambda,+}(L) = \lim_{T\to\infty} \langle - \rangle_{\lambda,+}(L\times T). \tag{3.57}$$

We define

$$F_\lambda^{(L)}(x) \equiv |\langle U'_\lambda(x) \rangle_{\lambda,+}(L) - \langle U'_\lambda(x) \rangle_{\lambda,+}|. \tag{3.58}$$

By (3.55)–(3.58) we have

$$L|\alpha_L(\lambda) - \alpha_\infty(\lambda)| \leqq L \int_0^\lambda d\lambda' \left| \frac{\partial}{\partial\lambda'} \alpha_L(\lambda') - \frac{\partial}{\partial\lambda'} \alpha_\infty(\lambda') \right| = \int_0^\lambda d\lambda' \int_{L\times 1} dx F_{\lambda'}^{(L)}(x). \tag{3.59}$$

Thus we are left with estimating $F_{\lambda'}^{(L)}(x)$. Let $d_L(x) \equiv \mathrm{dist}\left(x, \left\{y : y=(y,s), y= \pm \frac{L}{2}\right\}\right)$.

Lemma 21. *For* $0<\lambda<\lambda_0$ *and* L *sufficiently large, there exist finite, positive constants* α *and* β *independent of* λ *and* L *such that*

$$0 \leqq F_{\lambda'}^{(L)}(x) \leqq \alpha(\lambda')^{-1/2} e^{-\beta d_L(x)}.$$

Proof. Let Δ be a unit square in $L \times \mathbb{R}$, $d_L(\Delta) = \min_{x \in \Delta} d_L(x)$, and

$$A_\Delta = \begin{cases} \int_\Delta d^2x \quad :(\tilde{\phi} \cdot \tilde{\phi})^2:(x) \quad \text{or} \\ \int_\Delta d^2x \quad :\tilde{\phi}_1^3:(x). \end{cases}$$

We will prove that there exist finite constants $\beta > 0$ and γ independent of λ and L, for $0 \leqq \lambda < \lambda_0$, L large enough, such that

$$|\langle A_\Delta \rangle_{\lambda,+}(L) - \langle A_\Delta \rangle_{\lambda,+}| \leqq \gamma e^{-\beta d_L(\Delta)}. \tag{3.60}$$

To see this, we note that

$$|\langle A_\Delta \rangle_{\lambda,+}(L) - \langle A_\Delta \rangle_{\lambda,+}| = \left| \int_L^\infty \frac{\partial}{\partial L'} \langle A_\Delta \rangle_{\lambda,+}(L') dL' \right|$$

$$\leqq \int_{L'}^\infty dL' \left| \int_{-\infty}^{+\infty} dt \left\{ \left\langle A_\Delta ; \tilde{S}_I\left(-\frac{L'}{2}, t\right)\right\rangle_{\lambda,+} (L') + \left\langle A_\Delta ; \tilde{S}_I\left(\frac{L'}{2}, t\right)\right\rangle_{\lambda,+} (L') \right\} \right|,$$

where

$$\langle A ; B \rangle_{\lambda,+}(L') = \langle AB \rangle_{\lambda,+}(L') - \langle A \rangle_{\lambda,+}(L') \langle B \rangle_{\lambda,+}(L').$$

By Theorem 4.3.1 of [17] (existence of a mass gap uniform in L'), there exist positive constants δ and ε such that

$$\int_{-\infty}^{+\infty} dt \left\{ \left\langle A_\Delta ; \tilde{S}_I\left(-\frac{L'}{2}, t\right)\right\rangle_{\lambda,+} (L') + \left\langle A_\Delta ; \tilde{S}_I\left(\frac{L'}{2}, t\right)\right\rangle_{\lambda,+} (L') \right\} \leqq \delta e^{-\varepsilon d_{L'}(\Delta)}.$$

Integrating now over L' we obtain (3.60). Inequality (3.60) combined with (3.56) and (3.58) completes the proof of the lemma. Q.E.D.

From Lemma 21 and (3.59) we deduce

$$L|\alpha_L(\lambda) - \alpha_\infty(\lambda)| \leqq \alpha \int_0^\lambda d\lambda' (\lambda')^{-1/2} \int_{L \times 1} dx e^{-\beta d_L(x)}$$

$$\leqq 2\alpha \sqrt{\lambda} \int_0^\infty du e^{-\beta u} \leqq \delta \sqrt{\lambda},$$

for some finite constant δ independent of λ and L. This completes the proof of Theorem 17, (2).

Remark. The simple methods used in VI.2 of [42] to estimate quantities like $L|\alpha_L(\lambda) - \alpha_\infty(\lambda)|$ do not seem to be fine enough to yield inequality (3.54). (They appear to give a divergent estimate.)

4. Summary and Conclusions

We have now completed the proof of our main assertion that the mass gap $m_s(\lambda)$ on the soliton sectors $\mathscr{H}_s$ and $\mathscr{H}_{\bar{s}}$ of the anisotropic $|\phi|_2^4$-model satisfies

$$m_{\bar{s}}(\lambda) \geqq \tau(\lambda) = \mathcal{O}(\lambda^{-1}),$$

where

$$\tau(\lambda) = \overline{\lim_{L\to\infty}} \{E_{-+}(L) - E_{++}(L)\}$$

is the "surface tension" of the anisotropic $|\phi|_2^4$-quantum field theory.

Looking back into our proofs we find the following two features:

I. The inequality $m_s(\lambda) \geqq \tau(\lambda)$ can be proven in *any* two space-time dimensional quantum field model with the following properties:

1) There exist two (or more) disjoint ("orthogonal") clustering physical vacua ω_+ and ω_-.

2) There exists a bounded open double cone $\mathcal{O} \subset M_2$ and a *automorphism σ such that

$$\omega_+ \circ \sigma(A) = \omega_-(A), \text{ for all } A \in \mathscr{A}\ (\mathcal{O}_L),$$

where $\mathcal{O}_L$ is the space-like complement of $\mathcal{O}$ to the left of $\mathcal{O}$, and

$$\omega_+ \circ \sigma(A) = \omega_+(A), \text{ for all } A \in \mathscr{A}\ (\mathcal{O}_R),$$

where $\mathcal{O}_R$ is the space-like complement of $\mathcal{O}$ to the right of $\mathcal{O}$.

3) The *automorphism σ is unitarily implementable on the Fock space of the *free* fields corresponding to the basic, interacting fields of the model (which is assumed to be the limit of spatially cutoff models that can be constructed on Fock space).

The proof of the inequality $m_s(\lambda) \geqq \tau(\lambda)$ is therefore reduced, for a large class of models in two space-time dimensions (satisfying the "locally Fock property" of Theorem 7, see [39]) to a problem concerning free fields, namely 3). In general, this free field problem is non-trivial! But for some models other than the anisotropic $|\vec{\phi}|_2^4$-theory this problem can be solved.

The inequality $m_s(\lambda) \geqq \tau(\lambda)$ has also been proven by one of us (J.F.) for the quantum sine-Gordon equation by somewhat different methods similar to the ones used in [9] and for $\lambda\phi_2^4$, [10].

II. A non-trivial lower bound on $\tau(\lambda)$ [in the case of the ϕ_2^4- and the anisotropic $|\phi|_2^4$-models: $\tau(\lambda) \geqq 0(\lambda^{-1})$] is available in all models for which the Peierls argument for the "surface tension" converges, (i.e. an analogue of Theorem 17 holds). This includes the ϕ_2^4-model, the pseudoscalar Yukawa$_2$ model and a large class of lattice field theories, in the multiple phase region.

Two natural questions arise:

A. Is $m_s(\lambda) = \tau(\lambda)$?

We believe that this equation can be proven in the region of convergence of the expansion of [17] by making a more careful use of the powerful estimates of [17]. This is not attempted here. However, we emphasize that the equation $m_s(\lambda) = \tau(\lambda)$ will presumably be crucial in a proof of our conjecture that $m_s(\lambda)$ is the mass of a stable particle, the quantum soliton.

B. Is there a systematic, asymptotic expansion of $\tau(\lambda)$, e.g. of the form

$$\tau(\lambda) \approx a_{-1}\lambda^{-1} + a_0 + \sum_{n=1}^{\infty} a_{n/2}\lambda^{n/2} ?$$

Whereas it appears to be easy to guess such expansions and find recipes for computing the coefficients a_{-1}, a_0, $a_{1/2}, \ldots$ it is very non-trivial to *prove* that such guesses are correct in the sense that they give expansions asymptotic to the true $\tau(\lambda)$. Both, A and B deserve further investigations!

The most crucial question, however, is, whether $m_s(\lambda)$ is indeed an *isolated eigenvalue* of the mass operator on the soliton sectors, so that the soliton is a stable particle, and a Haag-Ruelle scattering theory (see [7, 10]) can be applied.

This paper essentially reduces this problem to analyzing detailed properties of the spectrum of $H_{-+}(L)$, for $L \to \infty$. We feel that this could be done by modifying known techniques.

Formal arguments indicate that $m_s(\lambda)$ is separated from the rest of the spectrum of the mass operator on the soliton sectors by an upper gap $\propto m(\lambda)$, where $m(\lambda)$ is the mass gap in the vacuum sector, and that the mass spectrum in the interval $[m_s(\lambda), m_s(\lambda) + m(\lambda))$ is discrete, (possibly containing eigenvalues corresponding to soliton-meson bound states).

Finally we should like to conjecture that the existence of quantum solitons in models like $\lambda|\boldsymbol{\phi}|_2^4$ and $\lambda\phi_2^4$ in the two phase region is *incompatible* with Borel summability of the perturbation series in $\lambda^{1/2}$ for the Schwinger functions set up and proven to be asymptotic in [17].

Acknowledgements. We thank the ZiF of the University of Bielefeld for hospitality extended to us in summer 1976 when we discovered our common interest in the problems solved in this paper. Two of us (J.B. and B.G.) would like to thank the DFG for financial support.

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Communicated by A. Jaffe

Received October 3, 1977