Commun. math. Phys. 58, 291—297 (1978)

Communications in
Mathematical
Physics

The Singular Holonomy Group

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Abstract. The "fibre" of the extension of the frame-bundle of a space-time over a b-boundary point p is a homogeneous space $\mathscr{L}/G_p$. It is shown that G_p can be found by a construction like that for a holonomy group, and that it contains a subgroup determined by the Riemann tensor. Near a curvature singularity one would expect $G_p = \mathscr{L}$.

1. Introduction

A singular space-time is one in which there is a curve γ (not necessarily causal) that cannot be extended further in the direction of increasing parameter—i.e. it does not stop at a point in space-time—and that has finite (Euclidean) length measured in a frame parallely propagated along it [6]. At the time when Hawking, Penrose and Geroch showed that being singular could be a general property of space-time, attempts were made to define singular points, endpoints of such incomplete curves, that formed a boundary to space-time. In particular, Schmidt [6] produced the elegant construction of the b-boundary involving a natural Cauchy completion of the bundle of all (pseudo-)orthonormal frames (i.e. frames with respect to the Lorentz metric). The usefulness of this as a means of providing a canonical boundary has been disputed; but it certainly can provide valuable insight into what is going wrong at a singularity, into why a curve is forced to be incomplete.

We shall see that a particular group arising in the course of Schmidt's construction contains information about the unbounded part of the curvature, and about the "topological" peculiarities that can complicate a singular situation. We thus have a tool for separating and classifying different aspects of singularities, a necessary step towards understanding their physical significance.

In forming the b-boundary one first forms the closure $\mathrm{Cl}_b L(M)$ of the (pseudo-)orthonormal frame bundle $L(M)$ with respect to a positive definite

0010-3616/78/0058/0291/$01.40

Riemannian metric g_b on $L(M)$ [6]. Then the Lorentz group $\mathscr{L}_+^\uparrow$ (we assume full orientability) acts on this closure, defining a projection

$$\pi: \mathrm{Cl}_b L(M) \to \mathrm{Cl}_b M := \mathrm{Cl}_b L(M)/\mathscr{L}_+^\uparrow$$

that extends the bundle projection $L(M) \to M$.

This paper analyses the structure of the "fibres" $L_p(M) := \pi^{-1}(p)$ for $p \in \partial_b M := \mathrm{Cl}_b M \backslash M$. It is well known ([8] and Theorem 1 below) that $L_p(M)$ is a homogeneous space of the Lorentz group, of the form $\mathscr{L}_+^\uparrow / G_p$. The group G_p is the *singular holonomy group* at p (determined up to conjugacy), whose construction is the subject of this paper. It will be shown to be the group of Lorentz transformations that can be generated by parallel propagation round "arbitrarily short" loops near p [see the definition of $G_p(\kappa)$ below].

The group G_p contains a subgroup $\tilde{G}_p$ (which usually is the whole of G_p) generated by groups that are completely determined by the Riemann tensor. When p is a curvature singularity [2, 3] then, except under very special conditions that are discussed in § 4, $G_p = \tilde{G}_p = \mathscr{L}_+^\uparrow$ and $L_p(M)$ is a single point.

2. Definition and Basic Properties

M is a fixed space- and time-orientable Hausdorff space-time.

The metric will be assumed to be at least C^2, although all the results can without difficulty be reformulated in the C^{2-}-case, using the techniques of [1]. Throughout we shall be considering a fixed horizontal curve $\kappa:[0, 1) \to L(M)$ ending over a point $p \in \mathrm{Cl}_b M$; i.e. κ extends to curve with $\kappa(1) \in L_p(M)$. All curves are assumed differentiable except where specified.

$$\Omega(x) := \{c:[0, 1] \to M \mid c(0) = c(1) = x\}$$

is the loop space at $x \in M$.

For $\gamma:[0, 1] \to L(M)$, $l(\gamma)$ is the b-metric length of γ.

If $c \in \Omega(x)$ and $u \in L_x(M)$, then we can lift c to a horizontal curve $\bar{c}_u$ with $\pi \circ \bar{c}_u = c$, $\bar{c}_u(0) = u$. Thus we can define

$$l(u, c) := l(\bar{c}_u)$$

$$L(u, c) \in \mathscr{L}_+^\uparrow : L(u, c)\bar{c}_u(0) = \bar{c}_u(1).$$

For $a \in \mathbb{R}$, $u \in L(M)$, we define

$$\Phi^a(u) := \{L(u, c) \mid c \in \Omega(\pi(u)) \wedge l(u, c) \leqq a\}$$

(the part of the holonomy group accessible along curves of length $\leqq a$).

Finally, we define a set $G_p(\kappa)$ that will be shown in Theorem 1 to be the singular holonomy group.

$$G_p(\kappa) := \bigcap_{a \in \mathbb{R}^+} \overline{\bigcup_{t \in [0, 1)} \Phi^a(\kappa(t))}.$$

Proposition 1. *If* $p \in M$ *then* $G_p(\kappa) = \{1\}$ ($1 =$ *identity in* $\mathscr{L}_+^\uparrow$).

Proof. For any u and $c \in \Omega(\pi(u))$, the point $c(t)$ will always lie in a normal neighbourhood of $\pi(u)$ provided $l(u,c)$ is small enough. Since the connection coefficients are bounded in such a neighbourhood we have, for $L(u,c) \in \Phi^a(u)$, that $\|L(u,c)\| \leqq f(a,u)$ for small enough a, where $f(a,u) \to 0$ as $a \to 0$. By continuity the range of t can be extended to $[0,1]$, if $p \in M$, so that $f(a,\kappa(t)) \to 0$ $(a \to 0)$ uniformly in t. Thus the result follows. □

Corollary 1. *For any $p \in \mathrm{Cl}_b M$ and κ ending over p, choose $0 \leqq u < 1$ and define $\kappa^u(t) := \kappa(u + t(1-u))$. Then $G_p(\kappa^u) = G_p(\kappa)$.*

Proof. Define κ'' by $\kappa''(t) := \kappa(ut)$. Clearly

$$G_p(\kappa) = G_p(\kappa^u) \cup G_{\pi(\kappa(u))}(\kappa'') .$$

But, by Lemma 1, $G_{\pi(\kappa(u))}(\kappa'') = \{1\} \subset G_p(\kappa^u)$. □

Proposition 2. *$G_p(\kappa)$ is a closed subgroup of $\mathscr{L}^\uparrow_+$.*

Proof. The only non-obvious requirement is closure under multiplication. Let $a \in \mathbb{R}^+$ and $L_1, L_2 \in G_p(\kappa)$. Choose u so that, with κ^u as in Corollary 1, $l(\kappa^u) \leqq a/4\|L_1 L_2\|$. Since by Corollary 1 $L_1, L_2 \in G_p(\kappa^u)$, we can choose $t_i^j, c_i^j (i=1,2; j=1,2,\ldots)$ so that $t_i^j \in [u,1)$, $c_i^j \in \Omega(\pi(\kappa(t_i^j)))$, $l(\kappa(t_2^j), c_2^j) \leqq a/4$, $l(\kappa(t_1^j), c_1^j) \leqq a/4\|L_2\|$, and $L(\kappa(t_i^j), c_i^j) \to L_i (j \to \infty; i=1,2)$. Then define $c^j \in \Omega(\pi(\kappa(t_2^j)))$ by

$$c^j(s) = \begin{cases} c_2^j(4s) & 0 \leqq s < \frac{1}{4} \\ \pi(\kappa(t_2^j + (t_1^j - t_2^j)(4s-1))) & \frac{1}{4} \leqq s < \frac{1}{2} \\ c_1^j(4s-2) & \frac{1}{2} \leqq s < \frac{3}{4} \\ \pi(\kappa(t_1^j + (t_2^j - t_1^j)(4s-3))) & \frac{3}{4} \leqq s < 1 . \end{cases}$$

By construction $l(\kappa(t_2^j), c^j) \leqq a$ and $L(\kappa(t_2^j), c^j) \to L_1 L_2$. Thus $L_1 L_2 \in G_p(\kappa)$. □

Proposition 3. *If κ_1, κ_2 are two curves ending over p, then $G_p(\kappa_1)$ and $G_p(\kappa_2)$ are conjugate in $\mathscr{L}^\uparrow_+$.*

Proof. There is a Lorentz transformation L such that κ_1 and $\kappa_2' := L\kappa_2$ end at the same point in $\mathrm{Cl}_b L(M)$. Hence there are connecting curves $\gamma_i (i=1,2,\ldots)$ joining $\kappa_1(t_{1i})$ and $\kappa_2'(t_{2i})$, where (t_{1i}) and (t_{2i}) tend to 1 and $l(\gamma_i) \to 0$; indeed, we can take γ_i to consist of a horizontal part γ_i^0, followed by a part γ_i^* in the fibre of $\kappa_2'(t_{2i})$, with γ_i^* connecting frames differing by a Lorentz transformation $L_i \to 1 (i \to \infty)$.

Let $a \in \mathbb{R}^+$ and $L_0 \in G_p(\kappa_1)$. Choose N so large that for $i \geqq N$, $\|L_i\| \leqq 2$, $l(\gamma_i^0) \leqq a/12$, $l(\kappa_1^{t_{1i}}) \leqq a/12$; then choose t_i', $t_{1i} \leqq t_i' < 1$, so that there is a $c_i \in \Omega(\kappa_1(t_i'))$ with $L(\kappa_1(t_i'), c_i) \to L_0 \in G_p(\kappa_1^{1i}) = G_p(\kappa_1)$ and $l(\kappa_1(t_i'), c) \leqq a/12$. Define $c_i' \in \Omega(\pi(\kappa(t_{2i})))$ to consist of the following curve segments (reparameterized as in Lemma 2): (i) $c_1 = \pi \circ \gamma_i^{0-}$, (ii) $c_2 = \pi \circ (\kappa_1 | [t_{1i}, t_i])$, (iii) c_i, (iv) c_2^-, (v) c_1^-; where the superscript $^-$ denotes describing a curve backwards.

This will give $L(\kappa_2'(t_{2i}), c_i') \to L_i L_0 L_i^{-1}$, $l(\kappa_2'(t_{2i}), c_i') \leqq a$. As $i \to \infty$, $L_i L_0 L_i^{-1} \to L_0$ and so $L_0 \in G_p(\kappa_2')$. Thus $G_p(\kappa_2') = G_p(\kappa_1)$.

But note now that, for any t and $c\in\Omega(\pi(\kappa_2(t)))$, $L(\kappa_2(t), c)=L^{-1}L(\kappa_2'(t), c)L$. Hence $G_p(\kappa_2)=L^{-1}G_p(\kappa_1)L$. □

In view of this Lemma we can usually drop the explicit reference to κ, and speak loosely of G_p as being "the" singular holonomy group at p.

Theorem 1. *$L_p(M)$ is isomorphic (as a homogeneous space) to $\mathscr{L}_+^\uparrow/G_p$.*

Remark. The spaces $\mathscr{L}_+^\uparrow/G_p(\kappa)$ for different κ are isomorphic.
Proof. By construction $\mathscr{L}_+^\uparrow$ acts transitively on $L_p(M)$. Any point $u\in L_p(M)$ can be represented as the limit of a sequence $(\kappa(t_i))$ $(t_i\to 1)$, where κ is a horizontal curve ending over p. Moreover, for any $g\in G_p(\kappa)$ we can choose (t_i) so that there are curves $c_i\in\Omega(\pi(\kappa(t_i)))$ with the transformations $g_i:=L(\kappa\ (t_i), x_i)\to g$ and $l(\kappa(t_i), c_i)\to 0$. Thus the sequence $(g\kappa(t_i))\to u$, and so $gu=u$.

The argument can be reversed to show that if $gu=u$ for some $g\in\mathscr{L}_+^\uparrow$, then $g\in G_p(\kappa)$. Thus $G_p(\kappa)$ is the isotropy group of u. Whence the result follows. □

3. The Generation G_p

The ordinary holonomy group $\Phi(u)$ contains various subgroups: $\Phi(u)\supset\Phi^0(u)\supset\Phi^*(u)\supset\Phi'(u)$—the restricted (curves homotopic to zero), local and infinitesimal holonomy groups, respectively [4]. In the singular case the situation is more complex. In particular, while there are analogues of the infinitesimal group generated solely by the values of the Riemann tensor and its derivatives at points on κ, or by its unbounded components, these analogous groups are not necessarily subgroups of G_p. The Riemann tensor can be unbounded, for instance, without thereby producing a non-trivial G_p.

To overcome this we must examine an integral of the curvature, rather than its values at points. This is reasonable on general grounds: one feels that the "severity" of a singularity should take into account the extent, as well as the intensity, of the curvature. With this in mind, we give a definition parallel to that of G, but using only the $\mathfrak{l}$-valued curvature 2-form Ω (where $\mathfrak{l}$ is the Lie algebra of $\mathscr{L}_+^\uparrow$). "Group S" denotes the Lie group generated by S and $B_\varepsilon(u)$ is the ε-ball in $L(M)$ centred on u.
Definition 1'. $G_p'(\kappa):=\bigcap_{k\in\mathbb{R}^+}\text{Group}\left[\bigcap_{a\in\mathbb{R}^+}\bigcup_{t\in[0,1)}\left\{\int_\Sigma\Omega\right|\Sigma$ is a compact 2-surface with boundary, contained in $B_{a'}(\kappa(t))$ with $l(\partial\Sigma)\leqq 2\pi a'$.

$$a'=\min\left(a, k\left(\sup_{B_a(\kappa(t))}|R_{\alpha\beta\gamma\delta}|\right)^{-1/2}\right)\Bigg\}\Bigg].$$

Proposition 4. $G_p'(\kappa)\subset G_p(\kappa)$.

Proof. The basis of the relation between curvature and holonomy is the formula $\left[\text{expressed in a suitable coordinate basis with } u=\left(\left.\frac{\partial}{\partial x^\alpha}\right|_0\right)\right]$

$$L(u,\partial\Sigma)^\alpha{}_\beta=(1+O(a^2\|R\|))\int_\Sigma R^\alpha{}_{\beta\gamma\delta}\,dx^\gamma\wedge dx^\delta$$

where a is the diameter of the region involved, the coordinate area of Σ is bounded by a^2 and $\|R\|$ is the supremum of the coefficient $R^\alpha{}_{\beta\gamma\delta}$. From the definition it is immediate that any element of $G'_p(\kappa)$ can be expressed as a limit of multiples of Lorentz transformations of the above form, with arbitrarily small total curve-length, and so is in $G_p(\kappa)$. $\square$

The group $G'_p(\kappa)$ is defined from the unbounded part of the curvature, as seen from κ. The bounded part may also contribute to G_p, however, in the following sense.

Let H be a subgroup of $G_p(\kappa)$. Analogously to the definitions of § 1, define

$$\Phi^a_H(u) := \{L \in \mathscr{L}^\uparrow_+ | (\forall h \in H)(\exists c \in \Omega(\pi(u))(L = L(u, c) \wedge l(hu, c) \leqq a)\} .$$

$\Phi^a_H(u)$ is, roughly speaking, the part of the holonomy group accessible along curves whose length, even when boosted by $h \in H$, is $\leqq a$.

We convert $\Phi^a_H(u)$ into a subset of $\mathfrak{l}$ by setting

$$\phi^a_H(u) := \{\lambda \in \mathfrak{l} | 0 \leqq \varepsilon \leqq 1 \quad \Rightarrow \exp \varepsilon\lambda \in \Phi^a(u)\}$$

and finally take the group generated by the superior limit of this as $t \to 1$ on κ, for arbitrarily small a:

$$\Phi_H(\kappa) := \bigcap_{a \in \mathbb{R}^+} \operatorname{Group}\left[\bigcap_{s \in [0,1)} \bigcup_{t \in [s,1)} \phi^a_H(\kappa(t))\right] .$$

This definition is admittedly somewhat contrived, but in practice it could well be simple to identify subgroups of Φ_H. For example, if H acts only in a timelike 2-plane, then we could examine curves c lying, as it were, "normal" to this plane, unaffected by H: the components of the Riemann tensor (whether bounded or not) in this normal direction could then generate a subgroup of $\Phi_H(\kappa)$.

Let G be a subgroup of $\Phi_H(\kappa)$, and $\mathfrak{g}$ its Lie algebra.

Proposition 5.

$$K := \operatorname{Group}\left[\bigcup_{\phi \in \mathfrak{g}} \bigcap_{\varepsilon \in \mathbb{R}^+} \{\operatorname{Ad} h(\varepsilon\phi) | h \in H\}\right]$$

is a subgroup of $G_p(\kappa)$.

Proof. Fix $a \in R^+$. Let k be an element of K for which there is a ϕ satisfying

(i) $\exists \delta > 0,\ \delta \cdot \phi \in \bigcap_s \bigcup_t \phi^{a/5}_H(\kappa(t))$.

(ii) $\forall \varepsilon,\ \exists h \equiv h(\varepsilon) \in H$ such that

$$k = \operatorname{Exp} \operatorname{Ad} h(\varepsilon\phi) = h^{-1}(\operatorname{Exp} \varepsilon\phi)h .$$

Choose s so that $\|h(\delta)\| \cdot l(\kappa^s) \leqq a/5$. Then we can find a t, $s \leqq t < 1$, and a $c \in \Omega(\pi(\kappa(t)))$ such that $L(\kappa(t), c) = \delta\phi$, and $l(h\kappa(t), c) \leqq a/6$.

Since $H \subset G_p(\kappa)$ we can also find $t_i (i = 1, 2, \ldots)$ with $t \leqq t_i < 1$, so that $L(\kappa(t_i), c_i) \to h$, $l(\kappa(t_i), c_i) \leqq a/5$.

As in Proposition 3, construct the curve c'_i consisting of (i) c_i, (ii) $(\pi \circ \kappa)^- |(t, t_i)$, (iii) c, (iv) $(\pi \circ \kappa)|(t, t_i)$, (v) c_i^-. This generates a Lorentz transformation tending to $k (i \to \infty)$ and so $k \in G_p(\kappa)$.

The case where k involves a product in G or a product of such k's can be reduced to the above case, showing that the entire group is in $G_p(\kappa)$. $\square$

4. Discussion

Various sources contribute to the group G_p that determines the degeneracy of the fibres $L_p(M)$. We have just described the contributions from

(i) *the singular curvature, via* G'_p;

(ii) the whole curvature, via $\Phi_H(\kappa)$ and K.

In addition we could have

(iii) "topological" contributions from accumulating singularities like those discussed in [3];

(iv) non-local contributions, arising from null geodesics whose tangent vector can be boosted to zero under G_p which link p with more distant parts of space-time.

We shall conclude by illustrating the role played by Proposition 5 [item (ii) above] when we have already identified a subgroup H of G_p, arising, say, from singular curvature or from topological effects, and a subgroup G of $\Phi_H(\kappa)$ arising from the whole of the curvature. Then one knows that G_p must contain the group $\tilde{G}$ generated by H and the "K" of Proposition 5. Table 1 lists the relevant possibilities for the special case $G \supset H$, with H non-compact, using the notation of [5]. It will be seen that a maximal $G_p = \mathscr{L}$ is obtained in most cases where the group $G = \Phi_H(\kappa) = \mathscr{L}$ (as would be expected in the presence of any generic regular curvature), provided that the singular group H is non-compact and not too small.

If a curvature singularity is to avoid being completely degenerate ($G_p = \mathscr{L}$), the groups involved must therefore be very specialised. We could compare the situation with that of the ordinary infinitesimal holonomy groups of general relativity, which should be analogous to Φ_H. Here restricted groups arise only *when there are relationships between the principle pressures/density and* corresponding components of the Weyl tensor. These are physically reasonable

Table 1. The subgroup $\tilde{G}$ of the singular holonomy group generated by groups H and K, in terms of H (due to singular behaviour) and G (a regular part). K is given from H and G by Proposition 5. The notation follows [5]

dim H	H	G	$\tilde{G}$	dim H	H	G	$\tilde{G}$
1	Q	P	Q	2	Ω	J, D	J
		D, K	K			SL(2, ℝ)	Ω
		$\mathscr{L}$	$\mathscr{L}$		Σ	$\mathscr{L}$	D
	Λ	Σ	Λ			J, D, Δ, K	Σ
		K, Δ, D	Σ		P	$\mathscr{L}$	$\mathscr{L}$
		$\mathscr{L}$	D			D	D
	Λ_0	J	Λ_0	3	J, K	$\mathscr{L}$	$\mathscr{L}$
	Λ_σ	J	Σ			D	D
	Π	P	Π		Δ	$\mathscr{L}$	D
		Ω, SL(2, ℝ)	Ω			D	Δ
		$D, \mathscr{L}, J$	D		SL(2, ℝ)	$\mathscr{L}$	$\mathscr{L}$
				4	D	$\mathscr{L}$	$\mathscr{L}$

only in the vacuum case, when groups Σ and D are the only non-trivial possibilities, with Petrov types N and III respectively [7].

One can thus expect that, in some sense, the "generic" situation for a curvature singularity is a completely degenerate $L_p(M)$; and that this will also occur when the singular part of the Riemann tensor is specialised, provided that the regular part remains sufficiently general.

Finally, we should note that the construction of the singular holonomy group provides much information about the singularity-structure of the space-time, which would be valuable in its own right whether or not one adopts the b-boundary as the "correct" boundary of space-time. Indeed, the degeneracy of the fibres shown here implies that the b-boundary is probably unsatisfactory, as was first explicitly realised by R. Johnson [8]. If one adopts instead an "enlargement" of the b-boundary [9] that removes at least some of its unsatisfactory features, then the construction described here transfers in an obvious and natural way to give singular holonomy groups for the points of the enlarged boundary.

Acknowledgements. This paper grew out of discussions, prompted by [8], with B. G. Schmidt, for whose suggestions I am most grateful.

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Communicated by R. Geroch

Received September 23, 1977