Commun. math. Phys. 58, 167—194 (1978)

Communications in
Mathematical
Physics

On the Characterization of Relativistic Quantum Field Theories in Terms of Finitely Many Vacuum Expectation Values. II

Erwin Brüning

Fakultät für Physik der Universität, D-4800 Bielefeld, Federal Republic of Germany

Abstract. The problem of uniqueness of monotone continuous linear extensions of

$$T_{(2N)}=\{1, T_1, \ldots, T_{2N}\}\in E'_{(2N)}=\prod_{n=0}^{2N} E'_n$$

is solved. A characterization of a relativistic QFT in terms of finitely many VEV's is derived. All results are illustrated by an explicit discussion of the extension problem for special cases of $T_{(4)}=\{1,0,T_2,T_3,T_4\}$. This discussion contains explicitly necessary and sufficient conditions on $T_{(4)}$ for the existence of minimal extensions and some convenient sufficient conditions.

1. Introduction

This note continues the discussion of the problem of characterizing a relativistic Quantum Field Theory by finitely many vacuum expectation values which we started in [1].

While the first part contains

(i) an exposition of the problem (which is shown to be the problem of monotone continuous linear extension with additional linear constraints),

(ii) a suggestion for constructing monotone continuous linear (m.c.l.) extensions,

(iii) the definition and some discussion on the relevance of the notion of minimal extensions,

(iv) necessary and sufficient conditions for the existence of minimal extensions,

(v) several applications to the simplest cases;

this part concentrates on

(i) the problem of uniqueness of m.c.l. extension,

(ii) minimal extensions in relativistic QFT,

(iii) the characterization of a relativistic QFT by $T_{(4)}=\{1,T_1,T_2,T_3,T_4\}$ (notation as in 1).

The problem of uniqueness of m.c.l. extension is solved in the following way (we use the notations of 1): The notion of a m.c.l. functional to be 'uniquely determined by $T_{(2N+1)}=\{1, T_1, \ldots, T_{2N+1}\}$" is introduced and a criterium for this is proven. The hypothesis of this criterium are conditions on $T_{(2N+2)} = \{1, T_1, \ldots, T_{2N+2}\}$. Thus a condition for the uniqueness of m.c.l. extension results. Clearly there are only three possibilities for a given $T_{(2N)} \in \underline{E}'_{(2N)}$:

A. There may be more than one m.c.l. extension.

B. There is at most one m.c.l. extension.

C. There is no m.c.l. extension at all.

These possibilities are expressed in terms of $T_{(2N)}$ (Section 2). In this context minimal extensions appear in a natural way: If we know that in Case B there is a m.c.l. extension this then is a minimal extension. The statements concerning the remarks on the relevance of minimal extensions in QFT (in Part I) are made explicit by showing.

(i) To construct minimal extensions is the easiest way of constructing m.c.l. extensions which are Poincaré-covariant and satisfy the spectral condition (Section 3).

(ii) If in Case B $T_{(2N)}$ satisfies the linear constraints of relativistic QFT and if we know that $T_{(2N)}$ has a m.c.l. extension then it follows that this m.c.l. extension automatically satisfies all linear constraints of relativistic QFT (Section 3).

As an application the extension problem for special cases of $T_{(4)}=\{1,0,T_2,T_3,T_4\}$ is discussed:

(i) necessary and sufficient conditions for the existence of minimal extensions are derived.

(ii) Some classes of convenient sufficient conditions are presented and then

(iii) some results on the structure of the n-point-functionals of some m.c.l. extension follow.

Our results show in particular that (at least for neutral fields) a characterization of a relativistic QFT by finitely many vacuum expectation values is possible!

2. On the Uniqueness of Monotone Continuous Linear Extension

The idea to answer the problem when a given functional $T_{(2N)}=\{1, T_1, \ldots, T_{2N}\} \in \underline{E}'_{(2N)}$ has at most one m.c.l. extension is as follows:

We prove a criterium which says when a m.c.l. functional $T \in \underline{E}'_{+,1}$ is "uniquely determined" by $T_1, T_2, \ldots, T_{2N-1}$. The hypotheses of this criterium are conditions on $\{T_1, \ldots, T_{2N}\}$. Thus a theorem on the uniqueness of m.c.l. extension of $T_{(2N)}$ follows.

The main technical tool is a generalization of a well-known relation for Gram-determinants. Given $T \in \prod_{n=0}^{\infty} E'_n$, $T(\underline{1})=1$ we may define a sequence of functions $G_n : \underline{E}_{(n)} \equiv \bigoplus_{i=0}^{n} E_i \to \mathbb{C}$, $n=0,1,2,\ldots$ by

$$G_n(\underline{x}_{(n)}) = \det \begin{bmatrix} \bar{x}_0 x_0 & \bar{x}_0 T_1(x_1) \ldots & \bar{x}_0 T_n(x_n) \\ T_1(x_1^*) x_0 & T_2(x_1^* \otimes x_1) \ldots & T_{n+1}(x_1^* \otimes x_n) \\ \vdots & \vdots & \vdots \\ T_n(x_n^*) x_0 & T_{n+1}(x_n^* \otimes x_1) \ldots & T_{2n}(x_n^* \otimes x_n) \end{bmatrix} \tag{2.1}$$

and we may characterize monotonicity of T by saying that all G_n take only non-negative real values. In terms of the canonical pre-Hilbertspace realization $V_T=(\Phi_T(\underline{E}), \langle\cdot,\cdot\rangle_T)$ of a given $T\in\underline{E}'_{+,1}$ by the nuclear quotient map $\Phi_T:\underline{E}\to V_T$, $\Phi=\Phi_T=\sum_{n=0}^{\infty}\Phi_n$ such that $T(\underline{x}^*\cdot\underline{y})=\langle\Phi_T(\underline{x}), \Phi_T(\underline{y})\rangle_T$ $\underline{x}, \underline{y}\in\underline{E}$ the functions G_n have the following realization as Gram-determinants

$$G_n(\underline{x}_{(n)})=\hat{G}_n(\Phi_0(x_0), \dots, \Phi_n(x_n)):$$
$$=\det\begin{bmatrix}\langle\Phi_0(x_0), \Phi_0(x_0)\rangle_T & \dots & \langle\Phi_0(x_0), \Phi_n(x_n)\rangle_T\\ \vdots & & \vdots\\ \langle\Phi_n(x_n), \Phi_0(x_0)\rangle_T & \dots & \langle\Phi_n(x_n), \Phi_n(x_n)\rangle_T\end{bmatrix} \tag{2.2}$$

By assumption $T(\underline{1})=1$ we know $G_0(\underline{x}_{(0)})=0$ iff $x_0=0$; define $S_0:=\{\underline{x}_{(0)}\in\underline{E}_{(0)}|G_0(\underline{x}_{(0)})=|x_0|^2=1\}\neq\emptyset$ and $\Delta_1:E_1\to\mathbb{R}_+$ by

$$\Delta_1(x_1)=\inf_{\underline{x}_{(0)}\in S_0}\{G_1(\underline{x}_{(0)};x_1)\}$$

If $\Delta_1\neq 0$, we know

$$S_1:=\{\underline{x}_{(1)}\in\underline{E}_{(1)}|G_1(\underline{x}_{(1)})=1\}\neq\emptyset$$

and thus

$$\Delta_2(x_2):=\inf_{\underline{x}_{(1)}\in S_1}\{G_2(\underline{x}_{(1)};x_2)\}$$

is welldefined as a function $\Delta_2:E_2\to\mathbb{R}_+$. In this way we proceed successively defining

$$\begin{aligned}S_n&:=\{\underline{x}_{(n)}\in\underline{E}_{(n)}|G_n(\underline{x}_{(n)})=1\}\\ \Delta_{n+1}(x_{n+1})&:=\inf_{\underline{x}_{(n)}\in S_n}\{G_{n+1}(\underline{x}_{(n)};x_{n+1})\}\,.\end{aligned} \tag{2.3}$$

In order to have an interpretation of these functions Δ_n and in order to investigate which possibilities are allowed, we introduce

$$\begin{aligned}&V_n=\Phi(\underline{E}_{(n)}), \quad \mathscr{H}_{(n)}=\bar{V}_n=\text{closure of } V_n \text{ in } \mathscr{H};\\ &\mathscr{H}=\mathscr{H}_T=\tilde{V}_T=\text{completion of } V_T;\\ &Q_n:\mathscr{H}\to\mathscr{H}_{(n)} \text{ orthogonal projection onto } \mathscr{H}_{(n)}.\end{aligned} \tag{2.4}$$

Evidently the following relation holds

$$\mathscr{H}_{(n)}\subseteqq\mathscr{H}_{(n+1)} \qquad n=0,1,2,\dots. \tag{2.5}$$

Suppose $\Delta_\nu\neq 0$, $\nu\leqq n$ and calculate

$$\begin{aligned}G_{n+1}(\underline{x}_{(n)}, x_{n+1})=&\hat{G}_{n+1}(\Phi_0(x_0), \dots, \Phi_n(x_n), Q_n\Phi_{n+1}(x_{n+1}))\\ &+G_n(\underline{x}_{(n)})\|Q_n^\perp\Phi_{n+1}(x_{n+1})\|^2\end{aligned} \tag{2.6}$$

where $Q_n^\perp = I - Q_n$. The appendix contains a formal proof of what is intuitively clear

$$\inf_{\underline{x}_{(n)} \in S_n} G_{n+1}(\Phi_0(x_0), \ldots, \Phi_n(x_n); Q_n \Phi_{n+1}(x_{n+1})) = 0. \tag{2.7}$$

Therefore we get the following interpretation of the functions Δ_n, $n \in \mathbb{N}$

$$\Delta_{n+1}(x_{n+1}) = \| Q_n^\perp \Phi_{n+1}(x_{n+1}) \|^2. \tag{2.8}$$

Proposition 2.1. *For a given* $T \in \underline{E}'_{+,1}$ *there are only two possibilities:*

(A) $\Delta_n \neq 0$, $n = 0, 1, 2, \ldots$, *corresponding to*

$$\mathcal{H}_{(n)} \subsetneqq \mathcal{H}_{(n+1)}, \qquad n = 0, 1, 2, \ldots.$$

(B) *There is* $N \in \mathbb{N}$ *such that* $\Delta_n \neq 0$, $n \leqq N$ *and* $\Delta_n = 0$, $n \geqq N+1$, *corresponding to*

$$\mathcal{H}_{(0)} \subsetneqq \mathcal{H}_{(1)} \subsetneqq \cdots \subsetneqq \mathcal{H}_{(N)} = \mathcal{H}_{(N+1)} = \cdots = \mathcal{H}.$$

Proof. If $\Delta_n \neq 0$ holds for all $n = 0, 1, 2, \ldots$ we see by Equation (2.8) that $\mathcal{H}_{(n)} \subsetneqq \mathcal{H}_{(n+1)}$, $n = 0, 1, 2, \ldots$ results. If on the other side we know $\mathcal{H}_{(n)} \subsetneqq \mathcal{H}_{(n+1)}$, $n = 0, 1, 2, \ldots$ then we get immediately $\Delta_n \neq 0$, $n = 0, 1, 2, \ldots$ by Equation (2.8). The negation of the first possibility is that there is $N \in \mathbb{N}$ and that $\Delta_n \neq 0$, $n \leqq N$ and $\Delta_{N+1} = 0$. Then by Equation (2.8)

$$\Phi_{N+1}(E_{N+1}) \subseteqq \mathcal{H}_{(N)}$$

and thus

$$\Phi_{N+1}(x_{N+1}) = \lim_{L \to +\infty} \Psi_L(x_{N+1}), \qquad x_{N+1} \in E_{N+1}$$

$$\Psi_L(x_{N+1}) = \sum_{j=1}^{L} \langle \Phi_{(N)}(\underline{h}^j_{(N)}), \Phi_{N+1}(x_{N+1}) \rangle \Phi_{(N)}(\underline{h}^j_{(N)}) \in V_N$$

$\{\Phi_{(N)}(\underline{h}^j_{(N)})\}_{j \in \mathbb{N}} \subseteqq V_N$ being an orthonormal basis of $\mathcal{H}_{(N)}$. The canonical GNS-$*$-representation $A = A_T$ of $\underline{E}$ by linear operators on $V_T = \bigcup_{n=0}^{\infty} V_n$ acts as a "shift" operator in V_T:

$$\Phi_{N+2}(x \otimes x_{N+1}) = A(x) \Phi_{N+1}(x_{N+1}), \qquad x \in E_1, \qquad x_{N+1} \in E_{N+1}.$$

Thus we have for all $\Psi \in V \cap \mathcal{H}_{(N)}^\perp$, all $x \in E_1$, all $x_{N+1} \in E_{N+1}$:

$$\begin{aligned} \langle \Psi, \Phi_{N+2}(x \otimes x_{N+1}) \rangle &= \langle A(x^*) \Psi, \Phi_{N+1}(x_{N+1}) \rangle \\ &= \lim_{L \to \infty} \langle A(x^*) \Psi, \Psi_L(x_{N+1}) \rangle \\ &= \lim_{L \to \infty} \langle \Psi, A(x) \Psi_L(x_{N+1}) \rangle \\ &= 0 \end{aligned}$$

because of $A(x) \Psi_L(x_{N+1}) \in \mathcal{H}_{(N)}$.

Therefore

$$\Phi_{N+2}(x \otimes x_{N+1}) \in (V \cap \mathcal{H}_{(N)}^\perp)^\perp = \mathcal{H}_{(N)}, \qquad x \in E_1, \qquad x_{N+1} \in E_{N+1}.$$

and thus

$$\Phi_{N+2}(E_{N+2}) \subseteqq \mathscr{H}_{(N)}, \quad Q_N^{\perp}\Phi_{N+2}(E_{N+2}) = \{0\}.$$

Similarly we get by induction

$$\Phi_{N+\nu}(E_{N+\nu}) \subseteqq \mathscr{H}_{(N)} \qquad \nu = 3, 4, \ldots$$

thus $\mathscr{H}_{(M)} = \mathscr{H}_{(N)}$, $M \geqq N$, follows.

That the relations $\mathscr{H}_{(0)} \subsetneqq \ldots \subsetneqq \mathscr{H}_{(N)} = \mathscr{H}_{(N+1)} = \ldots = \mathscr{H}$ for the Hilbertspace imply those claimed for the functions Δ_n is immediate by Equation (2.8).

Next we want to investigate the interpretation of Case (B) of Proposition (2.1) in terms of $T_n, n \in \mathbb{N}$. By Equation (2.8) we have in this case $Q_N \Phi_{N+1} = \Phi_{N+1}$ and thus

$$\begin{aligned} T_{2N+2}(x_{N+1} \otimes y_{N+1}) &= \langle \Phi_{N+1}(x^*_{N+1}), \Phi_{N+1}(y_{N+1}) \rangle \\ &= \langle \Phi_{N+1}(x^*_{N+1}), Q_N \Phi_{N+1}(y_{N+1}) \rangle \\ &= \sum_{j=1}^{\infty} \langle \Phi_{N+1}(x_{N+1}), \Phi_{(N)}(\underline{h}^j_{(N)}) \rangle \cdot \langle \Phi_{(N)}(\underline{h}^j_{(N)}), \Phi_{N+1}(y_{N+1}) \rangle \\ &= \sum_{j=1}^{\infty} (t^*_{N+1,j} \otimes t_{N+1,j})(x^*_{N+1} \otimes y_{N+1}) \end{aligned} \tag{2.9}$$

where we defined

$$\begin{aligned} t_{N+1,j}(x_{N+1}) &:= \langle \Phi_{(N)}(\underline{h}^j_{(N)}), \Phi_{N+1}(x_{N+1}) \rangle \\ &= \sum_{\nu=0}^{N} T_{\nu+N+1}(h^{j*}_{\nu} \otimes x_{N+1}). \end{aligned} \tag{2.10}$$

All the functionals $t_{N+1,j} \in E'_{N+1}$ are uniquely determined by

(i) $\{T_{N+1}, \ldots, T_{2N+1}\}$,

(ii) an orthonormal basis $\{\Phi_{(N)}(\underline{h}^j_{(N)})\}_{j \in \mathbb{N}} \subset V_N$ of $\mathscr{H}_{(N)}$.

Now any orthonormal basis of $\mathscr{H}_{(N)}$ in V_N is fixed by $\{T_1, \ldots, T_{2N}\}$ and according to (2.9) and (2.10) the series

$$\sum_{j=1}^{\infty} t^*_{N+1,j} \otimes t_{N+1,j}$$

give the same result for any orthonormal basis $\{\Phi_{(N)}(\underline{h}^j_{(N)})\}_{j \in \mathbb{N}}$ of $\mathscr{H}_{(N)}$. Thus we see that by Equation (2.9) and (2.10) T_{2N+2} is uniquely determined by $\{T_1, \ldots, T_{2N+1}\}$.

We proceed by induction. Assume that for some $m \in \mathbb{N}$, $m \geqq 2$ all the functionals $T_{2N+\mu}$, $\mu \leqq m$, are uniquely determined by $\{T_1, \ldots, T_{2N+1}\}$ in the sense of repeated application of Equation (2.9) and (2.10). Then it is enough to show that T_{2N+m+1} is uniquely determined by $\{T_1, \ldots, T_{2N+m}\}$ in the above sense to conclude that T_{2N+m+1} is uniquely determined by $\{T_1, \ldots, T_{2N+1}\}$ in the sense of repeated application of (2.9) and (2.10). For all $x_{N+m} \in E_{N+m}$ and all $y_{N+1} \in E_{N+1}$ we have

$$\begin{aligned} T_{2N+1+m}(x_{N+m} \otimes y_{N+1}) &= \langle \Phi_{N+m}(x^*_{N+m}), \Phi_{N+1}(y_{N+1}) \rangle \\ &= \langle \Phi_{N+m}(x^*_{N+m}), Q_N \Phi_{N+1}(y_{N+1}) \rangle \\ &= \sum_{j=1}^{\infty} t^*_{N+m,j}(x_{N+m}) t_{N+1,j}(y_{N+1}) \end{aligned} \tag{2.9'}$$

where $t_{N+m,j} \in E'_{N+m}$ is defined by

$$
\begin{aligned}
t_{N+m,j}(x_{N+m}) &= \langle \Phi_{(N)}(\underline{h}^j_{(N)}), \Phi_{N+m}(x_{N+m}) \rangle \\
&= \sum_{\nu=0}^{N} T_{\nu+N+m}(h^{j*}_{\nu} \otimes x_{N+m})
\end{aligned}
\tag{2.10'}
$$

and thus depends on $\{T_{N+m}, \ldots, T_{2N+m}\}$ and the choice of the orthogonal basis $\{\Phi_{(N)}(\underline{h}^j_{(N)})\}_{j\in\mathbb{N}}$ of $\mathcal{H}_{(N)}$. But again the sum $\sum_{j=1}^{\infty} t^*_{N+m,j} \otimes t_{N+1,j}$ does not depend on the special choice of such a basis. Therefore T_{2N+1+m} is determined by $\{T_1, \ldots, T_{2N+m}\}$ in the sense of Equation (2.9′) and (2.10′). So we define

Definition 2.2. We say that a m.c.l. functional $T=\{1, T_1, T_2, \ldots\} \in \underline{E}^*_{+,1}$ is "*uniquely determined*" by

$$T_{(2N+1)} = \{1, T_1, \ldots, T_{2N+1}\}, \quad N \quad \text{minimal},$$

if and only if all the functionals T_n, $n \geqq 2N+2$ are determined by $T_{(2N+1)}$ in the sense of repeated application of (2.9), (2.10), (2.9′) and (2.10′), and our discussion above shows

Theorem 2.3. *A m.c.l. functional $T \in \{1, T_1, T_2, \ldots\} \in \underline{E}'_{+,1}$ is uniquely determined by $T_{(2N+1)} = \{T_1, \ldots, T_{2N+1}\}$ if and only if $N \in \mathbb{N}$ is minimal such that*

$$\Delta_{N+1}(x_{N+1}) = \inf_{\substack{\underline{x}_{(N)} \in \underline{E}_{(N)} \\ G_N(\underline{x}_{(N)})=1}} \det \begin{bmatrix} \bar{x}_0 x_0, & \ldots, & \bar{x}_0 T_{N+1}(x_{N+1}) \\ \vdots & & \vdots \\ T_{N+1}(x^*_{N+1}) x_0, & \ldots, & T_{2N+2}(x^*_{N+1} \otimes x_{N+1}) \end{bmatrix} \equiv 0$$

in $x_{N+1} \in E_{N+1}$.

Remark. It is clear which part of the statement of Theorem 2.3 is not trivial. The other part is just a matter of definition (Definition 2.2).

By the way the hypotheses are formulated we see that Theorem 2.3 has an immediate application to the extension problem. Suppose we are given $T_{(2N)} \in \underline{E}'_{(2N)}$, $T_{(2N)}(\underline{1}) = 1$ such that $T_{(2N)} \upharpoonright \underline{E}_+ \cap \underline{E}_{(2N)} \geqq 0$. Then we are free to define for $0 \leqq n < \mu \leqq N$

$$G^{\mu}_n(\underline{x}_{(n)}; x_\mu) := \det \begin{bmatrix} \bar{x}_0 x_0, & \ldots, & \bar{x}_0 T_n(x_n), & \bar{x}_0 T(x_\mu) \\ \vdots & & \vdots & \vdots \\ T_n(x_n) x_0 & \ldots & T_{2n}(x^*_n \otimes x_n), & T_{n+\mu}(x^*_n \otimes x_\mu) \\ & & & \\ T_\mu(x^*_\mu) x_0 & \ldots & T_{n+\mu}(x^*_\mu \otimes x_n), & T_{2\mu}(x^*_\mu \otimes x_\mu) \end{bmatrix} \tag{2.11}$$

and

$$\Delta^{\mu}_n(x_\mu) = \inf_{\underline{x}_{(n)} \in S_n} \{G^{\mu}_n(\underline{x}_{(n)}; x_\mu\} \tag{2.12}$$

in the case of $\Delta_\nu \neq 0$, $\nu = 1, \ldots, n$. In terms of the canonical pre-Hilbertspace realization $V_N = (\Phi_{(N)}(\underline{E}_{(N)}), \langle \cdot, \cdot \rangle_{(N)})$ of $T_{(2N)}$ according to [Theorem 2.1; I] we have the following interpretation of the functions Δ_n^μ:

$$\Delta_n^\mu(x_\mu) = \|Q_n^\perp \Phi_\mu(x_\mu)\|_{(N)}^2 . \tag{2.12'}$$

This prepares

Theorem 2.4. *Given $T_{(2N)} \in \underline{E}'_{(2N)}$ such that $T_{(2N)} \upharpoonright \underline{E}_{(2N)} \cap \underline{E}_+ \geqq 0$ we have to distinguish the following cases:*

(A) *If $\Delta_n \neq 0$, $n = 1, \ldots, N$ then $T_{(2N)}$ may have more than one m.c.l. extension; if there is a m.c.l. extension at all there will be in general uncountably many.*

(B) *If there is $n \in \{1, \ldots, N-1\}$ such that*

(i) $\Delta_\nu \neq 0, \quad \nu = 1, \ldots, n$ *and*

(ii) $\Delta_n^\mu = 0, \quad \mu = n+1, \ldots, N$

then $T_{(2N)}$ has at most one m.c.l. extension.

(C) *If $N > 2$ and if there is $n \in \{1, \ldots, N-2\}$ such that*

(i) $\Delta_\nu \neq 0, \quad \nu = 1, \ldots, n$

(ii) $\Delta_{n+1} = 0$

(iii) $\Delta_n^\mu \neq 0$ *for some* $\mu \in \{n+2, \ldots, N\}$

then $T_{(2N)}$ has no m.c.l. extension at all.

Proof. (A) In Section 4 we will show by example that in Case A there will be uncountably many m.c.l. extensions if certain sufficient conditions guarantee the existence of at least one minimal extension of $T_{(2N)}$.

(B) Suppose $T \in \underline{E}'_{+,1}$ is any m.c.l. extension of $T_{(2N)}$. Then according to (2.1) and assumption (ii)

$$\Delta_{n+1}^T = \Delta_{n+1}^{T_{(2N)}} = 0 .$$

Therefore by Theorem 2.3 T is uniquely determined by $\{T_1, \ldots, T_{2n+1}\}$, that is there is at most one m.c.l. extension of $T_{(2N)}$.

(C) If there would be a m.c.l. extension T of $T_{(2N)}$ we had a contradiction to Proposition 2.1 which says in particular that $\Delta_n^\mu = 0$, $n+2 \leqq \mu \leqq N$.

Corollary 2.5. *If in Case* (B) *of Theorem 2.4 there is a m.c.l. extension T of $T_{(2N)}$, this is a minimal extension of $T_{(2N)}$ which is uniquely determined by $\{T_1, \ldots, T_{2n+1}\}$.*

Proof. Suppose $T \in \underline{E}'_{+,1}$ is an extension of $T_{(2N)}$. Then we get by (2.1)

$$\Delta_{n+1}^T = \Delta_{n+1}^{T_{(2N)}} = 0 .$$

By Theorem 2.3 we know that T is uniquely determined by $T_{(2n+1)}$ and Proposition 2.1 implies

$$\mathscr{H}_T = \mathscr{H}_{(n)}^T .$$

The assumptions of (B) imply: $\mathscr{H}_{(N)} = \mathscr{H}_{(n)}$. Therefore the canonical partial isometry $J : \mathscr{H}_{(N)} \to \mathscr{H}_T$ is a unitary transformation of $\mathscr{H}_{(N)}$ onto $\mathscr{H}_T$; e.g. T is a minimal extension of $T_{(2N)}$.

For illustration and further application we note:

Corollary 2.6. *Given* $T_{(4)}=\{1, T_1, T_2, T_3, T_4\}\in \underline{E}'_{(4)}$ *define*

$$T_{11}=T_2-T_1\otimes T_1\,,\quad T_{22}=T_4-T_2\otimes T_2\,,\quad T_{12}=T_3-T_1\otimes T_2$$

$$Q_2(x_1,x_2)=T_{22}(x_2^*\otimes x_2)T_{11}(x_1^*\otimes x_1)-|T_{12}(x_1^*\otimes x_2)|^2\,.$$

Then we have

a) *If* $T_{11}(x_1^*\otimes x_1)\geqq 0 \quad \forall x_1\in E_1\,, \quad T_{11}\neq 0$ *and if* $Q_2(x_1,x_2)\geqq 0 \quad \forall x_j\in E_j\,, \quad j=1,2$

and

$$\Delta_2(x_2):=\inf\{Q(x_1,x_2)|T_{11}(x_1^*\otimes x_1)=1\}\not\equiv 0$$

$$x_2\in E_2$$

then $T_{(4)}$ *may have more than one* m.c.l. *extension.*

b) *If* T_{11} *and* Q_2 *are as in* a) *and if* $\Delta_2=0$, *then there is at most one* m.c.l. *extension of* $T_{(4)}$.

c) *There is no* m.c.l. *extension of* $T_{(4)}$ *if one of the following conditions holds:*

(i) $T_{jj}(x_j^*\otimes x_j)<0$ *for some* $x_j\in E_j, \quad j=1$ *or* $j=2$

(ii) $T_{11}=0$ *and* $T_{22}\neq 0, \quad T_{22}(x_2^*\otimes x_2)\geqq 0$

(iii) $Q_2(x_1,x_2)<0$ *for some* $x_j\in E_j, \quad j=1,2$.

Proof. a) Note $G_2(x_0,x_1,x_2)=|x_0|^2 Q_2(x_1,x_2)$, thus Theorem 2.4 (A) applies.

b) Theorem 2.4 (B) applies.

c) Case (i) and (iii) imply that $T_{(4)}$ does not satisfy the necessary monotonicity conditions. Theorem 2.4 (C) applies to case (ii).

3. Minimal Extensions in QFT

In order to be able to formulate the linear constraints of relativistic QFT we have to assume in addition that E_1 is a space of testfunctions on Minkowski-space which has some further properties [such as the Schwartz-space $\mathscr{S}(\mathbb{R}^4)$].

(i) E_1 allows Fourier-transformation such that localization in "coordinate"- and "momentum-space" is possible.

(ii) The Poincaré-group $G=P_+$ acts linearly and continuously on E_1 by $*$-automorphisms. This action then induces the canonical action α_g of G on $\underline{E}$ by $*$-automorphisms of $\underline{E}$.

Then it is well-known [2] how to formulate the linear constraints of QFT in terms of a monotone continuous linear functional $T\in\underline{E}'_{+,1}$ such that the $*$-representation $A=A_T$ of $\underline{E}$ associated with T (via GNS-construction) describes a relativistic quantum field: T has to vanish on a suitable subspace $I\subseteqq\underline{E}$, e.g.

$$T\in I^0\subseteqq\underline{E}' \quad\text{with}\quad I^0=I_G^0\cap I_\Sigma^0\cap I_L^0$$

where we have used the following notation:

I^0 denotes the annihilator of the subspace $I \subseteqq \underline{E}$ in $\underline{E}'$.

$I_G = \{\underline{y} = \underline{x} - \alpha_g \underline{x} | \underline{x} \in \underline{E}, g \in G\}$

$I_\Sigma = \underline{E} \cdot (E_\Sigma * \underline{E}) \equiv$ spectral ideal

$I_L = \underline{E} \cdot L_2 \cdot \underline{E} \equiv$ locality ideal

$E_\Sigma = \{x \in E_1 | \operatorname{supp} \tilde{x} \subseteqq \Sigma^c = \mathbb{R}^4 \backslash \Sigma\}$

$\tilde{x}$ = Fourier transform of x; Σ a closed subset of the forward lightcone $\bar{V}_+$, $0 \in \Sigma$;

$$x * \underline{y} = \{\tilde{x}(0) y_0, x * y_1, \ldots, x * y_n, \ldots\}$$

$$(x * y_n)(\xi_1, \ldots, \xi_n) = \int_{\mathbb{R}^4} x(a) y_n(\xi_1 - a, \ldots, \xi_n - a) d^4 a$$

$L_2 = \{x_1 \otimes x_2 - x_2 \otimes x_1 \in E_2 | x_j \in E_1,$ $\operatorname{supp} x_1$ and $\operatorname{supp} x_2$ are spacelike separated$\}$.

In this section we want to discuss the problem of constructing m.c.l. extensions T of a given $T_{(2n)} \in \underline{E}'_{(2N)} \cap I^0$ which have the further property that they vanish on $I = I_G \cup I_\Sigma \cup I_L$, e.g. we are looking for extensions in $\underline{E}'_{+,1} \cap I^0$. Now the additional constraint for an extension $T \in \underline{E}'_{+,1}$ to vanish on I is rather strong so that in general it is very hard to do the construction of such an extension. Therefore we suggest the following strategy (compare Section 2 of I):

1) Construct first minimal extensions which are G-covariant and satisfy the spectral condition.

2) Then, if necessary construct local extension of the minimal extensions of Step 1 in a way that respects G-covariance and spectral conditions.

Remark. (a) The example of the generalized free field as discussed in I shows that this construction procedure for extensions in $\underline{E}'_{+,1} \cap I^0$ works in principle.

(b) As we will see in a moment Step 1 is not too hard.

(c) We will isolate the cases in which Step 1

(i) already yields an extension in $\underline{E}'_{+,1} \cap I^0$

(ii) only yields an extension in $\underline{E}'_{+,1} \cap I_G^0 \cap I_\Sigma^0$, which is not local.

At the moment we do not know to do Step 2 explicitly in the general case.

(d) In particular we will show that for a certain class of functionals $T_{(2N)} \in \underline{E}'_{(2N)} \cap I^0$ there are only two possibilities:

(i) $T_{(2N)}$ has no m.c.l. extension at all,

(ii) $T_{(2N)}$ has an extension in $\underline{E}'_{+,1} \cap I^0$.

To start we analyze the consequences of the additional constraint $T_{(2N)} \in \underline{E}'_{(2N)} \cap I^0$: Monotonicity $T_{(2N)} \geqq 0$ implies the canonical pre-Hilbertspace realization

$$T_{(2N)}(\underline{x}^* \cdot \underline{y}) = \langle \Phi_{(N)}(\underline{x}), \Phi_{(N)}(\underline{y}) \rangle_{(N)} \quad \underline{x}, \underline{y} \in \underline{E}_{(N)}$$

$$A_{(N-1)}(x) \Phi_{(N)}(\underline{y}) = \Phi_{(N)}(x \cdot \underline{y}) \quad \forall x \in E_1 \ \forall \underline{y} \in \underline{E}_{(N-2)}$$

of $T_{(2N)}$ according to [Theorem 2.1a, I and Proposition 2.3a, I]. If we assume in addition that $T_{(2N)}$ satisfies the linear constraints of QFT we get

Proposition 3.1. *Suppose $T_{(2N)}=\{1, T_1, T_2, \dots T_{2N}\}\in \underline{E}'_{(2N)}$ satisfies monotonicity:*

$$T_{(2N)}\upharpoonright \underline{E}_{(2N)}\cap \underline{E}_+ \geqq 0 \tag{3.1}$$

and the linear constraints of QFT:

$$T_{(2N)}\in \underline{E}'_{(2N)}\cap I^0 . \tag{3.2}$$

Then in addition to the statement of [Proposition 2.3a, I] we have

(a) *There exists a strongly continuous representation $U=U_{T_{(2N)}}$ of G by unitary operators on $\mathscr{H}_{(N)}=\tilde{V}_N$ such that*

(i) $U(g)\Phi_{(N)}(\underline{x})=\Phi_{(N)}(\alpha_g\underline{x})\quad \forall g\in G\quad \forall \underline{x}\in \underline{E}_{(N)}$.

(ii) $U(g)A_{(N-1)}(x)U(g)^{-1}=A_{(N-1)}(x_g)$ *on* $V_{N-2},\quad \forall x\in E_1\quad \forall g\in G$.

(iii) *The self-adjoint generator $P=P_U$ of the space-time translation group in this representation has its spectrum $\sigma(P)$ contained in Σ.*

(b) *If $N\geqq 3$ and if $x_1\otimes x_2-x_2\otimes x_1\in L_2$ then*

$$[A_{(N-1)}(x_1), A_{(N-1)}(x_2)]\upharpoonright V_{N-3}=0$$

(c) *If in addition the inequality*

$$|T_{(2N)}(\underline{y}^*\cdot \underline{x})|\leqq q_{(N+1)}(\underline{y})T_{(2N)}(\underline{x}^*\cdot \underline{x})^{1/2} \tag{3.3}$$

$\forall \underline{x}\in \underline{E}_{(N-1)}$, $\forall \underline{y}\in \underline{E}_{(N+1)}$, $q_{(N+1)}$ *some continuous seminorm on* $\underline{E}_{(N+1)}$ *holds the symmetric linear extension* $A_{(N)}:E_1\to L(V_{N-1}, V_N)$ *of* $A_{(N-1)}$ *according to [Proposition 2.3b, I] satisfies*

(i) $U(g)A_{(N)}(x)U(g)^{-1}=A_{(N)}(x_g)$ *on* $V_{N-1}\ \forall x\in E_1\ \forall g\in G$

(ii) $[A_{(N)}(x_1), A_{(N)}(x_2)]\upharpoonright V_{N-2}=0$, *if* $N\geqq 2$ *and*

$$x_1\otimes x_2-x_2\otimes x_1\in L_2 .$$

Proof. (a) The assumption $T_{(2N)}\in \underline{E}'_{(2N)}\cap I^0_G$ easily implies that $U(g)$ defined on

$$V_N=(\Phi_{(N)}(\underline{E}_{(N)}), \langle\cdot,\cdot\rangle_{(N)}) \text{ by}$$

$$U(g)\Phi_{(N)}(\underline{x}):=\Phi_{(N)}(\alpha_g\underline{x})\ \forall \underline{x}\in \underline{E}_{(N)}$$

is a unitary operator on the pre-Hilbertspace V_N which satisfies (a) (ii). The remaining part then follows by standard arguments. The equation

$$\langle\Phi_{(N)}(\underline{y}_1), \tilde{x}(P)\Phi_{(N)}(\underline{y}_2)\rangle_{(N)}=T_{(2N)}(\underline{y}_1^*\cdot(x*\underline{y}_2))\ \forall \underline{y}_j\in \underline{E}_{(N)}\ \forall x\in E_1$$

shows that $T_{(2N)}\in \underline{E}'_{(2N)}\cap I^0_\Sigma$ implies (a) (iii).

(b) If $N\geqq 3$ we have for all $\underline{y}\in \underline{E}_{(N-3)}$, all $\underline{x}\in \underline{E}_{(N)}$ and all $x_j\in E_1$:

$$\begin{aligned}&\langle\Phi_{(N)}(\underline{x}), [A_{(N-1)}(x_1), A_{(N-1)}(x_2)]\Phi_{(N)}(\underline{y})\rangle_{(N)}\\ &=T_{(2N)}(\underline{x}^*\cdot(x_1\otimes x_2-x_2\otimes x_1)\cdot \underline{y}).\end{aligned}$$

Therefore b) follows from $T_{(2N)}\in \underline{E}'_{(2N)}\cap I^0_L$.

(c) If the additional continuity property (3.3) holds Proposition 2.3b of [I] applies and the properties of the extension $A_{(N)}:E_1\to L(V_{N-1}, V_N)$ follow as above.

By Theorem 2.4 we know which possibilities can occur for a given $T_{(2N)}\in \underline{E}'_{(2N)}$, $T_{(2N)}\geqq 0$. In order to distinguish the case where we need the additional continuity property (3.3) and where not, we do the further distinction

(B$_1$) $\Delta_n \neq 0$, $\quad 0 \leqq n \leqq N-1$, $\quad \Delta_N = 0$

(B$_2$) $\Delta_n \neq 0$, $\quad 0 \leqq n \leqq M$, $\quad M \in \{1, \ldots, N-2\}$, $\quad \Delta_M^\mu = 0$, $\quad M+1 \leqq \mu \leqq N$.

By Proposition 2.1 the Hilbertspace $\mathscr{H}_{(N)}$ has the following structure

$$\mathscr{H}_{(0)} \subsetneqq \mathscr{H}_{(1)} \subsetneqq \ldots \subsetneqq \mathscr{H}_{(M)} = \mathscr{H}_{(N)}$$

in Case B$_1$ and Case B$_2$. Therefore Proposition 3.1 tells us that $A_{(N-1)}$ is densely defined in $\mathscr{H}_{(N)}$ in Case B$_2$. If in addition to (3.1) and (3.2) inequality (3.3) holds we know to construct $A_{(N)}: E_1 \to L(V_{N-1}, V_N)$ which is then densely defined in $\mathscr{H}_{(N)}$ in Case B$_1$. That is in Case B$_1$ and Case B$_2$ the problem of extending $A_{(N-1)}$ respectively $A_{(N)}$ to a proper subspace of $\mathscr{H}_{(N)}$ disappears. This allows to prove:

Theorem 3.2. *A functional* $T_{(2N)} \in \underline{E}'_{(2N)}$ *which satisfies* (3.1), (3.2) *and in addition*

(α) (B$_1$) *and* (3.3) *or*

(β) B$_2$

has at most one m.c.l. extension $T \in \underline{E}'_{+,1}$; *and in case of existence of such an extension this is a minimal extension which satisfies the linear constraints of QFT:* $T \in \underline{E}'_{+,1} \cap I^0$.

Proof. a) The first statement is just Theorem 2.4(B). Suppose $T \in \underline{E}'_{+,1}$ is an extension of $T_{(2N)}$. By Corollary 2.5 we know that T is a minimal extension of $T_{(2N)}$ which is uniquely determined by $T_{(2M+1)}$, $M \leqq N-1$. Thus we may assume that T has a pre-Hilbertspace realization $T(\underline{x} \cdot \underline{y}) = \langle \Phi(\underline{x}), \Phi(\underline{y}) \rangle$ such that $\Phi \restriction \underline{E}_{(N)} = \Phi_{(N)}$

$$\langle \cdot, \cdot \rangle \restriction V_N x V_N = \langle \cdot, \cdot \rangle_{(N)} \tag{3.4}$$

where $V_N = (\Phi_{(N)}(\underline{E}_{(N)}), \langle \cdot, \cdot \rangle_{(N)})$ is the canonical pre-Hilbertspace realization of $T_{(2N)}$.

b) First we prove G-invariance: We proceed by induction using the notation of Section 2

$$\begin{aligned}
T_{2N+1}(\alpha_g(x^*_{2N-M} \otimes x_{M+1})) &= \langle \Phi_{2N-M}(\alpha_g x_{2N-M}), \Phi_{M+1}(\alpha_g x_{M+1}) \rangle \\
&= \langle \Phi_{2N-M}(\alpha_g x_{2N-M}), Q_M \Phi_{M+1}(\alpha_g x_{M+1}) \rangle \\
&= \sum_{j=1}^{\infty} \langle \Phi_{2N-M}(\alpha_g x_{2N-M}), U(g)\Phi_{(M)}(\underline{h}^j_{(M)}) \rangle \langle U(g)\Phi_{(M)}(\underline{h}^j_{(M)}), \Phi_{M+1}(\alpha_g x_{M+1}) \rangle \\
&= \sum_{j=1}^{\infty} T_{(2N)}(\alpha_g(x^*_{2N-M} \otimes \underline{h}^j_{(M)})) T_{(2N)}(\alpha_g(\underline{h}^{j*}_{(M)} \otimes x_{M+1})) \\
&= \sum_{j=1}^{\infty} T_{(2N)}(x^*_{2N-M} \otimes \underline{h}^j_{(M)}) T_{(2N)}(\underline{h}^{j*}_{(M)} \otimes x_{M+1}) = T_{2N+1}(x^*_{2N-M} \otimes x_{M+1}).
\end{aligned}$$

By linearity and continuity of T_{2N+1} and of α_g the invariance of T_{2N+1} follows.

Assume that all $T_{2N+\mu}$, $1 \leqq \mu \leqq m$ are G-invariant. Then we get for T_{2N+m+1}:

$$T_{2N+m+1}(\alpha_g(x_{2N+m-M} \otimes x_{M+1})) = \langle \Phi_{2N+m-M}(\alpha_g x^*_{2N+m-M}), \Phi_{M+1}(\alpha_g x_{M+1})\rangle$$

$$= \sum_{j=1}^{\infty} \langle \Phi_{2N+m-M}(\alpha_g x^*_{2N+m-M}), \Phi_{(M)}(\alpha_g \underline{h}^j_{(M)})\rangle \langle \Phi_{(M)}(\alpha_g \underline{h}^j_{(M)}), \Phi_{M+1}(\alpha_g x_{M+1})\rangle$$

$$= \sum_{j=1}^{\infty} T_{(2N+m)}(\alpha_g(x_{2N+m-M} \cdot \underline{h}^j_{(M)})) T_{(2N)}(\alpha_g(\underline{h}^{j*}_{(M)} \cdot x_{M+1}))$$

$$= \sum_{j=1}^{\infty} T_{(2N+m)}(x_{2N+m-M} \cdot \underline{h}^j_{(M)}) T_{(2N)}(\underline{h}^{j*}_{(M)} \cdot x_{M+1})$$

$$= T_{2N+m+1}(x_{2N+m-M} \otimes x_{M+1}) \qquad \forall x_\nu \in E_\nu, \qquad \nu = M+1, 2N+m-M.$$

As above G-invariance of T_{2N+m+1} follows and thus

$$T \in \underline{E}'_{+,1} \cap I^0_G.$$

c) Proof of the spectral condition: Because of the G-invariance of $T \in \underline{E}'_{+,1}$ the Hilbertspace $\mathscr{H}_T$ of the canonical pre-Hilbertspace realization of T carries a strongly continuous unitary representation U_T of G. But T being a minimal extension of $T_{(2N)}$ we know that the canonical isometry $J: \mathscr{H}_{(N)} \to \mathscr{H}_T$ is a unitary map onto $\mathscr{H}_T$. Therefore $U_T(g) = JU(g)J^{-1}$ and thus T satisfies the spectral condition because $U = U_{T_{(2N)}}$ does according to Proposition 3.1a (iii).

d) Proof of locality: In case (α) of our assumptions we know that $A_{(N)}(x)$ is densely defined on $\mathscr{D}_N = \Phi_{(N)}\left(\bigoplus_{n=0}^{N-1} E_1^{\otimes n}\right)$, $A_{(N)}$ according to Proposition 3.1c. To prove locality of T we use the fact that T is a minimal extension of $T_{(2N)}$, e.g. T is defined in terms of a linear function $A: E_1 \to L(\mathscr{D}_A, \mathscr{D}_A)$ such that

(i) $\mathscr{D}_N \subseteqq \mathscr{D}_A$

(ii) $A(x)^* \upharpoonright \mathscr{D}_A = A(x^*) \upharpoonright \mathscr{D}_A$

(iii) $A(x) \upharpoonright \mathscr{D}_N = A_{(N)}(x) \upharpoonright \mathscr{D}_N$.

This follows from Proposition 2.5 of [I]. Locality of $T_{(2N)}$ implies ($M = N-1$):

$$\langle A_{(N)}(x_1^*)\Phi_{(M)}(\underline{y}_1), A_{(N)}(x_2)\Phi_{(M)}(\underline{y}_2)\rangle_{(N)} \qquad \forall \underline{y}_j \in \underline{E}_{(M)}$$

$$= \langle A_{(N)}(x_2^*)\Phi_{(M)}(\underline{y}_1), A_{(N)}(x_1)\Phi_{(M)}(\underline{y}_2)\rangle_{(N)} \qquad \forall x_1 \otimes x_2 - x_2 \otimes x_1 \in L_2.$$

Properties (i), (ii), (iii) above therefore imply

$$[A(x_1), A(x_2)] \upharpoonright \mathscr{D}_N = 0 \qquad \forall x_1 \otimes x_2 - x_2 \otimes x_1 \in L_2.$$

Then we use symmetry of A again. For all $\varphi \in \mathscr{D}_N$, all $\psi \in \mathscr{D}_A$ and all $x_1 \otimes x_2 - x_2 \otimes x_1 \in L_2$ we get:

$$\langle \varphi, [A(x_1), A(x_2)]\psi\rangle = \langle [A(x_2^*), A(x_1^*)]\varphi, \psi\rangle = 0$$

thus

$$[A(x_1), A(x_2)] \upharpoonright \mathscr{D}_A = 0 \qquad \forall x_1 \otimes x_2 - x_2 \otimes x_1 \in L_2.$$

In Case B_2 we know that already $A_{(N-1)}(x)$ is densely defined on $\mathscr{D}_{N-1} = \Phi_{(N)}\left(\bigoplus_{n=0}^{N-2} E_1^{\otimes n}\right)$; therefore we can proceed as above to prove locality. Thus in both cases $T \in \underline{E}'_{+,1} \cap I^0_L$.

Remarks. (a) The essential idea of the proof of Theorem 3.2 is contained in Corollary 2.6 of [I]. Indeed the ideas used in the proof of Theorem 2.7 of [I] could be used to give another proof of Theorem 3.2.

(b) In Case A of Theorem 2.4 where one expects more than one m.c.l. extension to exist under appropriate assumptions it is much harder to find conditions which imply the existence of an extension in $\underline{E}'_{+,1} \cap I^0$. In particular, one wants to isolate conditions on $T_{(2N)} \in \underline{E}'_{(2N)} \cap I^0$, $T_{(2N)} \geqq 0$ which ensure that in Case A there is at most one extension in $\underline{E}'_{+,1} \cap I^0$.

(c) Theorem 3.2 answers the question whether it is possible to characterize a relativistic quantum field theory in terms of finitely many vacuum expectation values or not.

Theorem 3.2 says that in principle it is possible.

4. Applications to the Extension Problem for $T_{(4)} = \{1, T_1, T_2, T_3, T_4\}$

In this section we want to discuss the problem of monotone continuous linear extension for particular cases of a given functional

$$T_{(4)} = \{1, 0, T_2, T_3, T_4\} \in \underline{E}'_{(4)} \quad \text{such that} \quad T_{(4)} \upharpoonright \underline{E}_{(4)} \cap \underline{E}_+ \geqq 0. \tag{4.1}$$

We restrict our discussion to the case of $T_{(4)}$ because

(i) most of the new problems (compared to the extension problem for $T_{(2)}$) appear already in this case (compare Theorem 2.4 and the discussion at the end of Section 3 of I)

(ii) we think that for applications to QFT, in particular in connection with the construction of nontrivial models for QFT knowing a fairly complete answer to this extension problem would be of some importance.

The assumption $T_1 = 0$ is no restriction. If we know how to treat the case (4.1) we also know how to treat the case

$$S_{(4)} = \{1, S_1, S_2, S_3, S_4\} \in \underline{E}'_{(4)}, \quad S_{(4)} \geqq 0, \quad S_1 \neq 0. \tag{4.1'}$$

Denoting $t_{(4)} = \{1, S_1, S_1^{\otimes 2}, S_1^{\otimes 3}, S_1^{\otimes 4}\}$ and using the Ⓢ-product of [2] we can solve the equations $(t_{(4)} \circledS T_{(4)})_n = S_n$ $n = 1, 2, 3, 4$ for T_n to get

$$\begin{aligned} T_{(4)} = \{1, 0, S_2 - S_1^{\otimes 2}, S_3 - S_1^{\otimes 3} - S_1 \circledS (S_2 - S_1^{\otimes 2}), S_4 - S_1^{\otimes 4} - S_1^{\otimes 2} \circledS (S_2 - S_1^{\otimes 2}) \\ + S_1 \circledS (S_3 - S_1^{\otimes 3} - S_1 \circledS (S_2 - S_1^{\otimes 2}))\}. \end{aligned}$$

Thus if $T \in \underline{E}'_{+,1}$ is a m.c.l. extension of this $T_{(4)}$ then $t \circledS T \in \underline{E}'_{+,1}$, $t = \{1, S_1, S_1^{\otimes 2}, S_1^{\otimes 3}, \ldots\} \in \underline{E}'_{+,1}$, is a m.c.l. extension of $S_{(4)} = (t \circledS T)_{(4)}$. Note that according to Corollary 2.6 this $t \in \underline{E}'_{+,1}$ is the unique m.c.l. extension of $\{1, S_1, S_1^{\otimes 2}\}$.

At first we investigate the canonical pre-Hilbertspace realization of $T_{(4)}$ in more detail. This prepares all further considerations.

Proposition 4.1. *If* $T_{(4)} \in \underline{E}'_{(4)}$ *satisfies* (4.1) *the canonical pre-Hilbertspace realization* $V_2 = (\Phi_{(2)}(\underline{E}_{(2)}), \langle \cdot, \cdot \rangle_{(2)})$ *of* $T_{(4)}$ *according to Theorem 2.1 and Proposition 2.3 of* [I] *has the following structure:*

a) *Structure of the Hilbertspace:*

The completion $\mathscr{H}_{(2)}=\tilde{V}_2$ *of* V_2 *has an orthogonal decomposition*

$$\mathscr{H}_{(2)}=\mathscr{H}_0\oplus\mathscr{H}_1\oplus\mathscr{H}_2$$

which results from a natural orthogonal decomposition of $\Phi_{(2)}=\Phi_0+\Phi_1+\Phi_2$*:*

$$\mathscr{H}_0=\mathbb{C}\Phi_0 \quad \Phi_0=\Phi_{(2)}(\underline{1}) \quad \mathscr{H}_1=\overline{\Phi_1(E_1)}=\text{closure in } \tilde{V}_2$$

Q_j *= orthogonal projection of* $\mathscr{H}_{(2)}$ *onto* $\mathscr{H}_j$, $j=0,1$

$$\Phi^2(x)=\Phi_2(x)-Q_0\Phi_2(x)-Q_1\Phi_2(x)=\Phi_2(x)-T_2(x)\Phi_0-\Phi_2^1(x) \quad \text{all} \quad x\in E_2$$

$\mathscr{H}_2=\overline{\Phi^2(E_2)}$ *= closure in* $\tilde{V}_2$.

b) *Realization by linear operators:*

α) *There are linear functions* $A_{ij}:E_1\to\mathscr{L}(\mathscr{H}_j,\mathscr{H}_i)$ $(ij)=(01)$, (10) *welldefined by*

$$A_{10}(x)\Phi_0=\Phi_1(x) \qquad A_{01}(x)\Psi_1=\langle\Phi_1(x^*),\Psi_1\rangle_1\Phi_0$$

for all $x\in E_1$, *all* $\Psi_1\in\mathscr{H}_1$ *such that*

(i) $\|A_{10}(x)\|_{10}=q_1(x)=T_2(x^*\otimes x)^{1/2}$

(ii) $\|A_{01}(x)\|_{01}=q_1(x^*)$

(iii) $A_{10}(x)^*=A_{01}(x^*)$

(iv) $T_2(x\otimes y)=\langle A_{10}(x^*)\Phi_0,A_{10}(y)\Phi_0\rangle_1=\langle\Phi_0,A_{01}(x)A_{10}(y)\Phi_0\rangle_1$

for all x,y *in* E_1.

β) *There is a linear function* $A_{11}:E_1\to\check{L}(\Phi_1(E_1),\mathscr{H}_1)$ *welldefined by* $A_{11}(x)\Phi_1(y)=\Phi_2^1(x\otimes y)=Q_1\Phi_2(x\otimes y)$ *all* $x,y\in E_1$, *such that*

(i) $q_2^1(x\otimes y)=\|A_{11}(x)\Phi_1(y)\|_1$ *is a continuous seminorm on* $E_1^{\otimes 2}$

(ii) $A_{11}(x)^*\restriction\Phi_1(E_1)=A_{11}(x^*)\restriction\Phi_1(E_1)$

(iii)
$$\begin{aligned}T_3(x\otimes y\otimes z)&=\langle A_{10}(x^*)\Phi_0,A_{11}(y)A_{10}(z)\Phi_0\rangle_1\\&=\langle\Phi_0,A_{01}(x)A_{11}(y)A_{10}(z)\Phi_0\rangle_1\\&=\langle A_{11}(y^*)A_{10}(x^*)\Phi_0,A_{10}(z)\Phi_0\rangle_1\end{aligned}$$

for all $x,y,z\in E_1$.

c) *Structure of the four-point-functional:*

$$T_4=T_4^0+T_4^1+T_4^2 \quad \text{with the definitions} \quad T_4^0=T_2\otimes T_2$$

$$T_4^1(x_1\otimes x_2\otimes x_3\otimes x_4)=\langle A_{11}(x_2^*)A_{10}(x_1^*)\Phi_0,A_{11}(x_3)A_{10}(x_4)\Phi_0\rangle_1 \quad \text{all} \quad x_j\in E_1$$

$$T_4^2(z\otimes w)=\langle\Phi^2(z^*),\Phi^2(w)\rangle_2 \quad \text{all} \quad z,w\in E_2.$$

d) *Uniqueness of m.c.l. extension:*

There is at most one m.c.l. extension if $T_4^2=0$, *e.g. if* $\Phi^2=0$.

Proof. a) $T_1=0$ implies $\Phi_0\perp\Phi_1(E_1)$, therefore $\mathscr{H}_0\perp\mathscr{H}_1$. By definition of Φ_2^i $i=1,2$ $\Phi_2(z)=T_2(z)\Phi_0\oplus\Phi_2^1(z)\oplus\Phi^2(z)$ for all $z\in E_2$ and thus $\mathscr{H}_2\perp\mathscr{H}_j$, $j=0,1$.

b) α) Clearly for all $x\in E_1$ $A_{10}(x)$ and $A_{01}(x)$ are welldefined linear operators as indicated and in addition $x\to A_{10}(x)$, $A_{01}(x)$ are linear functions. Furthermore

$$\|A_{10}(x)\|_{10}=\sup\{\|A_{10}(x)\Psi_0\|_1;\psi_0\in\mathscr{H}_0,\|\psi_0\|_0=1\}=q_1(x)$$

$$\|A_{01}(x)\|_{01}=\sup\{|\langle\Phi_1(x^*),\Psi_1\rangle_1|;\Psi_1\in\mathscr{H}_1,\|\Psi_1\|_1=1\}=q_1(x^*).$$

By definition of the adjoint for operators with respect to different spaces the symmetry relation $A_{10}(x)^* = A_{01}(x^*)$ is immediate as well as the relations expressing T_2 in terms of A_{10} and A_{01}.

β) Assume $\Phi_1(y)=0$, $y \in E_1$, then because of the consistency relations for $\Phi_{(2)}$

$$\|\Phi_2^1(x\otimes y)\|_1 = \sup\{|\langle \Phi_1(z), \Phi_2^1(x\otimes y)\rangle_1|; z\in E_1, q_1(z)=1\} = 0\,.$$

Thus $A_{11}(x)$ is a welldefined linear operator $\Phi_1(E_1) \to \mathscr{H}_1$ and

$$\|A_{11}(x)\Phi_1(y)\|_1 \leqq (T_{22}((x\otimes y)^* \otimes x \otimes y))^{1/2}\,,$$

therefore q_2^1 is a continuous seminorm on $E_1^{\otimes 2}$. The symmetry relation $T_3^* = T_3$ easily yields $A_{11}(x)^* \restriction \Phi_1(E_1) = A_{11}(x^*) \restriction \Phi_1(E_1)$ and thus the relations expressing T_3 in terms of A_{01}, A_{10}, and A_{11}.

c) The orthogonal decomposition of Φ_2 according to part a) and the definitions above easily yield the decomposition of T_4 as indicated.

d) This is implied by Corollary 2.6.

The previous results are applied to the following special cases of $T_{(4)} \in \underline{E}'_{(4)}$:

(a) $T_{(4)} = \{1, 0, T_2, T_3, T_4\}$ $\Delta_2 = 0$ (Theorem 4.2)

(b) $T_{(3)} = \{1, 0, T_2, T_3\}$ (Corollary 4.3–4.5)

(c) $T_{(4)} = \{1, 0, T_2, 0, T_4\}$ $\Delta_2 \neq 0$ (Proposition 4.6, Theorem 4.7, Corollary 4.8–4.9)

In Part I the case of a given $T_{(2)} = \{1, T_1, T_2\}$ has been treated (completely with respect to m.c.l. extensions); thus by Theorem 2.4 the only missing possibility of a given $T_{(N)} \in \underline{E}'_{(N)}$, $N \leqq 4$, is the case

(d) $T_{(4)} = \{1, 0, T_2, T_3, T_4\}$ $T_3 \neq 0$ and $\Delta_2 \neq 0$.

By the following discussion we will see why Case (d) causes additional technical difficulties concerning domain questions of the associated operators (compare remark b) of Section 5).

Case (c) is the simplest example (different from $T_{(2)}$) where one expects more than one m.c.l. extension to exist. Case (b) is an example where one fixes a minimal extension (here of $T_{(2)} = \{1, 0, T_2\}$) by fixing the next possible n-point-functional. This illustrates Corollary 2.6 and Theorem 2.7 of I. After all Case (a) represents a nontrivial example of a functional which has at most one m.c.l. extension and thus illustrates Theorem 2.4 and Theorem 3.2.

If we assume in addition to (4.1) that

$$\Delta_2 = 0 \quad \text{e.g.} \quad T_4^2 = 0 \tag{4.2}$$

holds we know that there is at most one m.c.l. extension of $T_{(4)}$ and there is a m.c.l. extension if and only if $A_{(1)}: E_1 \to L(V_1, \tilde{V}_2)$

$$A_{(1)}(x) = \begin{pmatrix} 0 & A_{01}(x) \\ A_{10}(x) & A_{11}(x) \end{pmatrix} \qquad x \in E_1 \tag{4.3}$$

as specified according to Proposition 4.1 has a symmetric linear extension $A: E_1 \to L(\mathscr{D}_A, \mathscr{D}_A)$ which has the properties listed in Proposition 2.5 of I. The properties of $A_{01}(x)$ and $A_{10}(x)$ as specified in Proposition 4.1 imply that this is the case if and only if $A_{11}: E_1 \to L(\Phi_1(E_1), \mathscr{H}_1)$ has a symmetric linear extension

$\bar{A}_{11}:E_1 \to L(\mathscr{D}_{\bar{A}_{11}}, \mathscr{D}_{\bar{A}_{11}})$, $\Phi_1(E_1) \subseteq \mathscr{D}_{\bar{A}_{11}}$ and this in turn can be expressed by a corresponding chain of inequalities $(K_{n,k})$ for T_3 (Theorem 2.7 of I). These inequalities read in this case

$$\begin{aligned}&\Bigg|\sum_{j=1}^{\infty}\Bigg\{\sum_{\nu_1=1}^{\infty}\Bigg(\sum_{\nu_2=1}^{\infty}\cdots\Bigg(\sum_{\nu_{n-1}=1}^{\infty} T_3(y_n^*\otimes y_{n-1}^*\otimes h_{\nu_{n-1}})T_3(h_{\nu_{n-1}}^*\otimes y_{n-2}^*\otimes h_{\nu_{n-2}})\Bigg)\\
&\ldots T_3(h_{\nu_2}^*\otimes y_1^*\otimes h_{\nu_1})\Bigg) T_3(h_{\nu_1}^*\otimes x\otimes h_j)\Bigg\}\cdot\Bigg\{\sum_{\mu_1=1}^{\infty} T_3(h_j^*\otimes x_1\otimes h_{\mu_1})\Bigg(\sum_{\mu_2=1}^{\infty} T_3(h_{\mu_1}^*\otimes x_2\otimes h_{\mu_2})\\
&\quad\cdots\Bigg(\sum_{\mu_{m-2}=1}^{\infty} T_3(h_{\mu_{m-3}}^*\otimes x_{m-2}\otimes h_{\mu_{m-2}})\, T_3(h_{\mu_{m-2}}^*\otimes x_{m-1}\otimes x_m)\Bigg)\cdots\Bigg)\Bigg\}\Bigg|\\
&\leq p_n(y_1\otimes\ldots\otimes y_n)p_{m+1}(x\otimes x_1\otimes\ldots\otimes x_m)\end{aligned} \tag{4.4}$$

for all $x, x_j, y_j \in E_1$ and $m=n-1, n$ and $n=2,3,\ldots$, where $\{\Phi_1(h_j)\}_{j\in\mathbb{N}}$ is an orthonormal basis of $\mathscr{H}_1$ and p_n; $n=2,3,\ldots$ are continuous seminorms on $E_1^{\otimes n}$ such that $p_2(z)=q_2^1(z)=(T_4^1(z^*\otimes z))^{1/2}$ for $z\in E_2$.

This essentially leads to

Theorem 4.2. *If $T_{(4)}\in \underline{E}'_{(4)}$ satisfies* (4.1) *and* (4.2) *then*

a) *$T_{(4)}$ has exactly one monotone continuous linear extension $T\in \underline{E}'_{+,1}$ if and only if the inequalities* (4.4) *hold with the specification given above.*

b) *If $T_{(4)}$ satisfies* (4.1), (4.2) *and* (4.4) *the structure of the n-point-functional T_n of the m.c.l. extension is as follows:*

$$\begin{aligned}T_n(x_1\otimes\ldots\otimes x_n) &= T_2^1(x_1\otimes x_2)T_{n-2}(x_3\otimes\ldots\otimes x_n)\\
&\quad + T_3^1(x_1\otimes x_2\otimes x_3)T_{n-3}(x_4\otimes\ldots\otimes x_n)\\
&\quad + \ldots + T_{n-2}^1(x_1\otimes x_2\ldots\otimes x_{n-2})T_2(x_{n-1}\otimes x_n)\\
&\quad + T_n^1(x_1\otimes\ldots\otimes x_n)\end{aligned} \tag{4.5}$$

where

$$T_2^1 = T_2 \qquad T_3^1 = T_3 \quad \textit{and for} \quad n\geqq 4$$

$$T_n^1(x_1\otimes\ldots\otimes x_n) = \langle \Phi_0, A_{01}(x_1)\bar{A}_{11}(x_2)\ldots\bar{A}_{11}(x_{n-1})A_{10}(x_n)\Phi_0\rangle_1$$

for all $x_j\in E_1$.

Proof. Assume first that all vectors

$$\bar{A}(x_1)\ldots\bar{A}(x_n)\Phi_0 = \begin{pmatrix}\varphi_n^0(x_1\otimes\ldots\otimes x_n)\\ \varphi_n^1(x_1\otimes\ldots\otimes x_n)\end{pmatrix} \in \begin{pmatrix}\mathscr{H}_0\\ \mathscr{H}_1\end{pmatrix} = \mathscr{H}_{(2)}$$

are welldefined in $\mathscr{H}_{(2)} = \mathscr{H}_0 \oplus \mathscr{H}_1$. Because of (4.3) we get the following recursion relations for the components φ_n^j, $j=0,1$ in $\mathscr{H}_0$ and $\mathscr{H}_1$ of these vectors:

$$\begin{aligned}\varphi_{n+1}^0(x_1\otimes\ldots\otimes x_{n+1}) &= A_{01}(x_1)\varphi_n^1(x_2\otimes\ldots\otimes x_{n+1})\\
\varphi_{n+1}^1(x_1\otimes\ldots\otimes x_{n+1}) &= A_{10}(x_1)\varphi_n^0(x_2\otimes\ldots\otimes x_{n+1})\\
&\quad + \bar{A}_{11}(x_1)\varphi_n^1(x_2\otimes\ldots\otimes x_{n+1}).\end{aligned} \tag{4.6}$$

By induction on n it follows for all $n \geqq 3$:

$$\begin{aligned}\varphi_n^1(x_1 \otimes \ldots \otimes x_n) = {} & \Phi_1(x_1) T_{n-1}(x_2 \otimes \ldots \otimes x_n) + A_{11}(x_1)\Phi_1(x_2) T_{n-2}(x_3 \otimes \ldots \otimes x_n) \\ & + \ldots + \bar{A}_{11}(x_1) \ldots \bar{A}_{11}(x_{n-3}) \Phi_1(x_{n-2}) T_2(x_{n-1} \otimes x_n) \\ & + \bar{A}_{11}(x_1) \ldots \bar{A}_{11}(x_{n-1}) \Phi_1(x_n). \end{aligned} \tag{4.7}$$

The special structure of Equation (4.6) and (4.7) (precisely the fact that in φ_n^1 there is only one term of highest possible degree in $\bar{A}_{11}(x_j)$) implies: $A_{(1)}: E_1 \to L(V_1, \tilde{V}_2)$ has a symmetric linear extension $A: E_1 \to L(\mathscr{D}_A, \mathscr{D}_A)$ which satisfies (i)–(iv) of Proposition 2.5 of I if and only if

$$A_{11}: E_1 \to L(\Phi_1(E_1), \mathscr{H}_1)$$

has a symmetric linear extension

$$\bar{A}_{11}: E_1 \to L(\mathscr{D}_{\bar{A}_{11}}, \mathscr{D}_{\bar{A}_{11}}), \qquad \Phi_1(E_1) \subseteqq \mathscr{D}_{\bar{A}_{11}},$$

with corresponding properties.

Thus by Theorem 2.7 I and Theorem 2.4 (B) part a) follows. Then Equation (4.5) is an easy consequence of (4.6) and (4.7). This proves b).

Corollary 4.3. *Suppose* $T_{(3)} = \{1, 0, T_2, T_3\} \in \underline{E}'_{(3)}$ *satisfies*

(i) $T_2(x_1^* \otimes x_1) \geqq 0$ *all* $x_1 \in E_1$, $T_2 \neq 0$,

(ii) $T_3^* = T_3$,

(iii) $|T_3(x_1^* \otimes x_2)| \leqq (T_2(x_1^* \otimes x_1))^{1/2} q_2(x_2)$ *all* $x_j \in E_j$, $j = 1, 2$

q_2 *some continuous seminorm on* E_2.

a) *If* T_3 *satisfies in addition the inequalities* (4.4) *with* $p_2 = q_2$ *then* $T_{(2)}$ *has exactly one minimal extension whose* 3-*point-functional is* T_3.

b) *The* 4-*point-functional* T_4 *of any m.c.l. extension of* $T_{(3)}$ *satisfies*

$$T_4(z_2^* \otimes z_2) - |T_2(z_2)|^2 - \sum_{j=1}^{\infty} T_3(z_2^* \otimes h_j) T_3(h_j^* \otimes z_2) \geqq T_4^2(z_2^* \otimes z_2) \geqq 0 \tag{4.8}$$

for all $z_2 \in E_2$ *and all* $\{h_j\}_{j \in \mathbb{N}} \subset E_1$ *such that* $T_2(h_i^* \otimes h_j) = \delta_{ij}$.

Proof. By assumption (i) $T_{(2)} = \{1, 0, T_2\}$ is monotone on $\underline{E}_{(2)}$. Therefore $T_{(2)}$ has a canonical pre-Hilbertspace realization $V_1 = (\Phi_{(1)}(\underline{E}_{(1)}), \langle \cdot, \cdot \rangle_{(1)})$ and the associated Hilbertspace $\mathscr{H}_{(1)} = \tilde{V}_1$ has the orthogonal decomposition $\mathscr{H}_{(1)} = \mathscr{H}_0 \oplus \mathscr{H}_1$ as in Proposition 4.1. Assumption (iii) implies $T_3(x_1 \otimes x_2) = \langle \Phi_1(x_1), \Phi_2^1(x_2) \rangle_1$ for all $x_j \in E_j$, $j = 1, 2$ with some continuous linear function $\Phi_2^1: E_2 \to \mathscr{H}_1$ and by assumption (ii) we know

$$\langle \Phi_1(x_1), \Phi_2^1(x_2 x_3) \rangle_1 = \langle \Phi_2^1(x_2^* \otimes x_1), \Phi_1(x_3) \rangle_1$$

for all $x_j \in E_1$. This implies that a strongly continuous linear function $A_{11}: E_1 \to L(\Phi_1(E_1), \mathscr{H}_1)$ is welldefined by $A_{11}(x)\Phi_1(y) = \Phi_2^1(x \otimes y)$ $x, y \in E_1$.

Furthermore there are linear functions $A_{ij}: E_1 \to L(\mathscr{H}_j, \mathscr{H}_i)$ $(ij) = (01)$ and (10) according to Proposition 4.1; there is a linear function $A_{(1)}: E_1 \to L(V_1, \mathscr{H}_{(1)})$ of the form (4.3) as in Theorem 4.2. The first statement then follows from Theorem 4.2.

Suppose $T \in \underline{E}'_{+,1}$ is an extension of $T_{(3)}$. Then by Corollary 2.6

$$\Delta_2(x_2) = \inf\{T_4 - T_2 \otimes T_2)(x_2^* \otimes x_2) - |T_3(x_1^* \otimes x_2)|^2 ; T_2(x_1^* \otimes x_1) = 1\} \geqq 0$$

and T is uniquely determined by $T_{(3)}$ if and only if $\Delta_2=0$. But

$$\Delta_2(x_2)=(T_4-T_2\otimes T_2)(x_2^*\otimes x_2)$$
$$-\sup\{|T_3(x_1^*\otimes x_2)|^2; x_1\in E_1, T_2(x_1^*\otimes x_1)=1\},$$

$$\|\Phi_2^1(x_2)\|_1^2 \geqq \sum_{j=1}^{\infty} \langle \Phi_2^1(x_2), \Phi_1(h_j)\rangle_1 \langle \Phi_1(h_j), \Phi_2^1(x_2)\rangle_1$$

for any orthonormal system $\{\Phi_1(h_j)\}_{j\in\mathbb{N}}$ in $\Phi_1(E_1)$ and all $x_2\in E_2$; therefore by definition of T_4^2 according to Proposition 4.1 the inequalities (4.8) result; and the first inequality becomes an equality iff $\{\Phi_1(h_j)\}_{j\in\mathbb{N}}$ is an orthonormal basis of $\mathscr{H}_1$.

Remark. (a) The conditions on $T_{(4)}$ expressed by inequalities (4.4) which are necessary and sufficient for the existence of a m.c.l. extension of $T_{(4)}$ are essentially no help in testing a given functional. But these conditions give hints for the construction of the most general two, three- and four-point-functionals which have at least minimal extensions and thus hopefully a lot of other extensions.

(b) In particular these conditions lead to some classes of more convenient conditions for the existence of (minimal) m.c.l. extensions. We discuss the simplest cases, but it is clear that they should work more general.

Corollary 4.4. *Suppose* $T_{(3)}=\{1,0,T_2,T_3\}\in \underline{E}'_{(3)}$ *satisfies*

(i) $T_2 \neq 0, T_2(x_1^*\otimes x_1)\geqq 0$ *for all* $x_1\in E_1$,

(ii) $T_3^*=T_3$

(iii) $|T_3(x_1^*\otimes x_2\otimes x_3)|\leqq (T_2(x_1^*\otimes x_1))^{1/2} p_1(x_2)(T_2(x_3^*\otimes x_3))^{1/2}$

for all $x_j\in E_1$ *and* p_1 *some continuous seminorm on* E_1.

Then we have

a) $T_{(2)}=\{1,0,T_2\}$ *has exactly one minimal extension which is fixed by* T_3.

b) *There are a lot of non-minimal extensions of* $T_{(2)}$. *The case which is most easily described is the following type of extensions: Let* $\mathscr{H}_0$ *and* $\mathscr{H}_1$ *be as in Corollary 4.3; assume* $\mathscr{H}_2$ *to be any separable Hilbertspace and assume that* $A_{ij}:E_1\to\mathscr{L}(\mathscr{H}_j,\mathscr{H}_i)$ *are linear functions such that*

(i) $A_{ij}(x)^*=A_{ji}(x^*)$,

(ii) $x\mapsto \|A_{ij}(x)\|_{ij}$ *is a continuous seminorm on* E_1 *for* $(ij)=(12), (21), (22)$

(iii) $\bigcup_{n=0}^{\infty} A_{(2)}(E_1)\dots A_{(2)}(E_1)\Phi_0$ *is total in* $\mathscr{H}_{(2)}=\mathscr{H}_0\oplus\mathscr{H}_1\oplus\mathscr{H}_2$

$$A_{(2)}(x)=\begin{pmatrix} 0 & A_{01}(x) & 0 \\ A_{10}(x) & A_{11}(x) & A_{12}(x) \\ 0 & A_{21}(x) & A_{22} \end{pmatrix} \quad \textit{all} \quad x\in E_1. \tag{4.9}$$

Then a non-minimal m.c.l. extension of $T_{(3)}$ *is defined by*

$$T_n(x_1\otimes\dots\otimes x_n)=\langle \Phi_0, A_{(2)}(x_1)\dots A_{(2)}(x_n)\Phi_0\rangle_{\mathscr{H}_{(2)}} \tag{4.9'}$$

for all $x_j\in E_1$ *and* $n=1,2,\dots$.

Proof. Using the same notation as in Corollary 4.3 we have $T_2(x\otimes y)=\langle A_{10}(x)\Phi_0, A_{10}(y)\Phi_0\rangle_1$ and $T_3(x^*\otimes z\otimes y)=\langle A_{10}(x)\Phi_0, A_{11}(z)A_{10}(y)\Phi_0\rangle_1$ for all $x, y, z\in E_1$. The more restrictive continuity assumption (iii) implies

$$\sup\{|\langle\Phi_1(x), A_{11}(z)\Phi_1(y)\rangle_1|;\, x, y\in E_1,\, \|\Phi_1(x)\|_1=\|\Phi_1(y)\|_1=1\}\leqq p_1(z)\,;$$

thus in this case it is easy to extend A_{11} to give $\bar{A}_{11}: E_1\to\mathscr{L}(\mathscr{H}_1,\mathscr{H}_1)$ such that $\|\bar{A}_{11}(x)\|\leqq p_1(x)$. $\mathscr{L}(\mathscr{H}_1,\mathscr{H}_2)$ denotes as usual the space of bounded linear operators from the Hilbertspace $\mathscr{H}_1$ into the Hilbertspace $\mathscr{H}_2$.

Therefore

$$A_{(1)}: E_1\to\mathscr{L}(\mathscr{H}_{(1)},\mathscr{H}_{(1)})\qquad A_{(1)}(x)=\begin{pmatrix}0 & A_{01}(x)\\ A_{10}(x) & \bar{A}_{11}(x)\end{pmatrix}$$

defines a minimal extension of $T_{(2)}$ which is also a m.c.l. extension of $T_{(3)}$ such that $T_4^2=0$ (Proposition 2.5, I), thus (Corollary 2.6 of I and Proposition 4.1) it is the minimal extension of $T_{(2)}$ which is fixed by T_3. This proves a).

To prove b) note that the assumptions above imply that formula (4.9) defines a linear function $A_{(2)}: E_1\to\mathscr{L}(\mathscr{H}_{(2)},\mathscr{H}_{(2)})$ such that $A_{(2)}(x)^*=A_{(2)}(x^*)$ and $x\mapsto\|A_{(2)}(x)\|$ is a continuous seminorm on E_1. Therefore a m.c.l. functional T on $\underline{E}$ is welldefined by (4.9′). It is clearly an extension of $T_{(3)}$.

Remark. (a) It is evident how to generalize part b) of Corollary 4.4 to include the case $A_{(2)}: E_1\to L(\mathscr{D}_{(2)},\mathscr{D}_{(2)})$, $A_{(2)}(x)$ not necessarily bounded in $\mathscr{H}_{(2)}$, $\mathscr{D}_{(2)}=\mathscr{H}_0\oplus\mathscr{H}_1\oplus\mathscr{D}_2$ a dense subspace of $\mathscr{H}_{(2)}$.

(b) The most general estimate each $T_3\in E_3'$ has to satisfy is

$$|T_3(x^*\otimes y\otimes z)|\leqq p_1(x)p_2(y)p_3(z)\quad\text{for all}\quad x, y, z\in E_1$$

where p_1, p_2, p_3 are continuous seminorms on E_1. If T_3 is symmetric, e.g. $T_3^*=T_3$ it is no restriction to assume an estimate of the form

$$|T_3(x^*\otimes y\otimes z)|\leqq p_1(x)p_2(y)p_1(z)\,.$$

Thus we are left with three possibilities for a given $T_{(3)}=\{1, 0, T_2, T_3\}$ such (i) and (ii) of Corollary 4.4 hold:

$$\left.\begin{array}{l}(\alpha)\ p_1(x)\leqq q_1(x)=(T_2(x^*\otimes x))^{1/2}\\ (\beta)\ p_1(x)\geqq q_1(x),\ p_1\neq q_1\\ (\gamma)\ p_1 \text{ and } q_1 \text{ are not comparable}\end{array}\right\}\quad\text{for all}\quad x\in E_1\,.$$

The first case is covered by Corollary 4.4. The second case however seems to be of more interest for applications in QFT. But it is harder to analyze. In [3] a special case of $p_1\geqq q_1$ and $p_1^{-1}(0)=q_1^{-1}(0)$ and $p_1\neq q_1$ is realized in the construction of a relativistic quantum field with control of the dynamics. A generalization along these lines seems to be possible.

We propose another kind of conditions on $T_{(3)}$ which imply the existence of the minimal extension of $T_{(2)}$ which is fixed by T_3 but which are more involved than those of Corollary 4.4. The idea simply is that according to Proposition 2.5 of I we get such an extension if $A_{11}(x)$ as specified in the proof of Corollary 4.3 maps $\Phi_1(E_1)$ into $\Phi_1(E_1)$ for all $x\in E_1$.

Corollary 4.5. *Assume* $T_{(3)} \in \underline{E}'_{(3)}$ *satisfies the conditions* (i), (ii), (iii) *of Corollary 4.3 and in addition the following estimate:*

$$\bigwedge_{x,y \in E_1} \bigvee_{\substack{y_1,\ldots,y_n \in E_1 \\ y_j = y_j(x,y)}} \bigwedge_{z \in E_1} |T_3(y^* \otimes x \otimes z)| \leqq \sup_{j=1,\ldots,n} |T_2(y_j^* \otimes z)|. \tag{4.10}$$

Then $T_{(2)}$ *has exactly one minimal extension which is fixed by* T_3 *and which is characterized in terms of a continuous linear map*

$$f:(E_1 \otimes E_1, q_2) \to (E_1, q_1) \quad \textit{such that}$$
$$T_3(x \otimes y \otimes z) = T_2(x \otimes f(y \otimes z)) \quad \textit{for all} \quad x, y, z \in E_1 \tag{4.11}$$

q_2 *is the continuous seminorm of condition* (iii) *of Corollary 4.3 and* q_1 *is defined as usual* $(q_1(x) = (T_2(x^* \otimes x))^{1/2})$.

Proof. We begin as in Corollary 4.3. Using the notation of the proof of Corollary 4.3 we have according to inequality (4.10):

$$|\langle A_{11}(x^*)\Phi_1(y), \Phi_1(z)\rangle_1| \leqq \sup\{|\langle \Phi_1(y_j), \Phi_1(z)\rangle_1|;\ j=1,\ldots,n\}$$

thus $\bigcap_{j=1}^{n} \operatorname{Ker} l_j \subseteqq \operatorname{Ker} l$ if we denote the linear functional on $\Phi_1(E_1)$

$$\Phi_1(z) \mapsto \langle \Phi_1(y_j), \Phi_1(z)\rangle_1 \quad \text{by } l_j, \text{ respectively}$$
$$\Phi_1(z) \mapsto \langle A_{11}(x^*)\Phi_1(y), \Phi_1(z)\rangle_1 \quad \text{by } l.$$

This implies $l = \sum_{j=1}^{n} \alpha_j l_j$, $\alpha_j \in \mathbb{C}$ and therefore

$$A_{11}(x^*)\Phi_1(y) = \sum_{j=1}^{n} \bar{\alpha}_j \Phi_1(y_j) = \Phi_1\left(\sum_{j=1}^{n} \bar{\alpha}_j y_j\right) \in \Phi_1(E_1)$$

because $\Phi_1(E_1)$ is dense in $\mathscr{H}_1$. So we see that (4.10) characterizes the statement

$$A_{11}: E_1 \to L(\Phi_1(E_1), \Phi_1(E_1)).$$

Thus there is some function $f_0: E_1 \times E_1 \to E_1$ such that $A_{11}(x)\Phi_1(y) = \Phi_1(f_0(x, y))$. The linearity of A_{11} and Φ_1 imply that f_0 is a bilinear function $\operatorname{mod} q_1^{-1}(0)$ and we may assume that f_0 is indeed a bilinear function $E_1 \times E_1 \to E_1$ and is thus given in terms of a linear function $f: E_1 \otimes E_1 \to E_1$, e.g. $A_{11}(x)\Phi_1(y) = \Phi_1(f(x \otimes y))$. By definition of the norms

$$q_1(f(x \otimes y)) = \|A_{11}(x)\Phi_1(y)\|_1 \leqq q_2(x \otimes y)$$

the continuity properties of f follow. The relation

$$T_3(x \otimes y \otimes z) = \langle \Phi_1(x^*), A_{11}(y)\Phi_1(z)\rangle_1$$

implies (4.11). Therefore there is a linear function $A_{(1)}: E_1 \to L(V_1, V_1)$ such that $A_{(1)}(x)^* \upharpoonright V_1 = A_{(1)}(x^*) \upharpoonright V_1$ and $p_n(x_1 \otimes \ldots \otimes x_n) = \|A_{(1)}(x_1) \ldots A_{(1)}(x_n)\Phi_0\|_{\mathscr{H}_{(1)}}$ are continuous seminorms on $E_1^{\otimes n}$. This function $A_{(1)}$ yields the minimal extension of $T_{(2)}$ which is fixed by T_3.

The content of the second part of this section is a discussion of the extension problem in Case A of Theorem 2.4. So we try to construct m.c.l. extensions of

$$\begin{aligned}&T_{(4)}=\{1,0,T_2,0,T_4\}\in \underline{E}'_{(4)} \quad \text{such that}\\ &0\leqq T_2(x_1^*\otimes x_1), \quad T_{22}(x_2^*\otimes x_2)\geqq 0 \quad \text{for all} \quad x_j\in E_j, \quad j=1,2\\ &T_2\neq 0, \quad T_{22}=T_4-T_2\otimes T_2\neq 0.\end{aligned} \tag{4.12}$$

Corollary 2.6 implies $T_{(4)}\restriction \underline{E}_{(4)}\cap \underline{E}_+\geqq 0$ and $\Delta_2\neq 0$. Our main interest will be to construct first minimal extensions.

By Proposition 4.1 it is evident that in the case $\Delta_2\neq 0$ we should take into account a further condition on $T_{(4)}$ which is obviously necessary for the existence of at least one m.c.l. extension; in this case this condition reads (compare Proposition 2.3b) of I)

$$|T_4(x_1^*\otimes x_3)|\leqq (T_2(x_1^*\otimes x_1))^{1/2} q_3'(x_3) \tag{4.13}$$

for all $x_j\in E_j$, $j=1,3$; q_3' some continuous seminorm on E_3. An easy consequence of (4.13) is that

$$q_3(x_3)=\sup\{|T_4^2(x^*\otimes x_3)|;\, x\in E_1,\, q_1(x)=1\} \tag{4.14}$$

defines a continuous seminorm on E_3, where according to Proposition (4.1) $T_4^2=T_{22}=T_4-T_2\otimes T_2$ and consequently

$$|T_4^2(x_1^*\otimes x_2\otimes x_3\otimes x_4)|\leqq q_1(x_1)q_3'(x_2\otimes x_3\otimes x_4)+q_1(x_1)q_1(x_2)q_1(x_3^*)q_1(x_4)$$

and thus (4.14). This allows to go beyond the results of Proposition 4.1 to get

Proposition 4.6. *Suppose* $T_{(4)}\in \underline{E}'_{(4)}$ *satisfies* (4.12) *and* (4.13). *Then in addition to the results of Proposition 4.1 the following holds:*

(a) $A_{11}=0$, *e.g.* $T_4^1=0$

(b) *There are linear functions*

$$A_{21}: E_1\to L(\Phi_1(E_1), \Phi^2(E_1\otimes E_1))$$

and

$$A_{12}: E_1\to L(\Phi^2(E_2), \mathscr{H}_1)$$

welldefined by

$$A_{21}(x)\Phi_1(y)=\Phi^2(x\otimes y), \qquad A_{12}(x)\restriction \Phi^2(E_2)=A_{21}(x^*)^*\restriction \Phi^2(E_2)$$

such that for all $x_j\in E_1$

$$\begin{aligned}T_4^2(x_1\otimes \ldots \otimes x_4)&=\langle A_{21}(x_2^*)\Phi_1(x_1^*), A_{21}(x_3)\Phi_1(x_4)\rangle_2\\ &=\langle \Phi_1(x_1^*), A_{12}(x_2)A_{21}(x_3)\Phi_1(x_4)\rangle_1\\ &=\langle A_{12}(x_3^*)A_{21}(x_2^*)\Phi_1(x_1^*), \Phi_1(x_4)\rangle_1\end{aligned}$$

and for all $x, y \in E_1$ *and all* $z \in E_2$

$$\|A_{21}(x)\Phi_1(y)\|_2 = q_2(x \otimes y), \quad \|A_{12}(x)\Phi^2(z)\|_1 = q_3(x \otimes z).$$

Proof. By Definition (4.14) $|T_4^2(x_1^* \otimes x_3)| \leqq q_1(x_1)q_3(x_3)$, $x_j \in E_j$, $j=1,3$. Therefore there is a linear function

$$\Phi_3^1 : E_3 \to \mathscr{H}_1 \quad \text{such that} \quad T_4^2(x_1 \otimes x_3) = \langle \Phi_1(x_1^*), \Phi_3^1(x_3)\rangle_1 .$$

$T_{22}(x_2^* \otimes x_2) \geqq 0$ for all $x_2 \in E_2$ implies $T_{22}^* = T_{22}$. Thus we get by Proposition 4.1 for all $x_j \in E_1$

$$\begin{aligned}\langle \Phi_1(x_1), \Phi_3^1(x_2 \otimes x_3 \otimes x_4)\rangle_1 &= \langle \Phi^2(x_2^* \otimes x_1), \Phi^2(x_3 \otimes x_4)\rangle_2 \\ &= \langle \Phi_3^1(x_3^* \otimes x_2^* \otimes x_1), \Phi_1(x_4)\rangle_1 . \end{aligned} \tag{4.15}$$

Then it is immediate that ($x, y \in E_1$, $z \in E_2$)

$$A_{21}(x)\Phi_1(y) = \Phi^2(x \otimes y), \quad A_{12}(x)\Phi^2(z) = \Phi_3^1(x \otimes z) \tag{4.16}$$

are welldefined and have the properties claimed in Proposition 4.6.

Proposition 4.6 shows in particular that $T_{(4)}$ is realized in terms of any linear function $A_{(2)} : E_1 \to L(\mathscr{D}_{(2)}, \mathscr{H}_{(2)})$,

$$\mathscr{D}_{(2)} = \mathscr{H}_0 \oplus \Phi_1(E_1) \oplus \mathscr{D}_2, \quad \mathscr{D}_2 \subseteqq \Phi^2(E_2)$$

of the form

$$A_{(2)}(x) = \begin{pmatrix} 0 & A_{01}(x) & 0 \\ A_{10}(x) & 0 & A_{12}(x) \\ 0 & A_{21}(x) & A_{22}(x) \end{pmatrix} \tag{4.17}$$

where A_{22} is any linear function $E_1 \to L(\mathscr{D}_2, \mathscr{H}_2)$.

In order to get a minimal extension of $T_{(4)}$ a linear function $A_{22} : E_1 \to L(\mathscr{D}_2, \mathscr{H}_2)$ has to be specified such that the resulting $A_{(2)} : E_1 \to L(\mathscr{D}_{(2)}, \mathscr{H}_{(2)})$ has a symmetric linear extension $\bar{A}_{(2)} : E_1 \to L(\mathscr{D}_{\bar{A}_{(2)}}, \mathscr{D}_{\bar{A}_{(2)}})$ which satisfies (i)–(iv) of Proposition 2.5 in I. Here we propose to discuss the case $A_{22} = 0$. This has the advantage that we can apply Theorem 2.7 of I directly to get necessary and sufficient conditions on $T_{(4)}$ alone to have minimal extensions in terms of a linear function $\bar{A}_{(2)} : E_1 \to L(\mathscr{D}_{\bar{A}_{(2)}}, \mathscr{D}_{\bar{A}_{(2)}})$ of the form

$$\bar{A}_{(2)}(x) = \begin{pmatrix} 0 & A_{01}(x) & 0 \\ A_{10}(x) & 0 & \bar{A}_{12}(x) \\ 0 & \bar{A}_{21}(x) & 0 \end{pmatrix} . \tag{4.18}$$

By Proposition 4.6 the operators $A_{01}(x)$ and $A_{10}(x)$ in (4.18) have the same properties as those in (4.3). Therefore we proceed similar. Suppose first that all vectors $\bar{A}_{(2)}(x_1) \dots \bar{A}_{(2)}(x_n)\Phi_0$, $x_j \in E_1$, $n \in \mathbb{N}$ are welldefined in $\mathscr{H}_{(2)}$. The orthogonal decomposition of $\mathscr{H}_{(2)} = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \mathscr{H}_2$ yields

$$\bar{A}_{(2)}(x_1) \dots \bar{A}_{(2)}(x_n)\Phi_0 = \begin{pmatrix} \varphi_n^0(x_1 \otimes \dots \otimes x_n) \\ \varphi_n^1(x_1 \otimes \dots \otimes x_n) \\ \varphi_n^2(x_1 \otimes \dots \otimes x_n) \end{pmatrix} . \tag{4.19}$$

The components φ_n^j of these vectors satisfy the following recursion relations

$$\begin{aligned} \varphi_{n+1}^0(x_1 \otimes \dots \otimes x_{n+1}) &= A_{01}(x_1)\varphi_n^1(x_2 \otimes \dots \otimes x_{n+1}) \\ \varphi_{n+1}^1(x_1 \otimes \dots \otimes x_{n+1}) &= A_{10}(x_1)\varphi_n^0(x_2 \otimes \dots \otimes x_{n+1}) + \bar{A}_{12}(x_1)\varphi_n^2(x_2 \otimes \dots \otimes x_{n+1}) \\ \varphi_{n+1}^2(x_1 \otimes \dots \otimes x_{n+1}) &= \bar{A}_{21}(x_1)\varphi_n^1(x_2 \otimes \dots \otimes x_{n+1}) \end{aligned} \tag{4.20}$$

and thus by induction on n

$$\begin{aligned}\varphi^1_{2n+1}(x_1\otimes\ldots\otimes x_{2n+1}) &= \bar{A}_{12}(x_1)\bar{A}_{21}(x_2)\ldots\bar{A}_{21}(x_{2n})\Phi_1(x_{2n+1})\\ &+\bar{A}_{12}(x_1)\bar{A}_{21}(x_2)\ldots\bar{A}_{21}(x_{2n-2})\Phi_1(x_{2n-1})T_2(x_{2n}\otimes x_{2n+1})+\ldots\\ &+\Phi_1(x_1)T_{2n}(x_2\otimes\ldots\otimes x_{2n+1})\quad\text{all}\quad x_j\in E_1,\qquad n=0,1,2,\ldots.\end{aligned}\tag{4.21}$$

$$\begin{aligned}&\varphi^0_{2n+1}=\varphi^2_{2n+1}=\varphi^1_{2n}=0\qquad n=0,1,2,\ldots\\ &\varphi^0_{2n}(x_1\otimes\ldots\otimes x_{2n})=A_{01}(x_1)\varphi^1_{2n-1}(x_2\otimes\ldots\otimes x_{2n})\\ &\varphi^2_{2n}(x_1\otimes\ldots\otimes x_{2n})=\bar{A}_{21}(x_1)\varphi^1_{2n-1}(x_2\otimes\ldots\otimes x_{2n}).\end{aligned}\tag{4.21'}$$

These equations show: If $A_{(2)}$ has a symmetric linear extension of the form (4.18) which satisfies (i)–(iv) of Proposition 2.5, I then

$$A:E_1\to L(\Phi_1(E_1)\oplus\Phi^2(E_2),\mathscr{H}_1\oplus\mathscr{H}_2)$$

$$A(x)=\begin{pmatrix}0 & A_{12}(x)\\ A_{21}(x) & 0\end{pmatrix}$$

has a symmetric linear extension $\bar{A}:E_1\to L(\mathscr{D}_{\bar{A}},\mathscr{D}_{\bar{A}})$ such that (i)–(iv) of Proposition 2.5, I hold with obvious modifications and $\Phi_1(E_1)\oplus\Phi^2(E_1\otimes E_1)\subseteqq\mathscr{D}_{\bar{A}}\subseteqq\mathscr{H}_1\oplus\mathscr{H}_2$. Conversely if such a function $\bar{A}$ exists the Equations (4.21) and (4.21′) imply the existence of a function $\bar{A}_{(2)}:E_1\to L(\mathscr{D}_{\bar{A}_{(2)}},\mathscr{D}_{\bar{A}_{(2)}})$ and this function $\bar{A}_{(2)}$ then satisfies the conditions (i)–(iv) as above. By [Theorem 2.7, I] such a symmetric linear function $\bar{A}$ exists if and only if the following chains of inequalities hold which result from the inequalities (K_{nk}) of [Theorem 2.7, I] by inserting an orthonormal basis $\{\Phi_1(h^1_j)\}_{j\in\mathbb{N}}$ of $\mathscr{H}_1$ and an orthonormal basis $\{\Phi^2(h^2_j)\}_{j\in\mathbb{N}}$ of $\mathscr{H}_2$:

$$\begin{aligned}&\Bigg|\sum_{j=1}^{\infty}\Bigg\{\sum_{\nu_1=1}^{\infty}\Bigg(\cdots\Bigg(\sum_{\nu_{2n-1}=1}^{\infty}T^2_4(y^*_{2n+1}\otimes y^*_{2n}\otimes y^*_{2n-1}\otimes h^1_{\nu_{2n-1}})\\ &\quad\cdot T^2_4(h^{1*}_{\nu_{2n-1}}\otimes y^*_{2n-2}\otimes h^2_{\nu_{2n-2}})\Bigg)\cdots\Bigg)\\ &\quad\cdot T^2_4(h^{2*}_{\nu_2}\otimes y^*_1\otimes h^1_{\nu_1})T^2_4(h^{1*}_{\nu_1}\otimes x^*\otimes h^2_j)\Bigg\}\\ &\quad\cdot\Bigg\{\sum_{\mu_1=1}^{\infty}T^2_4(h^{2*}_j\otimes x_1\otimes h^1_{\mu_1})\Bigg(\sum_{\mu_2=1}^{\infty}T^2_4(h^{1*}_{\mu_1}\otimes x_2\otimes h^2_{\mu_2})\ldots\\ &\quad\cdot\Bigg(\sum_{\mu_{2n-1}=1}^{\infty}T^2_4(h^{2*}_{\mu_{2n-2}}\otimes x_{2n-1}\otimes h^1_{\mu_{2n-1}})T^2_4(h^{1*}_{\mu_{2n-1}}\otimes x_2\otimes x_{2n+1}\otimes x_{2n+2})\Bigg)\cdots\Bigg)\Bigg\}\Bigg|\\ &\leqq p_{2n+1}(y_1\otimes\ldots\otimes y_{2n+1})p_{2n+3}(x^*\otimes x_1\otimes\ldots x_{2n+2}),\qquad n=0,1,2,\ldots\end{aligned}\tag{4.22a}$$

$$\begin{aligned}&\Bigg|\sum_{j=1}^{\infty}\Bigg\{\sum_{\nu_1=1}^{\infty}\Bigg(\sum_{\nu_2=1}^{\infty}\cdots\Bigg(\sum_{\nu_{2n-2}=1}^{\infty}T^2_4(y^*_{2n}\otimes y^*_{2n-1}\otimes y^*_{2n-2}\otimes h^1_{\nu_{2n-2}})\\ &\quad\cdot T^2_4(h^{1*}_{\nu_{2n-2}}\otimes y^*_{2n-3}\otimes h^2_{\nu_{2n-3}})\Bigg)\cdots\Bigg)\\ &\quad\cdot T^2_4(h^{1*}_{\nu_2}\otimes y^*_1\otimes h^2_{\nu_1})T^2_4(h^2_{\nu_1}\otimes x^*\otimes h^1_j)\Bigg\}\\ &\quad\cdot\Bigg\{\sum_{\mu_1=1}^{\infty}T^2_4(h^{1*}_j\otimes x_1\otimes h^2_{\mu 1})\Bigg(\sum_{\mu_2=1}^{\infty}T^2_4(h^{2*}_{\mu_1}\otimes x_2\otimes h^1_{\mu_2})\ldots\\ &\quad\cdot\Bigg(\sum_{\mu_{2n-1}=1}^{\infty}T^2_4(h^{1*}_{\mu_{2n-2}}\otimes x_{2n-1}\otimes h^2_{\mu_{2n-1}})T^2_4(h^{2*}_{\mu_{2n-1}}\otimes x_{2n}\otimes x_{2n+1})\Bigg)\cdots\Bigg)\Bigg\}\Bigg|\end{aligned}$$

$$\leqq p_{2n}(y_1 \otimes \ldots \otimes y_{2n}) p_{2n+2}(x^* \otimes x_1 \otimes \ldots \otimes x_{2n+1}), \qquad n=1,2,3,\ldots \tag{4.22b}$$

for all $x, x_j, y_j \in E_1$; p_n are continuous seminorms on $E_1^{\otimes n}$ such that $p_j = q_j$, $j=1,2,3$; q_j are the seminorms of Proposition 4.6.

As in the previous case the meaning of these inequalities (4.22) is that the extension

$$\bar{A}(x) = \begin{pmatrix} 0 & \bar{A}_{12}(x) \\ \bar{A}_{21}(x) & 0 \end{pmatrix}$$

is determined by the matrix-representation of

$$A(x) = \begin{pmatrix} 0 & A_{12}(x) \\ A_{21}(x) & 0 \end{pmatrix}$$

with respect to an orthonormal basis of $\mathscr{H}_1 \oplus \mathscr{H}_2$ in the domain of $A(x)$.

To summarize we formulate the analogue of Theorem 4.2:

Theorem 4.7. a) *If* $T_{(4)} = \{1, 0, T_2, 0, T_4\} \in \underline{E}'_{+,1}$ *satisfies* (4.12) *and* (4.13) *then* $T_{(4)}$ *has a minimal extension* $T \in \underline{E}'_{+,1}$ *which is determined by* $T_{(4)}$ *in the sense that all the functionals* $T_n, n \geqq 5$, *can be calculated in terms of a matrix-representation of* $A_{(2)}(x)$, *Equation* (4.18), *if and only if all the inequalities* (4.22) *hold.*

b) *If* $T_{(4)}$ *satisfies* (4.12), (4.13), (4.22) *the structure of the n-point-functionals* T_n *of the minimal extension which is determined by* $T_{(4)}$ *is as follows:*

$$T_2^2 = T_2 \qquad T_{2n+1} = 0 \qquad n=0,1,2,\ldots$$

$$T_{2(n+1)} = \sum_{j=1}^{n+1} T_{2j}^2 \otimes T_{2(n+1-j)} \qquad n=1,2,\ldots \tag{4.23}$$

$$T_{2j}^2(x_1 \otimes \ldots \otimes x_{2j}) = \langle \Phi_1(x_1^*), \bar{A}_{12}(x_2)\bar{A}_{21}(x_3)\ldots\Phi_1(x_{2j})\rangle_1$$

all $x_i \in E_1$ *and* $j=2,3,\ldots$.

Proof. Part a) has been proven above. The Equations (4.21) and (4.21′) then imply the relations (4.23) and thus b) is proven.

Concerning the usefulness of the conditions (4.22) the same remarks as those following Theorem 4.2 apply. Therefore we proceed similar and discuss the corresponding cases of sufficient conditions on $T_{(4)}$ to have m.c.l. extensions.

Corollary 4.8. *If* $T_{(4)} = \{1, 0, T_2, 0, T_4\}$ *satisfies* (4.12) *and*

$$|T_4^2(x_1 \otimes x_2 \otimes x_3 \otimes x_4)| \leqq q_1(x_1^*) p_1(x_2) p_1(x_3) q_1(x_4) \quad \text{all} \quad x_j \in E_1 \tag{4.24}$$

where p_1 *is some continuous seminorm on* E_1 *and* $q_1(x) = (T_2(x^* \otimes x))^{1/2}$, *then* $T_{(4)}$ *has a minimal extension which is determined by* $T_{(4)}$ *in terms of the bounded linear operators*

$$A_{(2)}(x) = \begin{pmatrix} 0 & A_{01}(x) & 0 \\ A_{10}(x) & 0 & A_{12}(x) \\ 0 & A_{21}(x) & 0 \end{pmatrix} \qquad \begin{array}{l} \|A_{12}(x)\|_{12} \leqq (p_1(x^*)p_1(x))^{1/2} \\ \|A_{21}(x)\|_{21} \leqq (p_1(x^*)p_1(x))^{1/2} \end{array}$$

according to Proposition 4.6.

$T_{(4)}$ has a lot of non minimal extensions. Those which are most easily described can be specified in terms of some linear function

$$A_{(3)}:E_1\to\mathscr{L}(\mathscr{H}_{(3)},\mathscr{H}_{(3)})\qquad \mathscr{H}_{(3)}=\mathscr{H}_0\oplus\mathscr{H}_1\oplus\mathscr{H}_2\oplus\mathscr{H}_3$$

$\mathscr{H}_3$ is any separable Hilbertspace and for all $x\in E_1$

$$A_{(3)}(x)=\begin{pmatrix} 0 & A_{01}(x) & 0 & 0\\ A_{10}(x) & 0 & A_{12}(x) & 0\\ 0 & A_{21}(x) & A_{22}(x) & A_{23}(x)\\ 0 & 0 & A_{32}(x) & A_{33}(x)\end{pmatrix} \tag{4.25}$$

$A_{ij}:E_1\to\mathscr{L}(\mathscr{H}_j,\mathscr{H}_i)$ are such that $A_{ij}(x)^*=A_{ji}(x^*)$ and $x\mapsto\|A_{ij}(x)\|_{ij}$, $i,j\in\{2,3\}$ are continuous seminorms on E_1 and such that $\mathscr{H}_0\cup\bigcup_{n=1}^{\infty}A_{(3)}(E_1)\dots A_{(3)}(E_1)\Phi_0$ is total in $\mathscr{H}_{(3)}$.

Proof. One has to start with Proposition 4.6 and then to proceed essentially in the same way as in the proof of Corollary 4.4.

Remark. (a) An immediate translation of remark (a) following Corollary 4.4 applies.

(b) By definition of the topology on E_4, each $T_4^2\in E_4'$, $T_4^{2*}=T_4^2$, admits an estimate

$$|T_4^2(x_1\otimes x_2\otimes x_3\otimes x_4)|\leqq p_1(x_1^*)\sigma_1(x_2)\sigma_1(x_3)p_1(x_4)\quad\text{all}\quad x_j\in E_1\,.$$

p_1 and σ_1 are continuous seminorms on E_1. That is for a given $T_{(4)}=\{1,0,T_2,0,T_2\otimes T_2+T_4^2\}\in \underline{E}'_{(4)}$ such that (4.12) holds there are only three possibilities:

$$\left.\begin{array}{l}(\alpha)\ p_1(x)\leqq q_1(x)=(T_2(x^*\otimes x))^{1/2}\\ (\beta)\ p_1(x)\geqq q_1(x),\quad p_1\neq q_1\\ (\gamma)\ p_1\text{ and }q_1\text{ are not comparable}\end{array}\right\}\ \text{for all}\quad x\in E_1\,.$$

Again the first case is covered by Corollary 4.8. Concerning the second possibility remark (b) following Corollary 4.4 applies.

The analogue of Corollary 4.5 is

Corollary 4.9. *If $T_{(4)}\in\underline{E}'_{(4)}$ satisfies* (4.12), (4.13) *and the following estimate*

$$\bigwedge_{\substack{x_j\in E_1\\ j=1,2,3}}\ \bigvee_{\substack{y_1\dots y_n\in E_1\\ y_j=y_j(x_i)}}\ \bigwedge_{x_4\in E_1}|T_4^2(x_1\otimes\dots\otimes x_4)|\leqq\sup_{j=1,\dots,n}|T_2(y_j\otimes x_4)| \tag{4.26}$$

then $T_{(4)}$ has a minimal extension which is determined by $T_{(4)}$ and this extension is characterized in terms of a continuous linear function $f:(E_1\otimes E_1,q_3)\to(E_1,q_1)$, q_3 as in (4.14), *such that*

$$T_4^2(x\otimes y\otimes z)=T_2(x\otimes f(y\otimes z))\quad\textit{for all}\quad x,y\in E_1\,,\qquad z\in E_2\,. \tag{4.27}$$

Proof. According to Proposition 4.6 $T_{(4)}$ is realized in terms of A_{01}, A_{10} and $A_{21}: E_1 \to L(\Phi_1(E_1), \Phi^2(E_1 \otimes E_1))$ and $A_{12}: E_1 \to L(\Phi^2(E_2), \mathscr{H}_1)$. And inequality (4.26) characterizes the fact that

$$A_{12}: E_1 \to L(\Phi^2(E_1 \otimes E_1), \Phi_1(E_1)).$$

Indeed by (4.26) we know

$$|\langle A_{12}(x_3^*)\Phi^2(x_2^* \otimes x_1^*), \Phi_1(x_4)\rangle_1| \leqq \sup_{j\in\{1\ldots n\}} |\langle \Phi_1(y_j), \Phi_1(x_4)\rangle_1|$$

and thus

$$A_{12}(x_3^*)\Phi^2(x_2^* \otimes x_1^*) = \Phi_1\left(\sum_{j=1}^{n} \alpha_j y_j\right) \in \Phi_1(E_1).$$

Then we can proceed as in the proof of Corollary 4.5 and the minimal extension of $T_{(4)}$ results from the fact that the range of $A_{12}(E_1)$ is contained in the domain of $A_{21}(E_1)$ such that the formulae (4.23) apply to define this minimal extension.

5. Conclusions and Further Problems

a) The problem of uniqueness of extensions of $T_{(2N)} \in \underline{E}'_{(2N)}$ in $\underline{E}'_{+,1}$ has been solved (Theorem 2.4). But the associated Hilbertspace has not the structure one expects in general in QFT (Proposition 2.1, Case B). For instance Fock-space has a structure which corresponds to case A of Proposition 2.1 and in this case one expects to have more than one extension in $\underline{E}'_{+,1}$ (Theorem 2.4A). Therefore in Case A the problem of uniqueness of extensions in $\underline{E}'_{+,1} \cap I^0$ arises.

b) It has been shown that the concept of minimal extensions leads to necessary and sufficient conditions on $T_{(2N)}$ for the existence of (at least a minimal) an extension in $\underline{E}'_{+,1}$ (Theorem 2.7 of I, Theorem 4.2, Theorem 4.7) and in a favourable situation it thus allows to construct many extensions in $\underline{E}'_{+,1}$ and in special cases in $\underline{E}'_{+,1} \cap I^0$. The problem which is still open is the question whether each extension in $\underline{E}'_{+,1}$ (or at least each extension in $\underline{E}'_{+,1} \cap I^0$) can be constructed via minimal extensions or not. This corresponds roughly to the problem whether each "field" $A: E_1 \to L(\mathscr{D}_A, \mathscr{D}_A)$ has a representation as a Jacobi-matrix or not.

c) The discussion of extensions in $\underline{E}'_{+,1}$ of $T_{(4)}$ supports to distinguish several types of extensions (extensions of $T_{(2N)}$ which are in some sense "generated by $T_{(2N)}$" and which are not) besides the minimal and non minimal extensions. One would like to have a precise definition of these notions.

d) We have shown that it is possible to characterize a relativistic QFT in terms of finitely many *VEV*'s. Collecting the various results we obtain in particular:

Theorem. *Each* $T_{(4)} = \{1, 0, T_2, T_3, T_4\} \in \underline{E}'_{(4)}$ *such that*

(i) $T_{(4)} \restriction \underline{E}_{(4)} \cap \underline{E}_+ \geqq 0$,

(ii) $T_2 \neq 0$,

(iii) $\Delta_2^{T_{(4)}} = 0$,

(iv) $T_{(4)} \in \underline{E}'_{(4)} \cap I^0$,

(v) $T_{(4)}$ *satisfies any of the sufficient conditions for the existence of a m.c.l. extension as discussed in Section 4* *characterizes exactly one relativistic* QFT *(For definiteness we assume here* $E_1 = \mathscr{S}(\mathbb{R}^4)$*).*

Appendix

This appendix contains a proof of Equation (2.7). Using the notation of Section 2 we want to show:

If $\Delta_j \neq 0, j=1, \ldots, n$, then for all $x_{n+1} \in E_{n+1}$

$$\inf\{\hat{G}_{n+1}(\Phi_0(x_0), \ldots, \Phi_n(x_n); Q_n \Phi_{n+1}(x_{n+1})); (x_0, x_1, \ldots, x_n) \in S_n\} = 0.$$

If $\varphi_j = \Phi_{(n)}(\underline{h}^j_{(n)}) \subseteqq \Phi_{(n)}(\underline{E}_{(n)})$, $j \in \mathbb{N}$ is any orthonormal basis of $\mathscr{H}_{(n)}$ then

$$\|Q_n \Phi_{n+1}(x_{n+1})\|^2 = \sum_{j=1}^{\infty} |\langle \varphi_j, \Phi_{n+1}(x_{n+1})\rangle|^2 .$$

Thus given $x_{n+1} \in E_{n+1}$ and $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that

$$\sum_{j=m+1}^{\infty} |\langle \varphi_j, \Phi_{n+1}(x_{n+1})\rangle|^2 < \varepsilon$$

S_n spans $\mathscr{H}_{(n)}$; therefore we may define

$$F_m = \{\underline{x}_{(n)} \in S_n; [\{\Phi_0(x_0), \ldots, \Phi_n(x_n)\}] = [\{\varphi_1, \ldots, \varphi_m\}]\}.$$

$[A]$ denotes the closed subspace generated by the set A in $\mathscr{H}_{(n)}$. Furthermore we denote by $P_{\underline{x}_{(n)}}$ the orthogonal projection onto the subspace spanned by $\Phi_0(x_0), \ldots, \Phi_n(x_n)$. Then for $\underline{x}_{(n)} \in F_m$ we obtain $P_{\underline{x}_{(n)}} = P =$ projection onto $[\{\varphi_1, \ldots, \varphi_m\}]$. The rules for determinants yield

$$\begin{aligned}&\hat{G}_{n+1}(\Phi_0(x_0), \ldots, \Phi_n(x_n); Q_n \Phi_{n+1}(x_{n+1}))\\ &= \hat{G}_n(\Phi_0(x_0), \ldots, \Phi_n(x_n)) \|P^{\perp}_{\underline{x}_{(n)}} Q_n \Phi_{n+1}(x_{n+1})\|^2 .\end{aligned}$$

This implies the following chain of inequalities respectively equalities:

$$\begin{aligned}&\inf\{\hat{G}_{n+1}(\Phi_0(x_0), \ldots, \Phi_n(x_n); Q_n \Phi_{n+1}(x_{n+1})); \underline{x}_{(n)} \in S_n\}\\ &\leqq \inf\{G_{n+1}(\Phi_0(x_0), \ldots, \Phi_n(x_n); Q_n \Phi_{n+1}(x_{n+1})); \underline{x}_{(n)} \in F_m\}\\ &= \|P^{\perp} Q_n \Phi_{n+1}(x_{n+1})\|^2 = \sum_{j=m+1}^{\infty} |\langle \varphi_j, \Phi_{n+1}(x_{n+1})\rangle|^2 < \varepsilon\end{aligned}$$

and thus proves Equation (2.7).

Acknowledgement. I would like to thank Prof. Dr. Haag and the Referee of Communications in Mathematical Physics for criticism and many helpful suggestions concerning a previous version of this paper. It is a great pleasure to thank Dr. P. Blanchard for stimulating discussion on the subject.

References

The list of references is mainly contained in

1. Brüning, E.: On the characterization of relativistic quantum field theories in terms of finitely many vacuum expectation values. I. Commun. math. Phys. **58**, 139—166 (1978). This is a revised version of [1'] and is usually quoted as I

1'. Brüning, E.: On the structure of monotone continuous linear functionals on tensoralgebras that occur in QFT and on a class of extension problems for these algebras. I. Bi-TP 77/06
2. Borchers, H. J.: Algebraic aspects of Wightman field theory. In: Statistical mechanics and field theory (ed.: R. N. Sen, C. Weil) (Haifa Summer School 1971). Jerusalem: Israel University Press 1972
3. Glimm, J., Jaffe, A.: Functional integral methods in quantum field theory. Preprint, to be published in the Proceedings of the 1976 Cargèse Summer School

Communicated by R. Haag

Received August 10, 1977