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Integrability of the Classical $[\bar{\psi}_i \psi_i]_2^2$ and $[\bar{\psi}_i \psi_i]_2^2 - [\bar{\psi}_i \gamma_5 \psi_i]_2^2$ Interactions*

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Abstract. We study the interaction of N classical two-dimensional massless Fermi fields through the symmetric couplings $[\bar{\psi}_i \psi_i]^2$ or $[\bar{\psi}_i \psi_i]^2 - [\bar{\psi}_i \gamma_5 \psi_i]^2$. We explicitly show complete integrability in the cases $N = 1, 2$, using the inverse scattering method. The fields occuring in the associated linear eigenvalue problem and evolution equation are simply related to the fundamental fields ψ_i that satisfy the original non-linear equations. For $N > 2$, calculations become very involved, but there is no doubt that the system remains completely integrable, reducing to appropriate generalizations of the sin$e$- and sin$h$-Gordon equation, a situation analogous to Pohlmeyer's discussion in a somewhat similar problem: the two-dimensional non-linear σ-model. Finally, all the explicit analytic solutions that we have worked out in the present framework are identical to those found by Dashen et al., and Shei, in a semiclassical treatment of the fully quantum mechanical version of these models. This leads us to conjecture that the quantum theory also shares most of the features of completely integrable systems, like the massive Thirring model.

I. Introduction

The theory of massless fermions with scalar contact interactions, first introduced by Nambu and Jona-Lasinio [1] as a model field theory for superconductors, is renormalizable in two space-time dimensions. It has been studied in [2], in the limit where the number N of fermion species goes to infinity, and was shown to be asymptotically free and exhibit dynamical spontaneous symmetry breaking. In the spirit of the so-called $1/N$ expansion, the above theory was further analyzed in [3] by partially integrating out the Fermi fields, reducing to an effective Lagrangian for the scalar composite field $\sigma = \bar{\psi}\psi$. To leading order in the adopted approximation scheme, the effective Lagrangian for the field σ was studied by semiclassical

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functional methods and a very large set of classical particle-like configurations was found, some having only space dependence in their rest frame, others having both space and time dependence. Whereas a systematic method was developed in [3] for finding the static solutions, time-dependent solutions were only found by guessing. Their remarkable soliton-like properties led the authors of this reference to conjecture that they were dealing with a completely integrable classical system.

It is this conjecture that we examine in the present article. Indeed, we shall find a whole new set of classical integrable systems, of which the Lagrangian of [2, 3] is only a limit. We will also find that the associated eigenvalue and evolution problem, which solves the dynamics by linear inverse scattering methods (for a review of these methods see [4]), involves fields that are intimately related to the original fields satisfying non-linear equations. In addition to the $(\bar{\psi}\psi)^2$ interaction, we analyze the chiral $(\bar{\psi}\psi)^2-(\bar{\psi}\gamma_5\psi)^2$ interaction, whose static solutions are known [5], and show that it reduces to completely integrable systems that can be studied systematically.

In [3], it was shown that the effective Lagrangian for the σ-field leads to the following equations of motion:

$$(i\partial\!\!\!/ - g\sigma)\psi_i = 0\,, \qquad \int dx\,\bar{\psi}_i\psi_i = 1\,, \tag{1.1a}$$

$$-(Z/g)\sigma = \sum_{\substack{\text{occupied}\\ \text{states}}} \bar{\psi}_i\psi_i\,, \tag{1.1b}$$

where g is the coupling constant and Z a suitable renormalization constant, which cancels the infinity of the righthand side of Equation (1.1b). The sum over occupied states usually represents the integration over the whole negative-energy Dirac sea, plus the sum over a discrete number of normalizable bound states. Examples of explicit solutions of (1.1) can be found in [3]. From the rather remarkable properties of the time-dependent solutions, one is led to conjecture the sum over the infinite set of modes of the Dirac sea is not crucial for the existence of soliton solutions, although it surely is crucial for their physical interpretation. Indeed, in Equations (1.1) the Dirac sea is essentially the only remanent of the fermionic character of the theory, the ψ_i's of (1.1) being treated as ordinary (commuting) functions. Hence, in the present paper, we shall mostly study the following simplified set of equations

$$(i\partial\!\!\!/ - \sigma)\psi_i = 0\,, \tag{1.2a}$$

$$\sigma = \sum_{i=1}^{N} \bar{\psi}_i\psi_i\,, \tag{1.2b}$$

where we have kept only a finite number N of modes of the Dirac Equation (1.2a). We do not impose any normalization condition on the ψ_i's and have scaled out the coupling constant g. With a finite number of modes, one can also drop the renormalization constant Z. We again stress the fact that the ψ_i's are ordinary c-number functions; thus we are dealing with a system of nonlinear partial differential equations, whose only particular limit (1.1) has physical significance for the description of fermions. However, the classical system (1.2) will be shown to possess a rather remarkable mathematical structure, from the point of view of inverse scattering theory, which we explore in this article.

An important element of our analysis of the non-linear system (1.2) is its local scale invariance. It also possesses an obvious global $U(N)$ invariance, which mixes all the N modes. We shall actually find that the global invariance group of (1.2) is $\mathrm{Sp}(2N,\mathbb{R})$, the non-compact symplectic group over the real numbers in $2N$-dimensions, $U(N)$ being its maximal compact subgroup. The fact that a non-compact symmetry group appears is related to the absence of positivity of the Hamiltonian of the Dirac equation, when the fermion fields commute. Of course, when one goes back to the prescription of [3] for the actual construction of physical solutions, positivity is recovered. The global $\mathrm{Sp}(2N,\mathbb{R})$ symmetry is the second important ingredient of our analysis.

The existence of the local scale invariance and the global $\mathrm{Sp}(2N,\mathbb{R})$ symmetry is reminiscent of the situation in the two-dimensional non-linear σ-model, studied by Pohlmeyer in [6] and shown to be completely integrable. Indeed, by a certain reduction procedure similar to that used in [6], we show the complete integrability, explicitly in the cases $N=1,2$, by using inverse scattering methods. Reduced integrable systems are obtained in terms of symplectic invariant functions, of which a typical example is the composite field σ defined in (1.2b). We derive the Lax representation for these systems and show that the wave-functions of the associated linear eigenvalue problem and evolution equation are intimately related to the original fields ψ_i, satisfying the non-linear system (1.2). We work out a few soliton and doublet solutions. They turn out to be identical to the solutions found in [3] for the system (1.1). For $N>2$, calculations become very cumbersome; we have shown that higher conserved currents exist for any N and have no doubt that the system is completely integrable for any N, in analogy with Pohlmeyer's case.

We also study the chirally symmetric model:

$$
\begin{aligned}
&[i\partial\!\!\!/-(\sigma+i\pi\gamma^5)]\psi_i=0\\
&\sigma\equiv\frac{1}{2}\sum_{i=1}^{N}\bar{\psi}_i\psi_i\\
&\pi\equiv\frac{i}{2}\sum_{i=1}^{N}\bar{\psi}_i\gamma^5\psi_i\,.
\end{aligned}
\tag{1.3}
$$

For $N=1$, this turns out to be a free theory. For $N=2$, it is completely integrable and equivalent to a generalization of the sin$e$-Gordon theory. For $N>2$, higher conservation laws also exist indicating that, apart from involved technical considerations, the system is completely integrable for any N.

In the course of our investigation of the solutions of the above non-linear systems, the close connection between the inverse scattering wave-functions and the original fields provided us with certain technical tools that we have subsequently applied to the two-dimensional non-linear σ-model of [6]. These new results on the reconstruction of the original fields of the non-linear σ-model in terms of inverse scattering wavefunctions are described in an appendix.

We close this introduction with a number of remarks concerning the quantum theory of the above systems. The complete integrability of the $(\bar{\psi}\psi)^2$ model of [2], in the large N approximation, is quite analogous to the complete integrability of the classical sin$e$-Gordon equation, $1/N$ playing the role of the coupling constant (or, of $\hbar$). For $N=1$, a large coupling case, the model reduces to the massless Thirring model, which is scale invariant with anomalous dimensions. For $N>1$, the theory exhibits non-trivial renormalization group behavior and mass-generation through dimensional transmutation. The only exact result known for the quantum theory is for the case $N=2$, which is eventually equivalent to a system of two sin$e$-Gordon equations [7]; although the interpretation of this result is unclear because of the peculiar value that one obtains for the coupling constant of those sin$e$-Gordon theories, it might still become possible to solve the above quantum system exactly. All these results, combined with those of the present article, make it reasonable to conjecture that the full quantum mechanical model of [2] is also solvable for any N, and that, much in analogy with the case of the sin$e$-Gordon theory, the semiclassical spectrum is actually exact, as discussed in the introduction of [3].

II. The $[\bar{\psi}_i\psi_i]_2^2$ Interaction—General

A. Preliminary Considerations for $N=1$

In the first part of this section, we study in some detail the simplest $N=1$ case of the system (1.2), whose many features generalize to arbitrary N. Using light cone coordinates

$$\xi=\tfrac{1}{2}(t-x), \qquad \eta=\tfrac{1}{2}(t+x), \tag{2.1}$$

and the representation of the Dirac matrices:

$$\gamma^0=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \gamma^1=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \tag{2.2}$$

the equations of motion, in terms of the components ψ_1 and ψ_2 of the spinor ψ, read:

$$\begin{aligned} &i\psi_{1,\xi}=\sigma\psi_2\\ &i\psi_{2,\eta}=\sigma\psi_1 \end{aligned} \tag{2.3a}$$

$$\sigma\equiv\psi_1\psi_2^*+\psi_2\psi_1^*, \tag{2.3b}$$

where we have adopted an obvious notation for the differentiation with respect to the light cone variables η and ξ, and * denotes complex conjugation. We remind the reader that all the symbols are c-number functions as explained in the introduction.

Three independent conservation laws follow immediately from (2.3):

$$\begin{aligned} &(\psi_1^*\psi_1)_{,\xi}+(\psi_2^*\psi_2)_{,\eta}=0\\ &(\psi_1^2)_{,\xi}=(\psi_2^2)_{,\eta} \end{aligned} \tag{2.4}$$

the last one being counted as two real equations. The content of (2.4) will be clarified later. In addition, energy-momentum conservation is contained in the following equations:

$$\begin{aligned} &h_{1,\xi}=0=h_{2,\eta}\\ &h_1(\eta)\equiv i(\psi_1^*\psi_{1,\eta}-\psi_1\psi_{1,\eta}^*),\\ &h_2(\xi)\equiv i(\psi_2^*\psi_{2,\xi}-\psi_2\psi_{2,\xi}^*). \end{aligned} \tag{2.5}$$

The equations of motion are form invariant under the local scale transformation:

$$d\xi' = |f_2(\xi)|^{1/2} d\xi\,, \qquad d\eta' = |f_1(\eta)|^{1/2} d\eta$$
$$\psi_1 = |f_1(\eta)|^{1/4} \psi_1'\,, \qquad \psi_2 = |f_2(\xi)|^{1/4} \psi_2'\,, \tag{2.6}$$

where f_1 and f_2 are arbitrary functions of η and ξ respectively.

It is a straightforward manipulation of the equations of motion to derive the following equation for the quantity σ, defined in (2.3b):

$$\sigma\sigma_{,\eta\xi} - \sigma_{,\eta}\sigma_{,\xi} = h_1 h_2 - \sigma^4\,. \tag{2.7}$$

Putting $\sigma \equiv \exp\theta$,

$$\theta_{,\eta\xi} = h_1 h_2 e^{-2\theta} - e^{2\theta}\,. \tag{2.8}$$

Exploring the local scale invariance of the theory, expressed in Equations (2.6), we can rescale h_1 and h_2 by arbitrary positive functions. Being interested in bounded real solutions for σ in (2.7), $h_1 h_2$ must approach asymptotically a non-negative constant. The above two observations lead us to consider the special conformal frame

$$h_1 = 1 = h_2\,, \tag{2.9}$$

where all dimensional parameters are set equal to 1 for convenience. Due to the fact that h_1 and h_2 are not positive definite functionals of the fields in the present theory, $h_1 h_2$ can in general be negative at finite distances; this class of solutions, if such solutions exist at all, requires special treatment that goes beyond the scope of the present work. We simply show in the following that the class (2.9) is highly non-trivial and likely contains the most interesting solutions of our system. With this choice, Equation (2.8) becomes the $\sin h$-Gordon equation that can be solved by inverse scattering techniques. In particular, we can easily derive the Backlünd transformation for the field σ:

$$(\sigma'/\sigma)_{,\xi} = \alpha(\sigma'^2 - \sigma^{-2})$$
$$(\sigma'\sigma)_{,\eta} = \frac{1}{\alpha}(\sigma^2 - \sigma'^2)\,. \tag{2.10}$$

$\sigma = 1$ and $\sigma = -1$ are solutions of the equation of motion; we shall call them vacuua. There is a soliton that connects these two vacuua: it is found by applying the Backlünd transformation (2.10). To find it at rest, we take $\alpha = 1$ to obtain

$$\sigma = \tan hx\,. \tag{2.11}$$

A second application of (2.10) gives the soliton-antisoliton solution:

$$\sigma = \frac{v\cos h(2x/\sqrt{1-v^2}) - \cos h(2vt/\sqrt{1-v^2})}{v\cos h(2x/\sqrt{1-v^2}) + \cos h(2vt/\sqrt{1-v^2})}\,, \qquad 0 < v < 1\,. \tag{2.12}$$

It is rather remarkable, at this stage, that these solutions were already found in [3], in connection with the complete quantum mechanical problem. Note also that although the sin$h$-Gordon equation has no bounded soliton solution, the equation for σ does, since $\theta = \ln\sigma$ is infinite for $\sigma = 0$[1]

We shall, of course, be interested in solving the model in terms of the fundamental fields. The actual integrability, the Backlünd transformation and the infinite set of conservation laws in terms of the fundamental fields ψ_1 and ψ_2 will be analyzed in the following sections. In the remainder of this section, we systematize certain formal properties of the theory, in order to establish a convenient framework for the following considerations.

B. Symplectic Symmetry

For $N=1$, the symmetry of the model is generated by the conservation laws (2.4). We construct the associated conserved charges:

$$\begin{aligned} Q_1 &= \frac{i}{2\sqrt{2}} \int dx(\pi_1^2 + \psi_1^2 - \pi_2^2 - \psi_2^2) \\ Q_2 &= \frac{1}{2\sqrt{2}} \int dx(\psi_1^2 - \pi_1^2 - \psi_2^2 + \pi_2^2) \qquad (2.13) \\ Q_3 &= -\frac{i}{2} \int dx(\pi_1\psi_1 + \pi_2\psi_2), \end{aligned}$$

with $\pi_1 = i\psi_1^*$, $\pi_2 = i\psi_2^*$ being the canonical momenta conjugate to ψ_1 and ψ_2. The Poisson brackets for the Q_i's are:

$$\{Q_1, Q_2\} = -Q_3, \quad \{Q_1, Q_3\} = -Q_2, \quad \{Q_2, Q_3\} = Q_1. \qquad (2.14)$$

Therefore, the charges Q_i generate a non-compact Lie group with three parameters. The nature of this group is found by looking at the most general global transformations that leave the Lagrangrian of the theory invariant. Splitting the fields ψ_1 and ψ_2 into real and imaginary components, and defining

$$\chi^T \equiv (\mathrm{Re}\,\psi_1, \mathrm{Im}\,\psi_1, \mathrm{Re}\,\psi_2, \mathrm{Im}\,\psi_2), \qquad (2.15)$$

where T denotes transposition of a matrix, we find that the most general transformation

$$\chi \to A\chi, \qquad (2.16)$$

that leaves the Lagrangian invariant, is a 4×4 real matrix of the form:

$$A = \begin{bmatrix} R & 0 \\ 0 & (R^T)^{-1} \end{bmatrix} \qquad (2.17)$$

where R is a 2×2 real matrix that leaves the two-dimensional symplectic form invariant:

$$R^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (2.18)$$

[1] We use the variable θ only for notational convenience

Hence, the symmetry group of the Lagrangian is the two dimensional symplectic group over the real numbers, Sp(2, ℝ), whose Lie algebra is indeed isomorphic to (2.14). Recall that Sp(2, ℝ) ∼ SO(2, 1).

We observe that both the covariant and the contravariant representations R and $(R^T)^{-1}$, come into play. This suggests the following interpretation and choice of notation, which we call the symplectic spinor formulation: the set of real numbers $\mathrm{Re}\,\psi_1$, $\mathrm{Im}\,\psi_1$ can be chosen as the contravariant components of a vector u,

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \equiv \sqrt{2}\begin{pmatrix} \mathrm{Re}\,\psi_1 \\ \mathrm{Im}\,\psi_1 \end{pmatrix}, \tag{2.19}$$

whereas the set $\mathrm{Re}\,\psi_2$, $\mathrm{Im}\,\psi_2$ can be chosen as the covariant components of a vector v:

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \equiv \sqrt{2}\begin{pmatrix} \mathrm{Re}\,\psi_2 \\ \mathrm{Im}\,\psi_2 \end{pmatrix}. \tag{2.20}$$

The metric tensor $\varepsilon_{\alpha\beta}$ is

$$(\varepsilon_{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \alpha, \beta = 1, 2 \tag{2.21a}$$

$$(\varepsilon^{\alpha\beta}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.21b}$$

Raising and lowering of indices goes as usual. For example:

$$(u_\alpha) = \sqrt{2}\begin{pmatrix} \mathrm{Im}\,\psi_1 \\ -\mathrm{Re}\,\psi_1 \end{pmatrix}, \qquad (v^\alpha) = \sqrt{2}\begin{pmatrix} -\mathrm{Im}\,\psi_2 \\ \mathrm{Re}\,\psi_2 \end{pmatrix}. \tag{2.22}$$

Notice properties of the form

$$u^2 = 0 = v^2, \qquad u^\alpha v_\alpha = -u_\alpha v^\alpha. \tag{2.23}$$

This last property suggests the following convention, useful for calculations:

$$uv \equiv u^\alpha v_\alpha = -v^\alpha u_\alpha \equiv -vu. \tag{2.24}$$

The calculations of the first part of this section can be expressed, and actually simplified, in this notation. In particular, the equations of motion are:

$$u_{,\xi} = -\sigma v, \qquad v_{,\eta} = \sigma u, \qquad \sigma = uv, \tag{2.25}$$

and the energy-momentum densities read

$$h_1 = -uu_{,\eta}, \qquad h_2 = -vv_{,\xi}. \tag{2.26}$$

The generalization of the above discussion to arbitrary $N > 1$ is straightforward. It is not difficult to find that the symmetry group in the general case is Sp(2N, ℝ). Equations (2.25) and (2.26) retain the same form except now u and v are interpreted as N-component symplectic spinors defined in terms of the fundamental fields ψ_i as follows: Let $\psi_{1,i}$ and $\psi_{2,i}$ be the upper and lower components of the Lorentz spinors ψ_i, $i=1,\dots N$. The components of the symplectic spinors u and v read

$$\begin{aligned}(u^\alpha)^T &= \sqrt{2}(\mathrm{Re}\psi_{1,1}\dots\mathrm{Re}\psi_{1,N},\mathrm{Im}\psi_{1,1}\dots\mathrm{Im}\psi_{1,N})\\ (v_\alpha)^T &= \sqrt{2}(\mathrm{Re}\psi_{2,1}\dots\mathrm{Re}\psi_{2,N},\mathrm{Im}\psi_{2,1}\dots\mathrm{Im}\psi_{2,N}).\end{aligned} \tag{2.27}$$

The metric tensor $\varepsilon_{\alpha\beta}$, $\alpha,\beta=1,\dots N$, generalizes to

$$(\varepsilon_{\alpha\beta})=\begin{pmatrix}0 & I_N\\ -I_N & 0\end{pmatrix},\quad (\varepsilon^{\alpha\beta})=\begin{pmatrix}0 & -I_N\\ I_N & 0\end{pmatrix}, \tag{2.28}$$

with I_N the unit N-dimensional matrix. Equations (2.23) and (2.24) remain valid. Notice that for $N=2$, the group $\mathrm{Sp}(4,\mathbb{R})$ is locally isomorphic to $\mathrm{SO}(3,2)$. No such correspondence exists for higher N.

Armed with the above general framework, we now come to discuss two important points, before closing this section. First, the existence of higher conservation laws. In the special conformal frame $h_1=1=h_2$,

$$(u_{,\eta}u_{,\eta\eta})_{,\xi}=(-2\sigma^2)_{,\eta} \tag{2.29a}$$

$$\begin{aligned}&[u_{,\eta\eta}u_{,\eta\eta\eta}+\tfrac{5}{4}(u_{,\eta}u_{,\eta\eta})^2]_{,\xi}\\ &=[(\sigma^2)_{,\eta}-\sigma\sigma_{,\eta\eta}-\sigma^2(u_{,\eta}u_{,\eta\eta})]_{,\eta}\end{aligned} \tag{2.29b}$$

and similar ones with $u\leftrightarrow v$ and $\eta\leftrightarrow\xi$. Equations (2.29) are valid for arbitrary N. These are only the first two of an infinite set of conservation laws, involving higher and higher derivatives of the fields and expressing the integrability of our systems.

The second point to make is to indicate an important difference that arises as we go to higher and higher values of N. To be precise, we rederive here the sin$h$-Gordon equation associated with the $N=1$ system, using Equations (2.25). A simple calculation shows that for arbitrary N the following equation is valid:

$$\sigma_{,\eta\xi}=u_{,\eta}v_{,\xi}-\sigma^3. \tag{2.30}$$

For $N=1$, however, the symplectic invariant quantity $\omega\equiv u_{,\eta}v_{,\xi}$ can be further reduced. In fact, the solution vectors u and v can in this case be taken as a complete basis. The following identities then hold:

$$\begin{aligned}\sigma u_{,\eta}&=\sigma_{,\eta}u-h_1v\\ \sigma v_{,\xi}&=h_2u+\sigma_{,\xi}v,\end{aligned} \tag{2.31}$$

implying that

$$\omega\equiv u_{,\eta}v_{,\xi}=(\sigma_{,\eta}\sigma_{,\xi}+h_1h_2)/\sigma, \tag{2.32}$$

which combined with (2.30) yields:

$$\sigma\sigma_{,\eta\xi}-\sigma_{,\eta}\sigma_{,\xi}=h_1h_2-\sigma^4. \tag{2.33}$$

This is precisely Equation (2.7) that was subsequently transformed into the sin$h$-Gordon equation. The reduction of the invariant ω in (2.32) in terms of the invariants σ, h_1, h_2 and their derivatives is a special case of the important fact that for $N=1$ all relevant symplectic invariant quantities can be reduced in terms of the fundamental set $\{\sigma, h_1, h_2\}$. In particular, the conservation laws (2.29), expressed in terms of symplectic invariants, can for $N=1$ be reduced in terms of the above fundamental set, leading to forms that could otherwise be obtained directly from the known conservation laws of the sin$h$-Gordon equation. We have explicitly checked that this is indeed so. Turning our discussion to $N>1$, we immediately realize that the reduction (2.32) fails, for the simple reason that u and v do not form a complete basis. The appropriate fundamental set of invariants is now larger, depending on the particular value of N. We shall see in Section V that the construction of the above set of invariants proceeds in a systematic manner for higher N.

III. Coordinate Dependent Symplectic Transformation

The discussion of this section is valid for any N. This becomes manifest in the symplectic notation.

Although the choice $h_1 = 1 = h_2$ breaks Lorentz covariance, the final equation for the scalar field σ in Section II is manifestly Lorentz invariant. In a Lorentz transformation, h_1 is scaled by a constant γ, h_2 by $1/\gamma$. We now exhibit a spacetime dependent symplectic transformation $R(\eta, \xi; \gamma)$ under which σ remains, of course, invariant and h_1 and h_2 are scaled in a manner identical to a Lorentz transformation, in close analogy with a similar transformation constructed in Section V of [6].

Given a solution u and v of (2.25), it is straightforward to verify that the following equations are compatible:

$$\begin{aligned} \partial_\eta R^\alpha_\beta &= (1-\gamma) R^\alpha_\varrho u^\varrho u_\beta \\ \partial_\xi R^\alpha_\beta &= (1-\gamma^{-1}) R^\alpha_\varrho v^\varrho v_\beta , \end{aligned} \tag{3.1}$$

and define a symplectic matrix R. Similarly, one checks that:

$$\begin{aligned} u(\ ;\gamma) &\equiv \sqrt{\gamma}\, Ru , \\ v(\ ;\gamma) &\equiv \frac{1}{\sqrt{\gamma}} Rv , \end{aligned} \tag{3.2}$$

also satisfy the equation of motion with the same σ, and that the associated energy-momentum densities h_1 and h_2 are scaled by γ and $1/\gamma$. Hence the product $h_1 h_2$ and the equation of motion for σ remain invariant.

The implications and possibly a deep connection of the transformation $R(\eta, \xi; \gamma)$ with the Poincaré group are not completely understood. We include here, however, some remarks that eventually explain the fact that explicit solutions for the invariant σ constructed in the present framework (finite N) coincide with those found in [3] in the limit $N\to\infty$. At spatial infinity $\sigma\to\pm 1$, and the corresponding field ψ is a plane wave. The effect of R on this wave is to simply change its energy-momentum, thus R relates to one another all the modes of the continuum of the Dirac equation. It is then not surprising that the infinite sum over all these modes, which occurs in [3] and Equation (1.1), can actually give very simple results: in practice, R appears to reduce the above infinite sum over an essentially finite number of modes, which is precisely the object of the present work. Of course, the detailed description of the evolution of the system at finite distances requires the determination of an increasing number of fundamental invariants besides σ, with increasing N, as we shall explicitly see in the following.

IV. Complete Integrability of the Sp(2, ℝ) System

In this section, we study the inverse problem for the $N=1$ case. In particular, we solve the problem of the determination of the fundamental fields u and v associated with any solution of the sin$h$-Gordon equation, formulated in terms of the symplectic invariant function $\sigma=uv$. Throughout this section, the conformal frame is fixed to be

$$h_1=\gamma, \quad h_2=1/\gamma; \quad h_1h_2=1. \tag{4.1}$$

The equations of motion (2.25) along with the identities (2.31) are written here in a compact form as

$$\begin{bmatrix} u \\ v \end{bmatrix}_{,\eta}=C_1\begin{bmatrix} u \\ v \end{bmatrix}, \quad \begin{bmatrix} u \\ v \end{bmatrix}_{,\xi}=C_2\begin{bmatrix} u \\ v \end{bmatrix}$$

$$C_1\equiv\begin{bmatrix} \sigma^{-1}\sigma_{,\eta} & -\gamma\sigma^{-1} \\ \sigma & 0 \end{bmatrix}, \quad C_2\equiv\begin{bmatrix} 0 & -\sigma \\ \gamma^{-1}\sigma^{-1} & \sigma^{-1}\sigma_{,\xi} \end{bmatrix}. \tag{4.2}$$

The integrability condition for the system (4.2) reads

$$C_{1,\xi}-C_{2,\eta}+[C_1, C_2]=0 \tag{4.3}$$

resulting in the equation for σ:

$$\sigma\sigma_{,\eta\xi}-\sigma_{,\eta}\sigma_{,\xi}=1-\sigma^4 \tag{4.4}$$

which is precisely the reduced system obtained in Section II. It is therefore clear that the associated linear problem is essentially provided by the system (4.2), the role of the eigenvalue parameter being played by the energy-momentum parameter introduced in (4.1). It is not difficult to transform (4.2) into a more familiar form. Observe that Equations (4.2) are valid for both real components of the symplectic spinors u and v. For the actual solution of the inverse problem, it is more convenient to work with appropriate complex combinations of the above real components, the most obvious choice being the inverse transform (2.19) and (2.20). Thus in terms of the original complex fields ψ_1 and ψ_2, and the change of variables $\psi\to\chi$:

$$\psi_1=\gamma^{1/4}(\chi_1+\chi_2), \quad \psi_2=\gamma^{-1/4}\sigma(\chi_1-\chi_2) \tag{4.5}$$

Equations (4.2) read:

$$\chi_{,\eta}=C_1'\chi, \quad \chi_{,\xi}=C_2'\chi$$

$$C_1'=\begin{bmatrix} -i\zeta & \theta_{,\eta} \\ \theta_{,\eta} & i\zeta \end{bmatrix}, \quad C_2'=\frac{1}{i\zeta}\begin{bmatrix} \cos h2\theta & -\sin h2\theta \\ \sin h2\theta & -\cos h2\theta \end{bmatrix}$$

$$\zeta\equiv\gamma^{1/2}, \quad \sigma\equiv\exp\theta, \tag{4.6}$$

where we recognize the familiar Lax representation used for the solution of the sin$h$-Gordon equation [4].

It should be noted that the Backlünd transformation (4.9) and (4.10) does not change the conformal frame (4.1), for the Backlünd parameter α is not related to the energy momentum parameter γ. The proof of the above results can, of course, be obtained by a direct calculation. It is also a simple and interesting exercise to show that, up to a trivial symplectic rotation, the explicit solution (4.8) can rapidly be reconstructed by using the Backlünd transformation.

Further, Equations (4.9) and (4.10) imply the conservation equation (set $\gamma = 1$):

$$(u' u_{,\eta})_{,\xi} = \frac{\alpha^2}{\sqrt{1+\alpha^2}} (\sigma' \sigma)_{,\eta} . \tag{4.11}$$

Expanding this equation in powers of α, with the aid of (4.9) and (4.10), we obtain an infinite set of conservation laws expressed directly in the conformal frame $h_1 = 1 = h_2$. The first few special cases have been already given in Section II. The generalization of these conservation laws to arbitrary energy-momentum densities h_1 and h_2 is not difficult, but presents no interest for our considerations. Finally, we note the trivial fact that a second Backlünd transformation is obtained by interchanging $u \leftrightarrow v$, $u' \leftrightarrow v'$, and $\eta \leftrightarrow \xi$.

We conclude this section with a brief discussion of the singular nature of the inverse problem (4.6), at space-time points where σ becomes equal to zero. Such a singular behavior was well expected, for the associated linear problem was established by using the identities (2.31) based on the assumption that the solution vectors u and v are linearly independent, which ceases to be true when $\sigma \equiv uv = 0$. Fortunately, the above singularity cannot occur in the asymptotic region thanks to the fact that at large distances any bounded solution for σ must reach the asymptotic values ± 1, as dictated by the equation of motion (4.4), in which case the matrices C_1 and C_2 in (4.6) are well tempered. The inverse problem is in general not expected to meet any serious difficulty, as was already evident in the explicit example (4.8). Such a singularity might, however, cause peculiar behavior of the eigenvalue problem and certainly deserves closer attention. In this connection, the following remarks are due to Kaup[2], which we briefly outline. The argument is most conveniently phrased in a representation that is obtained from (4.6) by changing variables to $\phi_1 = \chi_1 + \chi_2$, $\phi_2 = \chi_1 - \chi_2$:

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}_{,\eta} = \begin{bmatrix} \sigma^{-1}\sigma_{,\eta} & -i\zeta \\ -i\zeta & -\sigma^{-1}\sigma_{,\eta} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \quad \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}_{,\xi} = \frac{1}{i\zeta} \begin{bmatrix} 0 & \sigma^2 \\ \sigma^{-2} & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} . \tag{4.12}$$

It is then easy to show that ϕ_1 satisfies the system of equations, $V \equiv \sigma^{-1} \sigma_{,\eta\eta}$:

$$\phi_{1,\eta\eta} + (\zeta^2 - V)\phi_1 = 0, \qquad \phi_{1,\xi} = \frac{1}{\zeta^2} (\sigma^2 \phi_{1,\eta} - \sigma \sigma_{,\eta} \phi_1) . \tag{4.13}$$

Eventually, Equations (4.13) may be used for the formulation of the inverse problem. From the equation of motion (4.4) and the definition of the "potential" V follows that the latter is bounded at vanishing σ. Hence, the (Schrödinger) eigenvalue problem in (4.13) is well behaved, but in turn shows the following

[2] D. J. Kaup, private communication. We thank Dr. Kaup for his kind interest

Hence, we arrive at the important conclusion that the inverse problem based on Equations (4.6) serves not only to determine the symplectic invariant function σ, but also the fundamental fields ψ_1 and ψ_2. Few qualifications are necessary, however. In solving the inverse problem (4.6), the "potential" σ and the wave-function χ are determined simultaneously. Further, for a given solution for σ there are two independent wave-functions satisfying the "linear" Dirac problem (4.6); the most general wave-function solving (4.6) is, therefore, an arbitrary linear superposition of any two special solutions, involving two arbitrary complex coefficients, thus four real parameters. The above arbitrariness carries over to the wave-functions ψ_1 and ψ_2, see Equation (4.5). The relation $\sigma = \bar{\psi}\psi = \psi_1\psi_2^* + \psi_1^*\psi_2$, necessary in order to interpret ψ as a solution of the original problem, is in general violated. The resolution of this question is actually immediate: It is not difficult to prove that for any solution σ and χ of the inverse problem (4.6) the wave-function ψ defined in (4.5) satisfies the equation

$$\sigma = K(\bar{\psi}\psi), \tag{4.7}$$

where K is a space-time independent real constant. Clearly, K can be set equal to 1 by appropriately choosing the arbitrary parameters entering the linear superposition mentioned above. With this choice, ψ becomes a solution of the original equation of motion in the conformal frame (4.1). After imposing the $K=1$ condition, there remains a freedom of three arbitrary real constants corresponding to trivial symplectic rotations.

As an illustration we have worked out the one soliton case. Generalizations to more complicated solutions present no difficulty. The explicit expressions read

$$\begin{aligned}
\psi_1 &= \left(\frac{\zeta}{2}\right)^{1/2}\left[1 + \frac{i\alpha}{\zeta + i\alpha}(\tanh\varrho - 1)\right]e^{-i\theta} \\
\psi_2 &= \pm\frac{1}{(2\zeta)^{1/2}}\left[\tanh\varrho + \frac{i\alpha}{\zeta + i\alpha}(1 - \tanh\varrho)\right]e^{-i\theta} \\
\sigma &= \pm\tanh\varrho\,, \qquad h_1 = \gamma\,, \qquad h_2 = \gamma^{-1} \\
\theta &\equiv \zeta\eta + \zeta^{-1}\xi \qquad \varrho \equiv \alpha^{-1}\xi - \alpha\eta\,.
\end{aligned} \tag{4.8}$$

Recall that $\zeta = \gamma^{1/2}$; α is an arbitrary constant.

The Backlünd transformation in terms of the fundamental fields is most easily formulated in the symplectic notation. Let u and v be a solution with $h_1 = \gamma$, $h_2 = 1/\gamma$ and $uv \equiv \sigma$. Then u' and v' defined as

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \frac{1}{\sqrt{1+\gamma\alpha^2}}\begin{bmatrix} \sigma'/\sigma & \gamma\alpha/\sigma \\ -\alpha\sigma' & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}, \tag{4.9}$$

with α an arbitrary real parameter, in general different from γ, is also a solution with $u'v' = \sigma'$, $h_1' = h_1 = \gamma$, $h_2' = h_2 = 1/\gamma$, provided that σ and σ' are related by the Backlünd transformation, quoted already in Equation (2.10):

$$\begin{aligned}
(\sigma'/\sigma)_{,\xi} &= \alpha[\sigma'^2 - \sigma^{-2}] \\
(\sigma'\sigma)_{,\eta} &= \frac{1}{\alpha}[\sigma^2 - \sigma'^2]\,.
\end{aligned} \tag{4.10}$$

peculiarity: from the very definition of V it is clear that (4.13) admits a zero eigenvalue. In particular,

$$\sigma=\phi_1(\zeta=0)\,. \tag{4.14}$$

More importantly, Equation (4.14) can be used in conjunction with the inverse problem (4.6) for a simple determination of the invariant σ. Up to trivial ξ-dependent phases and overal factors the explicit expressions (4.8) may be used to illustrate this result.

V. The Sp(4, ℝ) System

We proceed with the reduction of the $N=2$ case. We shall show that the system of differential equations in terms of symplectic invariants is formulated with the variables

$$\sigma=uv\,,\qquad \omega=u_{,\eta}v_{,\xi}\,,\qquad h_1=-uu_{,\eta}=\gamma\,,\qquad h_2=-vv_{,\xi}=1/\gamma \tag{5.1a}$$

$$H_1=u_{,\eta}u_{,\eta\eta}\,,\qquad H_2=v_{,\xi}v_{,\xi\xi}\,, \tag{5.1b}$$

which turns out to be the fundamental set of invariants, extending the set $\{\sigma, h_1, h_2\}$ encountered in the $N=1$ case. It is noticeable that the above extension possesses a systematic character. The invariants (5.1a) appear in the symplectic contractions of the vectors of the now extended complete basis

$$e_1=u\,,\qquad e_2=v\,,\qquad e_3=u_{,\eta}\,,\qquad e_4=v_{,\xi}\,, \tag{5.2}$$

or else, they are the independent elements of the antisymmetric metric tensor $g_{\alpha\beta}=e_\alpha e_\beta$:

$$(g_{\alpha\beta})=\begin{bmatrix} 0 & \sigma & -h_1 & \sigma_{,\xi} \\ -\sigma & 0 & -\sigma_{,\eta} & -h_2 \\ h_1 & \sigma_{,\eta} & 0 & \omega \\ -\sigma_{,\xi} & h_2 & -\omega & 0 \end{bmatrix}. \tag{5.3}$$

H_1 and H_2 in (5.1b) are the invariants appearing in the lefthand side of the first non-trivial conservation law, Equation (2.29a) and its symmetrical with $u\leftrightarrow v$, $\eta\leftrightarrow\xi$. This pattern is trivially verified in the $N=1$ case, where σ is the only independent element of the metric and h_1 and h_2 appear in the very first (energy-momentum) conservation law of the system, preceeding Equations (2.29).

The differential equations obeyed by the H_1 and H_2 are already known. We shall only have to generalize Equation (2.29a) to arbitrary value γ of the energy-momentum parameter:

$$\begin{aligned} H_{1,\xi}&=(-2\gamma\sigma^2)_{,\eta} \\ H_{2,\eta}&=\left(-\frac{2}{\gamma}\sigma^2\right)_{,\xi}. \end{aligned} \tag{5.4}$$

The equation for σ is also known, Equation (2.30):

$$\sigma_{,\eta\xi}=\omega-\sigma^3\,. \tag{5.5}$$

The derivation of the equation for ω is lengthy and results in a complicated second order differential equation, which, nevertheless, contains only the invariants appearing in (5.1) and their derivatives. We shall not give here the details of the derivation but rather outline its main points and, finally, introduce a different but equivalent set of variables in which the system takes over a very simple form. A direct application of the equation of motion yields

$$\omega_{,\eta\xi}=u_{,\eta\eta}v_{,\xi\xi}+\gamma\sigma_{,\xi\xi}+\frac{1}{\gamma}\sigma_{,\eta\eta}-\sigma^2\omega-\sigma^2\sigma_{,\eta\xi}-5\sigma\sigma_{,\eta}\sigma_{,\xi}\,. \tag{5.6}$$

We are thus led to the reduction of the invariant $u_{,\eta\eta}v_{,\xi\xi}$. This is achieved by first expanding the symplectic vector $u_{,\eta\eta}$ in the basis (5.2)

$$u_{,\eta\eta}=c^{\alpha}e_{\alpha}\,, \tag{5.7}$$

and determining the coefficients:

$$c^{\alpha}=g^{\alpha\beta}(e_{\beta}u_{,\eta\eta})\,, \tag{5.8}$$

where

$$(g^{\alpha\beta})=(g_{\alpha\beta})^{-1}=\frac{1}{\varDelta}\begin{bmatrix} 0 & -\omega & -h_2 & \sigma_{,\eta} \\ \omega & 0 & -\sigma_{,\xi} & -h_1 \\ h_2 & \sigma_{,\xi} & 0 & -\sigma \\ -\sigma_{,\eta} & h_1 & \sigma & 0 \end{bmatrix} \tag{5.9a}$$

$$\varDelta\equiv[\det(g_{\alpha\beta})]^{1/2}=\sigma\omega-\sigma_{,\eta}\sigma_{,\xi}-1\,. \tag{5.9b}$$

Hence,

$$u_{,\eta\eta}v_{,\xi\xi}=c^{\alpha}(e_{\alpha}v_{,\xi\xi})=g^{\alpha\beta}(e_{\beta}u_{,\eta\eta})(e_{\alpha}v_{,\xi\xi})\,. \tag{5.10}$$

In order to complete the reduction, we also need the identities (recall that h_1 and h_2 are constants)

$$\begin{aligned} &vv_{,\xi\xi}=0=uu_{,\eta\eta}\,, \quad H_1=u_{,\eta}u_{,\eta\eta}\,, \quad H_2=v_{,\xi}v_{,\xi\xi} \\ &uv_{,\xi\xi}=\sigma_{,\xi\xi}-h_2\sigma\,, \quad vu_{,\eta\eta}=h_1\sigma-\sigma_{,\eta\eta} \\ &u_{,\eta}v_{,\xi\xi}=\omega_{,\xi}-h_2\sigma_{,\eta}+\sigma^2\sigma_{,\xi}\,, \quad v_{,\xi}u_{\eta\eta}=-\omega_{,\eta}+h_1\sigma_{,\xi}-\sigma^2\sigma_{,\eta} \end{aligned} \tag{5.11}$$

that can easily be verified. Substitution of (5.9) and (5.11) into (5.10) yields the desired result, unfortunately in a complicated form. We have succeeded in simplifying the above expression by a suitable choice of variables, obtained by direct experimentation and some guidance offered from the existence of the coordinate dependent symplectic transformation discussed in Section III. In fact, under the transformation $R=R(\ ;\gamma')$ σ is obviously invariant and so turns out to be true for ω; whereas h_1, h_2, H_1, and H_2 transform according to:

$$\begin{aligned} &h_1\to\gamma' h_1\,, \quad h_2\to\gamma'^{-1}h_2 \\ &H_1\to\gamma' H_1+\gamma'(1-\gamma')h_1^2\,, \quad H_2\to\gamma'^{-1}H_2+\gamma'^{-1}(1-\gamma'^{-1})h_2^2\,. \end{aligned} \tag{5.12}$$

It appears preferable to introduce R-invariant variables, so that the energy-momentum parameter drops out from our system of equations. Of course, such a choice is not unique. At any rate, the following choice of R-invariant variables results in a tremendous simplification of the equations of motion:

$$\begin{aligned}
&\sigma,\ \Delta=\sigma\omega-\sigma_{,\eta}\sigma_{,\xi}-1\\
&Q_1=h_2H_1+h_1-\sigma^{-1}\sigma_{,\eta\eta}\\
&Q_1=h_1H_2+h_2-\sigma^{-1}\sigma_{,\xi\xi},
\end{aligned}\tag{5.13}$$

in terms of which the desired system of equations reads:

$$\begin{aligned}
&\sigma\sigma_{,\eta\xi}-\sigma_{,\eta}\sigma_{,\xi}=1+\Delta-\sigma^4\\
&\Delta\Delta_{,\eta\xi}-\Delta_{,\eta}\Delta_{,\xi}=\sigma^2(Q_1Q_2-\Delta^2)\\
&\sigma^2Q_{1,\xi}+\Delta_{,\eta}=0\\
&\sigma^2Q_{2,\eta}+\Delta_{,\xi}=0.
\end{aligned}\tag{5.14}$$

Observe that the energy-momentum parameter has disappeared in (5.14), through the R-invariant constraint $h_1h_2=1$. Equations (5.14) constitute the first non-trivial generalization of the sin$h$-Gordon system analyzed in the preceeding sections. Clearly, the above reduction procedure can in principle be generalized to arbitrary $N>2$, but with rapidly increasing amount of technical work.

The derivation of the associated linear problem follows the pattern of Section IV, but is tedious. We skip the technical details and state the result. In terms of the symplectic vectors appearing in (5.2), we form the linear combinations:

$$\begin{aligned}
&\hat{Z}_1=\gamma^{-1}e_1, && \hat{Z}_2=\sigma^{-1}e_2\\
&\hat{Z}_3=\gamma^{-1}(e_3-\sigma^{-1}\sigma_{,\eta}e_1), && \hat{Z}_4=\Delta^{-1}(\sigma e_4-\sigma_{,\xi}e_2-\gamma^{-1}e_1).
\end{aligned}\tag{5.15}$$

We can then derive the system of first order differential equations

$$\hat{Z}_{\alpha,\eta}=C_{1,\alpha\beta}\hat{Z}_\beta,\quad \hat{Z}_{\alpha,\xi}=C_{2,\alpha\beta}\hat{Z}_\beta$$

$$C_1=\begin{bmatrix}\sigma^{-1}\sigma_{,\eta} & 0 & 1 & 0\\ \gamma & -\sigma^{-1}\sigma_{,\eta} & 0 & 0\\ -\gamma & \Delta^{-1}\Delta_{,\eta} & (\Delta^{-1}\Delta_{,\eta}-\sigma^{-1}\sigma_{,\eta}) & Q_1\\ 0 & -\Delta^{-1}(1+\Delta) & -\Delta^{-1} & (\sigma^{-1}\sigma_{,\eta}-\Delta^{-1}\Delta_{,\eta})\end{bmatrix}$$

$$C_2=\begin{bmatrix}0 & -\gamma^{-1}\sigma^2 & 0 & 0\\ \sigma^{-2} & 0 & 0 & \Delta\sigma^{-2}\\ -(1+\Delta)\sigma^{-2} & 0 & 0 & 0\\ 0 & -\Delta^{-2}\sigma^2Q_2 & -\Delta^{-2}\sigma^2Q_2 & 0\end{bmatrix}\tag{5.16}$$

By construction, the integrability condition of the system (5.16) leads to the system of Equations (5.14).

We call attention to certain peculiar, though well expected, features of the inverse problem (5.16). It becomes singular at vanishing values of the invariant Δ. Recall that Δ is the square root of the determinant of the metric tensor associated with the moving frame (5.2), therefore the basic vectors e_α cease to be linearly independent at vanishing Δ. A similar phenomenon was already observed in the end of Section IV, in the considerably simpler Sp(2, ℝ) case. In the present problem, the above singularities are more serious in that they affect the behavior of the matrices at large distances. To be more precise, for bounded solutions the invariants σ, Δ, Q_1, Q_2 appearing in (5.14) must reach asymptotic values, denoted here by the same symbols, satisfying the algebraic system:

$$1+\Delta-\sigma^4=0, \qquad Q_1Q_2-\Delta^2=0. \tag{5.17}$$

In particular, the choice of asymptotic values according to $\Delta=0$, $Q_1Q_2=0$, $\sigma=\pm 1$ is compatible with (5.17), and thus with (5.14), but leads to degeneracy of C_1 and C_2 at large distances, causing obvious problems in the solution of the inverse problem. Clearly, such configurations must be reached by a careful limiting procedure. Eventually, (5.16) may be transformed into less singular representations by suitable coordinate dependent transformations. We have not studied this problem in detail. Nevertheless, we have succeeded in constructing a non-trivial static solution of the system, showing obvious soliton characteristics:

$$\begin{aligned}
&\sigma=1+y\tanh\left[yx-\frac{1}{4}\ln\left(\frac{1+y}{1-y}\right)\right]-y\tanh\left[yx+\frac{1}{4}\ln\left(\frac{1+y}{1-y}\right)\right]\\
&\Delta=\frac{4(\sigma-1)}{\cosh^2(2A)}, \qquad Q_1=Q_2=\frac{4(\sigma^{-1}-1)}{\cosh^2(2A)},
\end{aligned} \tag{5.18}$$

where x is the position variable, A a constant and $y\equiv\tanh 2A$. The expression for σ in (5.18) is precisely identical to the kink-antikink solution of [3], obtained in the semi-classical ($N\to\infty$) limit. Although it now becomes clear that the complete specification of the dynamical behavior of the system requires the determination of additional symplectic invariant quantities, such as Δ, Q_1, and Q_2, it should be noted that the invariant σ plays a distinguished role in the semi-classical approximation.

VI. The Two-dimensional Chiral Interaction

In analogy with the symplectic case, a reduction procedure can also be applied to the chiral interaction defined in Equation (1.3). We first establish a convenient notation and discuss important general features of the theory. We denote by u_i and v_i the upper and lower components of the Lorentz spinors ψ_i, with $i=1,2,\ldots N$. With the representation (2.2) for the Dirac matrices and $\gamma^5=\gamma^0\gamma^1$, we obtain

$$\begin{aligned}
&\phi\equiv\sigma-i\pi=u_i^*v_i\equiv u^*v\\
&\phi^*=\sigma+i\pi=v_i^*u_i\equiv v^*u,
\end{aligned} \tag{6.1}$$

and the equations of motion (1.3) read:

$$iu_{,\xi}=\phi^*v, \qquad iv_{,\eta}=\phi u. \tag{6.2}$$

Using (6.2), we immediately obtain the conservation laws:

$$\begin{aligned}
&(u^*u)_{,\xi}=0=(v^*v)_{,\eta},\\
&(iu^*u_{,\eta}+\text{c.c.})_{,\xi}=0=(iv^*v_{,\xi}+\text{c.c.})_{,\eta}
\end{aligned} \tag{6.3}$$

where c.c. stands for complex conjugation. We shall use the abbreviated notation:

$$
\begin{aligned}
&g_1 \equiv u^*u = g_1(\eta), \qquad g_2 \equiv v^*v = g_2(\xi)\\
&h_1 \equiv \tfrac{1}{2}(iu^*u_{,\eta} + \text{c.c.}) = h_1(\eta),\\
&h_2 \equiv \tfrac{1}{2}(iv^*v_{,\xi} + \text{c.c.}) = h_2(\xi).
\end{aligned}
\tag{6.4}
$$

Our subsequent analysis is based on the following two important properties of the present theory. First, there exists a symmetry group, which, in contrast with the interaction analyzed in previous sections, coincides with the obvious $U(N)$ invariance group. Second, the local scale invariance of Section II takes now on a generalized form including chiral transformations: For arbitrary complex functions $f_1 = f_1(\eta)$ and $f_2 = f_2(\xi)$, the transformation:

$$
\begin{aligned}
u &= f_1(\eta)u',\\
v &= f_2(\xi)v',\\
d\xi' &= (f_2^* f_2)(\xi)d\xi,\\
d\eta' &= (f_1^* f_1)(\eta)d\eta,
\end{aligned}
\tag{6.5}
$$

leaves the equations of motion (6.2) form invariant. Under this transformation, the local current densities appearing in (6.3) and (6.4) transform as follows:

$$
\begin{aligned}
&g_1 = (f_1^* f_1)g_1'\\
&h_1' = (f_1^* f_1)^{-2}h_1 + g_1'(\arg f_1)_{,\eta'},
\end{aligned}
\tag{6.6}
$$

and similar expressions for g_2 and h_2. It is now clear that, without loss of generality, the absolute value of f_1 and f_2 can be chosen such that the charges take over positive constant values, which we denote with the same symbols g_1 and g_2, whereas the energy-momentum densities h_1 and h_2 can be set equal to zero, by appropriately choosing the arguments of f_1 and f_2:

$$
\begin{aligned}
&u^*u = g_1 = \text{const}, \qquad v^*v = g_2 = \text{const}\\
&h_1 = 0 = h_2
\end{aligned}
\tag{6.7}
$$

Equations (6.7) also imply:

$$
u^*u_{,\eta} = 0 = v^*v_{,\xi}\,. \tag{6.8}
$$

After these preliminaries, we examine the equation of motion for the $U(N)$-invariant complex field ϕ defined in (6.1). Without using the special frame (6.7) we derive, directly from (6.2):

$$
\phi_{,\eta\xi} = u_{,\eta}^* v_{,\xi} + i(g_2\phi_{,\eta} - g_1\phi_{,\xi}) - (\phi\phi^*)\phi\,. \tag{6.9}
$$

This equation is valid for any N. For $N=1$, and taking into account (6.7), the system leads to free field behavior. The situation is not trivial for $N=2$, in which case, however, Equation (6.9) can be further reduced, by reducing the invariant $u_{,\eta}^* v_{,\xi}$ in terms of the fundamental set $\{\phi, g_1, g_2\}$. The solution vectors u and v can be taken as a complete basis. The following identities are then valid, in the special frame (6.7)

$$u^*_{,\eta}=\frac{\phi_{,\eta}+ig_1\phi}{g_1g_2-\phi\phi^*}[-\phi^*u^*+g_1v^*]$$
$$v_{,\xi}=\frac{\phi_{,\xi}-ig_2\phi}{g_1g_2-\phi\phi^*}[g_2u-\phi^*v]\,. \tag{6.10}$$

Substitution of (6.10) into (6.9) yields the result:

$$(g_1g_2-\phi\phi^*)\phi_{,\eta\xi}=-(\phi^*\phi_{,\eta}\phi_{,\xi}+g_1g_2\phi^*\phi\phi)+ig_1g_2(g_2\phi_{,\eta}-g_1\phi_{,\xi})$$
$$-\phi\phi^*\phi(g_1g_2-\phi\phi^*)\,. \tag{6.11}$$

This equation can be simplified further by an additional chiral transformation:

$$u\to ue^{-ig_1\eta}\,,\qquad v\to ve^{-ig_2\xi}\,, \tag{6.12}$$

which, of course, changes the frame (6.7) into:

$$u^*u=g_1\,,\qquad v^*v=g_2\,,\qquad h_1=g_1^2\,,\qquad h_2=g_2^2\,. \tag{6.13}$$

Equation (6.11) now becomes (we keep the same symbol ϕ for the variable):

$$(g_1g_2-\phi\phi^*)\phi_{,\eta\xi}=-\phi^*\phi_{,\eta}\phi_{,\xi}+\phi(g_1g_2-\phi\phi^*)^2\,. \tag{6.14}$$

Observe that only the combination g_1g_2 enters this equation, a situation analogous to the symplectic case. Note the Schwartz inequality:

$$|\phi|\equiv|u^*v|\leqq(u^*u)^{1/2}(v^*v)^{1/2}=\sqrt{g_1g_2}\,, \tag{6.15}$$

which permits the following choice of real variables:

$$\phi\equiv\sqrt{g_1g_2}\,\sin\left(\frac{\alpha}{2}\right)e^{i\beta/2}\,. \tag{6.16}$$

Substitution into (6.14) yields the system of equations

$$\alpha_{,\eta\xi}=\sin\alpha+\frac{\mathrm{tg}^2\left(\frac{\alpha}{2}\right)}{\sin\alpha}\beta_{,\eta}\beta_{,\xi}$$
$$\sin\alpha\beta_{,\eta\xi}+\alpha_{,\xi}\beta_{,\eta}+\alpha_{,\eta}\beta_{,\xi}=0\,, \tag{6.17}$$

a generalization of the sin *e*-Gordon equation. It is rather remarkable that (6.17) is identical to the system derived by Pohlmeyer in [6], by reducing the 0(4)-invariant non-linear σ-model (see, also, our appendix), and very similar to a theory derived by Lund and Regge in [9]. In the following considerations, we shall need to write the identities (6.10) in the new frame (6.13):

$$
\begin{aligned}
u_{,\eta}+ig_1u&=\frac{\phi^*_{,\eta}}{1-\phi\phi^*}[-\phi u+g_1v]\\
v_{,\xi}+ig_2v&=\frac{\phi_{,\xi}}{1-\phi\phi^*}[g_2u-\phi^*v],
\end{aligned}
\tag{6.18}
$$

where, for simplicity, g_1g_2 is set equal to one:

$$
g_1=\gamma, \qquad g_2=1/\gamma,
\tag{6.19}
$$

with γ a positive constant.

The starting point for the derivation of the associated linear problem is again the original equation of motion, Equation (6.2), complemented with the identities (6.18):

$$
\begin{aligned}
\begin{bmatrix}u\\v\end{bmatrix}_{,\eta}&=\begin{bmatrix}-i\gamma-\dfrac{\phi\phi^*_{,\eta}}{1-\phi\phi^*}, & \gamma\dfrac{\phi^*_{,\eta}}{1-\phi\phi^*}\\ -i\phi & 0\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}\\
\begin{bmatrix}u\\v\end{bmatrix}_{,\xi}&\begin{bmatrix}0 & -i\phi^*\\ \dfrac{1}{\gamma}\dfrac{\phi_{,\xi}}{1-\phi\phi^*}, & -\dfrac{i}{\gamma}-\dfrac{\phi^*\phi_{,\xi}}{1-\phi\phi^*}\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}.
\end{aligned}
\tag{6.20}
$$

By construction, the integrability condition of (6.20) leads to the system (6.17). However, for the actual solution of the inverse problem alternative representations will be needed, obtained from (6.20) by coordinate dependent transformations. Before embarking into technical details concerning the actual inverse problem, we explain here in general terms how representations originating from (6.20) can be used for the reconstruction of both the $U(2)$-invariant ϕ and the fundamental fields u and v, a phenomenon already observed in the symplectic theory. For our present purposes, it is more convenient to work with the orthonormal $U(2)$-basis:

$$
\begin{aligned}
&\hat{u}\equiv\{2[1+\sin(\alpha/2)]\}^{-1/2}\{\gamma^{-1/2}u+\gamma^{1/2}e^{-i\beta/2}v\}\\
&\hat{v}\equiv\{2[1-\sin(\alpha/2)]\}^{-1/2}\{\gamma^{-1/2}u-\gamma^{1/2}e^{-i\beta/2}v\}\\
&\hat{u}^*\hat{u}=1, \qquad \hat{v}^*\hat{v}=1, \qquad \hat{u}^*\hat{v}=0.
\end{aligned}
\tag{6.21}
$$

Now let U_α, $\alpha=1,2,3,4$, denote the anti-hermitean generators of $U(2)$:

$$
U_1=\begin{bmatrix}0&1\\-1&0\end{bmatrix}, \quad U_2=\begin{bmatrix}0&i\\i&0\end{bmatrix}, \quad U_3=\begin{bmatrix}i&0\\0&0\end{bmatrix}, \quad U_4=\begin{bmatrix}0&0\\0&i\end{bmatrix}.
\tag{6.22}
$$

It is tedious but straightforward to derive the equations

$$
\begin{bmatrix}\hat{u}\\\hat{v}\end{bmatrix}_{,\eta}=\omega_1^\alpha U_\alpha\begin{bmatrix}\hat{u}\\\hat{v}\end{bmatrix}, \qquad \begin{bmatrix}\hat{u}\\\hat{v}\end{bmatrix}_{,\xi}=\omega_2^\alpha U_\alpha\begin{bmatrix}\hat{u}\\\hat{v}\end{bmatrix},
\tag{6.23}
$$

where

$$\omega_1^1 = -\frac{\alpha_{,\eta}}{4}, \quad \omega_1^2 = -\frac{\gamma}{2}\cos(\alpha/2) + \frac{\beta_{,\eta}}{4\cos(\alpha/2)}$$
$$\omega_1^3 = -\frac{\gamma}{2}\left[1+\sin\left(\frac{\alpha}{2}\right)\right] - \frac{1+A}{4}\beta_{,\eta} \tag{6.24a}$$
$$\omega_1^4 = -\frac{\gamma}{2}\left[1-\sin\left(\frac{\alpha}{2}\right)\right] - \frac{1-B}{4}\beta_{,\eta}$$

$$\omega_2^1 = \frac{\alpha_{,\xi}}{4}, \quad \omega_2^2 = \frac{1}{2\gamma}\cos(\alpha/2) + \frac{\beta_{,\xi}}{4\cos(\alpha/2)}$$
$$\omega_2^3 = -\frac{1}{2\gamma}\left[1+\sin\left(\frac{\alpha}{2}\right)\right] - \frac{1-A}{4}\beta_{,\xi} \tag{6.24b}$$
$$\omega_2^4 = -\frac{1}{2\gamma}\left[1-\sin\left(\frac{\alpha}{2}\right)\right] - \frac{1+B}{4}\beta_{,\xi}$$

$$A \equiv \frac{1-\sin\left(\frac{\alpha}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)}\,\mathrm{tg}\left(\frac{\alpha}{2}\right), \quad B \equiv \frac{1+\sin\left(\frac{\alpha}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)}\,\mathrm{tg}\left(\frac{\alpha}{2}\right). \tag{6.24c}$$

Strictly speaking, the only interesting and non-trivial property of (6.23) and (6.24) is that ω's turn out to be real. The integrability condition of (6.23) reads

$$\omega^\alpha_{1,\xi} - \omega^\alpha_{2,\eta} + C^\alpha_{\beta\gamma}\omega_1^\beta\omega_2^\gamma = 0\,, \tag{6.25}$$

where $C^\alpha_{\beta\gamma}$ are the structure constants of the $U(2)$ Lie Algebra in the representation (6.22); (6.25) leads, by construction, to the system (6.17). Let now ψ be a two component complex spinor solving the equations

$$\psi_{,\eta} = \omega_1^\alpha U_\alpha \psi\,, \quad \psi_{,\xi} = \omega_2^\alpha U_\alpha \psi\,. \tag{6.26}$$

Thanks to the anti-hermiticity of the $U(2)$ generators and the reality of ω's, we easily derive

$$(\psi^+\psi)_{,\eta} = 0 = (\psi^+\psi)_{,\xi}\,; \quad \psi^+\psi \equiv \psi_1^*\psi_1 + \psi_2^*\psi_2 \tag{6.27}$$

stating that $\psi^+\psi$ is a space-time independent constant. In analogy with the discussion of Section IV, ψ is parametrized by four arbitrary real constants. Let χ be a second solution of (6.23) corresponding in general to a different choice of the above constants. Furthermore, this set of eight arbitrary numbers is restricted by imposing the four real equations:

$$\psi^+\psi = 1 = \chi^+\chi\,, \quad \psi^+\chi = 0\,. \tag{6.28}$$

With this choice, the 2×2 matrix

$$U \equiv \begin{bmatrix} \psi_1 & \chi_1 \\ \psi_2 & \chi_2 \end{bmatrix} \tag{6.29}$$

contains four arbitrary real parameters and satisfies the system of equations:

$$U_{,\eta}=\omega_1^\alpha U_\alpha U, \qquad U_{,\xi}=\omega_2^\alpha U_\alpha U. \tag{6.30}$$

Equations (6.28) imply that U is an element of $U(2)$. It is now clear that the rows of this matrix are identified with the $U(2)$ components of the orthonormal vectors $\hat{u}$ and $\hat{v}$ defined in (6.21). They provide a solution in terms of the fundamental fields, since the invariants α and β are obtained independently in the course of the solution of the inverse problem. The solution is parametrized by four arbitrary real parameters corresponding to trivial $U(2)$ rotations.

However, the above analysis is too general to account for the technical complexities of the actual solution of the inverse problem. In practice, neither (6.20) nor (6.21) are particularly useful representations. Nevertheless, alternative and more convenient forms of the inverse problem can be obtained by transformations of the type $\psi\to\psi'$; $\psi'=C\psi$, where C is a 2×2 matrix depending on the $U(2)$ invariant functions α and β. Therefore, the inverse transformation $\psi=C^{-1}\psi'$ is possible, in principle, before attempting to realize the points of the preceeding discussion.

A more direct procedure is established in the remainder of our considerations, exploring the existence of the coordinate depending $U(2)$-invariance of the theory, analogous to the symplectic R-transformation discussed in Section III. It is based on a representation of the inverse problem in the basis u', v'.

$$\begin{aligned} u' &\equiv -\gamma^{-1} u e^{i\gamma\eta/2} \\ v' &\equiv e^{i\omega}\left[\frac{e^{-i\beta/2}}{\cos\left(\frac{\alpha}{2}\right)} v-\gamma^{-1}\,\mathrm{tg}\left(\frac{\alpha}{2}\right) u\right] e^{i\gamma\eta/2}. \end{aligned} \tag{6.31}$$

where ω is defined through the compatible equations (in analogy with a similar construction in [6]):

$$\omega_{,\eta}=\frac{\beta_{,\eta}\cos\alpha}{2\cos^2\left(\frac{\alpha}{2}\right)}, \qquad \omega_{,\xi}=\frac{\beta_{,\xi}}{2\cos^2\left(\frac{\alpha}{2}\right)}. \tag{6.32}$$

Using (6.20) and (6.31) and (6.32), we derive the first order differential equations for the basis u' and v':

$$\begin{aligned} \begin{bmatrix} u' \\ v' \end{bmatrix}_{,\eta} &= \begin{bmatrix} -i\zeta & q \\ -q^* & i\zeta \end{bmatrix}\begin{bmatrix} u' \\ v' \end{bmatrix} \\ \begin{bmatrix} u' \\ v' \end{bmatrix}_{,\xi} &= \frac{i}{4\zeta}\begin{bmatrix} -1+\cos\alpha & \sin\alpha\, e^{-i\omega} \\ \sin\alpha\, e^{i\omega} & -1-\cos\alpha \end{bmatrix}\begin{bmatrix} u' \\ v' \end{bmatrix} \\ \zeta &\equiv \frac{\gamma}{2}, \qquad q\equiv -\frac{1}{2}(\alpha_{,\eta}-i\,\mathrm{tg}\,\alpha\,\omega_{,\eta})e^{-i\omega}. \end{aligned} \tag{6.33}$$

An additional trivial transformation of the form $(u', v')\to(u', v')e^{(i/2\gamma)\xi}$, brings (6.33) into the system derived in [6]. The integrability condition for (6.33) results in the non-trivial constraint

$$q_{,\xi}+\tfrac{1}{2}\sin\alpha\, e^{-i\omega}=0 \tag{6.34}$$

that encompasses both equations in (6.17). In the above form, the inverse problem essentially reduces to the analysis of Zakharov and Shabat [8]. The reconstruction of the "potential" q is straightforward and the determination of the potential $\phi \equiv \sin\left(\frac{\alpha}{2}\right) e^{i\beta/2}$, and thereby of the original fields, is effected by exploring (6.34) and simple additional integrations. It is this problem that we simplify further by using, as mentioned earlier, the coordinate dependent $U(2)$ invariance transformation, which we derive next. It is defined through the compatible equations $(R = R(\eta, \xi; \gamma'))$:

$$\begin{aligned} \partial_\eta R_{ij} &= i(1-\gamma')R_{ik}u_k u_j^* \\ \partial_\xi R_{ij} &= i(1-\gamma'^{-1})R_{ik}v_k v_j^* \end{aligned} \tag{6.35}$$

where γ' is an arbitrary parameter and u_i, v_i are the $U(N)$ components of the solution u and v. (Notice that this construction is valid for any N.) It is easy to verify that (6.35) is compatible with R being an element of $U(N)$ and that the vectors

$$u(\ ;\gamma') \equiv \sqrt{\gamma'}\, Ru, \qquad v(\ ;\gamma') \equiv \frac{1}{\sqrt{\gamma'}} Rv \tag{6.36}$$

are also solutions. Under (6.36) the potential $\phi = u^* v$ remains invariant, whereas the charges and the energy-momentum densities transform according to:

$$\begin{aligned} &g_1(;\gamma') = \gamma' g_1, \qquad g_2(;\gamma') = \frac{1}{\gamma'} g_2 \\ &h_1(\ ;\gamma') = \gamma' h_1 - \gamma'(1-\gamma')g_1^2, \qquad h_2(\ ;\gamma') = \frac{1}{\gamma'} h_1 - \frac{1}{\gamma'}\left(1 - \frac{1}{\gamma'}\right) g_2^2 . \end{aligned} \tag{6.37}$$

The novel second term in Equations (6.37) originates from the chiral invariance of the present theory. In fact, it can be compensated by a trivial chiral transformation.

With this in mind, we finally derive an explicit and remarkably simple prescription for the solution of the inverse problem (6.33). We first remark that the second of Equations (6.2) reads explicitly:

$$v_{i,\eta} = -iu_i u_j^* v_j \tag{6.38}$$

which should be compared with the first of (6.35), written in terms of the hermitean conjugate matrix R^+:

$$\partial_\eta R_{ik}^+ = -i(1-\gamma')u_i u_j^* R_{jk}^+ . \tag{6.39}$$

The analogy with (6.38) is obvious at $\gamma' = 0$. Eventually up to trivial ξ-dependent phases and an overall constant, the existence of $R(\gamma' = 0)$ will be shown in the following by explicit construction. Let now

$$\psi(\zeta) = \begin{bmatrix} \psi_1(\zeta) \\ \psi_2(\zeta) \end{bmatrix}, \qquad \psi\left(\zeta = \frac{\gamma}{2}\right) \to \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i\gamma\eta/2}, \qquad \eta \to -\infty \tag{6.40}$$

be the Jost function for the system (6.33) with the boundary condition indicated above—details about its explicit construction can be found in [8]. We only mention here that $(\psi_2^*, -\psi_1^*)$ is also a solution and that

$$\psi_1 \psi_1^* + \psi_2 \psi_2^* = 1 . \tag{6.41}$$

The space-time dependence is suppressed. The first of (6.31) suggests that the original $U(2)$ vector u can immediately be constructed in terms of ψ: fixing the normalization from $u^*u=\gamma$, it is given, up to a trivial $U(2)$ rotation, by

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \sqrt{\gamma} \begin{bmatrix} \psi_1\left(\zeta=\dfrac{\gamma}{2}\right) \\ \psi_2^*\left(\zeta=\dfrac{\gamma}{2}\right) \end{bmatrix} e^{-\frac{i}{2}\gamma\eta} \tag{6.42}$$

We are now able to construct R essentially uniquely. One finds:

$$R^+(\gamma'=0)= \begin{bmatrix} \psi_2\left(\zeta=\dfrac{\gamma}{2}\right) & \psi_1\left(\zeta=\dfrac{\gamma}{2}\right) \\ -\psi_1^*\left(\zeta=\dfrac{\gamma}{2}\right) & \psi_2^*\left(\zeta=\dfrac{\gamma}{2}\right) \end{bmatrix} \begin{bmatrix} \psi_2^*(\zeta=0) & -\psi_1(\zeta=0) \\ \psi_1^*(\zeta=0) & \psi_2(\zeta=0) \end{bmatrix} e^{-\frac{i}{2}\gamma\eta}. \tag{6.43}$$

Comparing (6.38) and (6.39) with $\gamma'=0$, we conclude that the first column of $R^+(\gamma'=0)$ is a candidate for the vector v, up to an overall scale, that is fixed from the requirement $v^*v=1/\gamma$, and a trivial $U(2)$ rotation. The latter is uniquely fixed, consistently with the choice (6.42), by recalling the definition $\phi\equiv u^*v$, and inforcing the assumed boundary condition for the invariant ϕ, which is here taken to be

$$\phi(\eta\to-\infty)=1\,. \tag{6.44}$$

This boundary condition is compatible with the equation of motion (6.14), with $g_1g_2=1$. We thus consider the $U(2)$ contraction.

$$u^*R^+(\gamma'=0)=\sqrt{\gamma}\{\psi_1^*(\zeta=0), \quad \psi_2(\zeta=0)\} \tag{6.45}$$

obtained by using (6.41)–(6.43). Taking now into account the asymptotic behavior (6.40), the scale of v is fixed as $\dfrac{1}{\sqrt{\gamma}}x$ the first column of $R^+(\gamma'=0)$, and the above mentioned trivial $U(2)$ rotation is equal to the identity. Hence, with the boundary condition (6.44) the solution of the original problem is completely determined in terms of the Jost function defined in (6.40). We summarize here the explicit expressions:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \sqrt{\gamma} \begin{bmatrix} \psi_1\left(\zeta=\dfrac{\gamma}{2}\right) \\ \psi_2^*\left(\zeta=\dfrac{\gamma}{2}\right) \end{bmatrix} e^{-\frac{i}{2}\gamma\eta}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{1}{\sqrt{\gamma}} \begin{bmatrix} \psi_1\left(\zeta=\dfrac{\gamma}{2}\right)\psi_1^*(\zeta=0)+\psi_2\left(\zeta=\dfrac{\gamma}{2}\right)\psi_2^*(\zeta=0) \\ -\psi_1^*\left(\zeta=\dfrac{\gamma}{2}\right)\psi_2^*(\zeta=0)+\psi_2^*\left(\zeta=\dfrac{\gamma}{2}\right)\psi_1^*(\zeta=0) \end{bmatrix} e^{-\frac{i}{2}\gamma\eta}$$

$$\phi\equiv u^*v=\psi_1^*(\zeta=0)\,. \tag{6.46}$$

The above results along with the analysis of [8] reduce the construction of the solutions of the present chiral theory to straightforward quadratures. We have explicitly verified this scheme in the one and two-soliton cases. Our one soliton example led to an expression for the $U(2)$ invariant ϕ that is equivalent to the solution previously obtained by Shei [5], in the semiclassical limit of the chiral $U(N)$ theory.

Needless to add that this construction applies to the symplectic theory as well as to the non-linear σ-model, both possessing an R-transformation. Details about the application to each particular example are not given here. We only recall that the analog of the third of Equations (6.46) in the simple $\mathrm{Sp}(2,\mathbb{R})$ system was derived in Section IV by a simple manipulation of the equations of motion, see Equation (4.14).

We finally derive the Backlünd transformation for the present theory. Let ψ be a two component spinor solving the inverse problem:

$$\begin{aligned} &\psi_{,\eta}=C_1\psi, \quad \psi_{,\xi}=C_2\psi, \\ &C_1=C_1(\phi)\equiv\begin{bmatrix} -i\zeta & q \\ -q^* & i\zeta \end{bmatrix}, \\ &C_2=C_2(\phi)\equiv\frac{i}{4\zeta}\begin{bmatrix} \cos\alpha & \sin\alpha e^{-i\omega} \\ \sin\alpha e^{i\omega} & -\cos\alpha \end{bmatrix}. \end{aligned} \tag{6.47}$$

A spinor ψ' related to ψ by a coordinate dependent unitary transformation B:

$$\psi'=B\psi, \tag{6.48}$$

satisfying the system of equations

$$\begin{aligned} B_{,\eta}&=C_1(\phi')B-BC_1(\phi) \\ B_{,\xi}&=C_2(\phi')B-BC_2(\phi), \end{aligned} \tag{6.49}$$

is also a solution associated with the invariant ϕ' and characterized by the same energy-momentum parameter $\zeta=\gamma/2$. The integrability condition for (6.49) is simply the statement that both ϕ and ϕ' satisfy the reduced system (6.14). Since C_1 and C_2 are traceless and anti-hermitian, (6.49) admits a solution B that is actually an element of SU(2). Having guaranteed the existence of a solution, we now turn to its explicit construction. This task is facilitated by the following considerations. First, we explore the information contained in the third of Equations (6.46). Denoting the associated Jost function by the same symbol ψ, we have:

$$\begin{aligned} \psi_1(\zeta=0)&=\sin(\alpha/2)e^{-i\beta/2} \\ \psi_2(\zeta=0)&=-\cos(\alpha/2)e^{i(\omega-\beta/2)}. \end{aligned} \tag{6.50}$$

Using the fact that there is only one unitary 2×2 matrix that transforms a given spinor into another [which was already used in Equation (6.43), for example], the above information suffices to determine B at $\zeta=0$.

It is also trivial to find B for $\zeta\to\infty$, in which limit it is independent of ϕ and ϕ'. A reasonable Ansatz for the solution of (6.49) is then an affine combination that interpolates between the above two limiting cases. Indeed, the full solution is obtained in this manner and is given by

$$|\varepsilon|\sqrt{\gamma^2+\varepsilon\varepsilon^*}\,B_{11}=i\gamma\varepsilon^*+\varepsilon\varepsilon^*\left[\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha'}{2}\right)e^{i(\beta-\beta')/2}+\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha'}{2}\right)e^{i(\beta'-\beta+2\omega-2\omega')/2}\right]$$

$$|\varepsilon|\sqrt{\gamma^2+\varepsilon\varepsilon^*}\,B_{12}=\varepsilon\varepsilon^*\left[\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha'}{2}\right)e^{i(\beta'-\beta-2\omega')/2}-\sin\left(\frac{\alpha'}{2}\right)\cos\left(\frac{\alpha}{2}\right)e^{i(\beta-\beta'-2\omega)/2}\right]$$

$$B_{21}=-B_{12}^*$$

$$B_{22}=B_{11}^*, \tag{6.51}$$

provided that the invariants are connected by the Backlünd transformation:

$$\left[\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha'}{2}\right)e^{i(\beta-\beta')/2}+\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha'}{2}\right)e^{i(\beta'-\beta+2\omega-2\omega')/2}\right]_{,\xi}=-\frac{1}{\varepsilon}\sin\left(\frac{\alpha+\alpha'}{2}\right)\sin\left(\frac{\alpha-\alpha'}{2}\right),$$

$$\left[\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha'}{2}\right)e^{i(\beta-\beta')/2}-\frac{\varepsilon^*}{\varepsilon}\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha'}{2}\right)e^{i(\beta-\beta'-2\omega+2\omega')/2}\right]_{,\eta}=\varepsilon^*\sin\left(\frac{\alpha+\alpha'}{2}\right)\sin\left(\frac{\alpha-\alpha'}{2}\right). \tag{6.52}$$

ε is an arbitrary complex constant. For real ε and $\beta=\beta'=\omega=\omega'=0$, (6.52) reduces to the Backlünd transformation of the sine-Gordon theory. From (6.52) and the asymptotic condition (6.44), one derives a number of relations, among which

$$\mathrm{Im}\left\{\varepsilon\left[\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha'}{2}\right)e^{i(\beta-\beta')/2}+\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha'}{2}\right)e^{i(\beta'-\beta-\omega'+\omega)/2}-1\right]\right\}=0. \tag{6.53}$$

This relation ensures that the matrix B is an element of SU(2) for real γ, as it was expected. It is gratifying that a Backlünd transformation exists for the generalized sine-Gordon system, however, it appears not to be practical for the explicit computation of multisoliton solutions. Perhaps, an algebraic reduction of the type derived for the sine-Gordon system in Appendix A will prove more efficient for practical purposes.

Appendix—Inverse Scattering Wave-Functions and the Solution of the Two-dimensional Non-linear σ-Model

A. The 0(3)*-Case*

Having been inspired by the methods developed in the non-linear σ-model, we now go back to it and complete its solution for fundamental fields, in terms of the wave-functions appearing in the inverse problem of the reduced systems obtained in [6]. We first study the 0(3)-invariant model. q^i, $i=1,2,3$ are real variables obeying the equations of motion:

$$q_{,\eta\xi}+(q_{,\eta}q_{,\xi})q=0, \qquad q^2=1. \tag{A.1}$$

Due to the local scale invariance of (A.1), we can set

$$q_{,\eta}^2=\gamma^2, \qquad q_{,\xi}^2=\gamma^{-2}; \qquad (q_{,\eta}q_{,\xi})\equiv\cos\alpha. \tag{A.2}$$

Further, consider the moving orthonormal trihedral:

$$Z=\{\hat{z}_1,\hat{z}_2,\hat{z}_3\}$$
$$\hat{z}_1=\frac{\gamma^{-1}q_{,\eta}+\gamma q_{,\xi}}{2\cos\left(\frac{\alpha}{2}\right)}, \qquad \hat{z}_2=\frac{\gamma^{-1}q_{,\eta}-\gamma q_{,\xi}}{2\sin\left(\frac{\alpha}{2}\right)}, \qquad \hat{z}_3=q. \tag{A.3}$$

By using the equation of motion and elementary completeness arguments, we find

$$Z_{,\eta}=C_1Z, \qquad Z_{,\xi}=C_2Z$$

$$C_1=\begin{bmatrix} 0 & \frac{\alpha_{,\eta}}{2} & -\gamma\cos\left(\frac{\alpha}{2}\right) \\ -\frac{\alpha_{,\eta}}{2} & 0 & -\gamma\sin\left(\frac{\alpha}{2}\right) \\ \gamma\cos\left(\frac{\alpha}{2}\right) & \gamma\sin\left(\frac{\alpha}{2}\right) & 0 \end{bmatrix} \tag{A.4}$$

$$C_2=\begin{bmatrix} 0 & \frac{-\alpha_{,\xi}}{2} & -\gamma^{-1}\cos\left(\frac{\alpha}{2}\right) \\ \frac{\alpha_{,\xi}}{2} & 0 & \gamma^{-1}\sin\left(\frac{\alpha}{2}\right) \\ \gamma^{-1}\cos\left(\frac{\alpha}{2}\right) & -\gamma^{-1}\sin\left(\frac{\alpha}{2}\right) & 0 \end{bmatrix}.$$

It is an observation due to Lund and Regge [9] that the above matrices can be written as real superpositions of the standard anti-hermitean generators of the rotation group:

$$C_1=\omega_1^iI_i, \qquad C_2=\omega_2^iI_i$$
$$\bar{\omega}_1=\left\{\gamma\sin\left(\frac{\alpha}{2}\right), \quad -\gamma\cos\left(\frac{\alpha}{2}\right), \quad -\frac{\alpha_{,\eta}}{2}\right\} \tag{A.5}$$
$$\bar{\omega}_2=\left\{-\gamma^{-1}\sin\left(\frac{\alpha}{2}\right), \quad -\gamma^{-1}\cos\left(\frac{\alpha}{2}\right), \quad \frac{\alpha_{,\xi}}{2}\right\}.$$

Hence, the integrability condition for the system (A.4) reads:

$$\bar{\omega}_{1,\xi}-\bar{\omega}_{2,\eta}+\bar{\omega}_1\times\bar{\omega}_2=0, \tag{A.6}$$

and results in the sin *e*-Gordon equation for $\alpha : \alpha_{,\eta\xi} + \sin\alpha = 0$. Notice that Equation (A.6) depends on the structure constants than the particular representation of the rotation group. Therefore, the same integrability condition would be realized in any representation, in particular the two-dimensional spinor representation [9]. Let τ_i be the two-dimensional anti-hermitean generators of SU(2) (see footnote 3) and ϕ a two-component spinor field. Concerning solutions for the invariant α, the following inverse problem can be considered:

$$\phi_{,\eta} = \omega_1^i \tau_i \phi, \qquad \phi_{,\xi} = \omega_2^i \tau_i \phi. \tag{A.7}$$

Our ultimate aim is, however, to obtain solutions for the variable q; we shall show here that this is actually straightforward. Thanks to the fact that τ_i's are anti-hermitean and ω's real:

$$(\phi^+\phi)_{,\eta} = 0 = (\phi^+\phi)_{,\xi}. \tag{A.8}$$

It is therefore possible to adjust the arbitrary constants appearing in the superposition of the two independent solutions of the Dirac problem (A.7) such that

$$\phi^+\phi \equiv \phi_1^*\phi_1 + \phi_2^*\phi_2 = 1, \tag{A.9}$$

leaving the freedom of three arbitrary real parameters corresponding to the 0(3)-invariance. The SU(2) matrix:

$$r \equiv \begin{bmatrix} \phi_1 & -\phi_2^* \\ \phi_2 & \phi_1^* \end{bmatrix}, \tag{A.10}$$

satisfies the system of Equations (A.7), and its three-dimensional representation is a real orthogonal matrix, whose rows are identified with the vectors $\hat{z}_i$. We are interested in the last row $\hat{z}_3 = q$:

$$q = \{-(\phi_1\phi_2 + \phi_1^*\phi_2^*), i(\phi_1\phi_2 - \phi_1^*\phi_2^*), (\phi_1\phi_1^* - \phi_2\phi_2^*)\} \tag{A.11}$$

providing a solution in the original variables.

Concerning the Backlünd transformation of the present model, it is not difficult to find that if ϕ defines a solution in the manner described above, then ϕ':

$$\phi' = B\phi$$

$$B \equiv \frac{1}{\sqrt{\gamma^2+\varepsilon^2}} \begin{bmatrix} \gamma e^{i\frac{\alpha'-\alpha}{4}} & \varepsilon e^{-i\frac{\alpha'+\alpha}{4}} \\ -\varepsilon e^{i\frac{\alpha'+\alpha}{4}} & \gamma e^{-i\frac{\alpha'-\alpha}{4}} \end{bmatrix}. \tag{A.12}$$

also defines a solution with the same energy-momentum parameter γ, provided that the invariants α and α' are connected by the standard Backlünd transformation:

$$\begin{aligned} \left(\frac{\alpha-\alpha'}{2}\right)_{,\eta} &= \varepsilon \sin\left(\frac{\alpha'+\alpha}{2}\right) \\ \left(\frac{\alpha+\alpha'}{2}\right)_{,\xi} &= \frac{1}{\varepsilon}\sin\left(\frac{\alpha'-\alpha}{2}\right) \end{aligned} \tag{A.13}$$

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3 Throughout the Appendix we use the particular representation:

$$\tau_1 = \frac{i}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tau_2 = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \tau_3 = \frac{i}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \quad [\tau_i, \tau_j] = \varepsilon_{ijk}\tau_k$$

for arbitrary ε, in general different from γ. The matrix B in (A.12) is an element of SU(2). The original trihedral Z transforms according to its three dimensional representation. In particular,

$$q' = \frac{1}{\gamma^2+\varepsilon^2}\left\{2\varepsilon\gamma\left[\cos\left(\frac{\alpha'}{2}\right)\hat{z}_1 - \sin\left(\frac{\alpha'}{2}\right)\hat{z}_2\right] + (\gamma^2-\varepsilon^2)\hat{z}_3\right\}, \tag{A.14}$$

a Backlünd transformation for the original variable q. In words, if q is a solution with energy-momentum parameter γ, and $(q_{,\eta}q_{,\xi}) = \cos\alpha$, q' is also a solution with the same energy-momentum and $(q'_{,\eta}q'_{,\xi}) = \cos\alpha'$, for arbitrary ε. For the special choice $\varepsilon = 1 = \gamma$, Equation (A.14) reduces to the explicit expression given by Pohlmeyer [6], and is also known in differential geometry, in the theory of pseudo-spherical surfaces (Bianchi transformation—see, e.g. [10], page 291).

An interesting aspect of (A.12) is that it leads to a completely algebraic Backlünd transformation, through the following considerations: we first transform (A.7) into a more convenient representation that can be actually used for the solution of the inverse problem. It is obtained with the change of variables $\phi = C\psi$:

$$C = \frac{1}{\sqrt{2}}\begin{bmatrix} ie^{i\alpha/4} & e^{i\alpha/4} \\ -e^{-i\alpha/4} & -ie^{-i\alpha/4}\end{bmatrix}; \quad CC^+ = I, \tag{A.15}$$

in terms of which:

$$\begin{aligned}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}_{,\eta} &= \frac{1}{2}\begin{bmatrix} i\gamma & -\alpha_{,\eta} \\ \alpha_{,\eta} & -i\gamma\end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} \\ \begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}_{,\xi} &= \frac{i}{2\gamma}\begin{bmatrix}\cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha\end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}.\end{aligned} \tag{A.16}$$

In these new variables the Backlünd transformation (A.12) reads:

$$\psi' = C^+(\alpha')BC(\alpha)\psi \equiv \bar{B}\psi, \tag{A.17}$$

$$\bar{B} = \frac{1}{\sqrt{\gamma^2+\varepsilon^2}}\begin{bmatrix} \gamma + i\varepsilon\cos\left(\frac{\alpha'+\alpha}{2}\right) & i\varepsilon\sin\left(\frac{\alpha'+\alpha}{2}\right) \\ i\varepsilon\sin\left(\frac{\alpha'+\alpha}{2}\right) & \gamma - i\varepsilon\cos\left(\frac{\alpha'+\alpha}{2}\right)\end{bmatrix}. \tag{A.18}$$

The observation now is simply that, given a solution $\{\psi, \alpha\}$, the construction of $\{\psi', \alpha'\}$ can be effected without solving the Riccati differential equations (A.13), but with purely algebraic steps. In fact, setting $\gamma = i\varepsilon$ in (A.17) and (A.18), we get:

$$\begin{bmatrix} 1+\cos\left(\frac{\alpha'+\alpha}{2}\right) & \sin\left(\frac{\alpha'+\alpha}{2}\right) \\ \sin\left(\frac{\alpha'+\alpha}{2}\right) & 1-\cos\left(\frac{\alpha'+\alpha}{2}\right)\end{bmatrix}\begin{bmatrix}\psi_1(\gamma=i\varepsilon)\\ \psi_2(\gamma=i\varepsilon)\end{bmatrix} = 0. \tag{A.19}$$

The determinant of this system vanishes, (compatible with the fact that ψ_1 and ψ_2 are in general different from zero). (A.19) leads to the following non-trivial algebraic solution:

$$\sin\left(\frac{\alpha'+\alpha}{2}\right) = \frac{-2\lambda}{1+\lambda^2}, \quad \cos\left(\frac{\alpha'+\alpha}{2}\right) = -\frac{1-\lambda^2}{1+\lambda^2}, \quad \lambda \equiv (\psi_2/\psi_1)(\gamma = i\varepsilon), \tag{A.20}$$

that determines α' in terms of α and ψ. The specification of ψ', and thereby of the original variable q', is then immediate. Algebraic prescriptions of the form (A.20) have previously appeared in the literature [11].

The remaining question is to examine whether λ can be real, as it is obviously required in (A.20). The answer is provided by the observation that the solution $\{\alpha, \psi(\gamma=i\varepsilon)\}$ satisfies the system (A.16) with $\gamma=i\varepsilon$, in which case the above system has real coefficients. It is therefore clear that (A.16) admits real solutions for $\lambda=(\psi_2/\psi_1)(\gamma=i\varepsilon)$, with an appropriate choice for the arbitrary parameters entering the superposition of any two special solutions. Hence, the algebraic equation (A.20) reduces the construction of arbitrary multisoliton solutions to a series of simple quadratures. As an example, the one soliton solution is immediately obtained starting with the trivial solution $\alpha=0$, $\lambda=\exp(\varepsilon\eta-\xi/\varepsilon)$.

We finally note that the above procedure is not special to the sin$e$-Gordon system and can be applied with obvious modifications to the systems analyzed in Sections IV and VI.

B. The $0(4)\simeq SU(2)\otimes SU(2)$ *Theory*

We now generalize Equation (A.11) to the 0(4) case. In terms of the original variables q^i, $i=1,\dots 4$, we form the orthonormal tetrahedral:

$$Z=\{\hat{z}_1,\hat{z}_2,\hat{z}_3,\hat{z}_4\}$$

$$\hat{z}_1=\frac{\gamma^{-1}q_{,\eta}+\gamma q_{,\xi}}{2\cos\left(\dfrac{\alpha}{2}\right)},\qquad \hat{z}_2=\frac{\gamma^{-1}q_{,\eta}-\gamma q_{,\xi}}{2\sin\left(\dfrac{\alpha}{2}\right)}$$

$$\hat{z}_3=\frac{[q_{,\eta},q_{,\xi},q]}{\sin\alpha},\qquad \hat{z}_4=q, \tag{B.1}$$

where [, ,] denotes the vector product in the four-dimensional Euclidean space. As usual, $q_{,\eta}^2=\gamma^2$, $q_{,\xi}^2=\gamma^{-2}$, $\cos\alpha\equiv(q_{,\eta}q_{,\xi})$. Z obeys the system of equations:

$$Z_{,\eta}=C_1Z,\qquad Z_{,\xi}=C_2Z$$

$$C_1=\omega^i_{1,(+)}J_i+\omega^i_{1,(-)}K_i,\qquad C_2=\omega^i_{2,(+)}J_i+\omega^i_{2,(-)}K_i, \tag{B.2}$$

where

$$\bar{\omega}_{1,(\pm)}\equiv\left\{-\left[\frac{v}{2\sin\left(\dfrac{\alpha}{2}\right)}\pm\gamma\cos\left(\frac{\alpha}{2}\right)\right],\ \left[\frac{v}{2\cos\left(\dfrac{\alpha}{2}\right)}\mp\gamma\sin\left(\frac{\alpha}{2}\right)\right],\ \frac{-\alpha_{,\eta}}{2}\right\}$$

$$\bar{\omega}_{2,(\pm)}\equiv\left\{\left[\frac{u}{2\sin\left(\dfrac{\alpha}{2}\right)}\mp\gamma^{-1}\cos\left(\frac{\alpha}{2}\right)\right],\ \left[\frac{u}{2\cos\left(\dfrac{\alpha}{2}\right)}\pm\gamma^{-1}\sin\left(\frac{\alpha}{2}\right)\right],\ \frac{\alpha_{,\xi}}{2}\right\}$$

$$u\equiv\gamma(q_{,\xi\xi}\hat{z}_3),\qquad v\equiv\gamma^{-1}(q_{,\eta\eta}\hat{z}_3). \tag{B.3}$$

J_i and K_i, $i=1,2,3$, are the generators of 0(4) defined as follows: Let $\varepsilon_{\alpha\beta}$, $\alpha,\beta=1,\dots 4$ be a 4×4 matrix whose only non-vanishing element is located at the α^{th} row and β^{th}

column, and is equal to unity:

$$\begin{aligned} &J_i=\tfrac{1}{2}(M_i+N_i), \quad K_i=\tfrac{1}{2}(M_i-N_i) \\ &M_1=\tfrac{1}{2}(-\varepsilon_{23}+\varepsilon_{32}), \quad M_2=\tfrac{1}{2}(\varepsilon_{13}-\varepsilon_{31}), \quad M_3=\tfrac{1}{2}(-\varepsilon_{12}+\varepsilon_{21}) \\ &N_1=\tfrac{1}{2}(\varepsilon_{14}-\varepsilon_{41}), \quad N_2=\tfrac{1}{2}(\varepsilon_{24}-\varepsilon_{42}), \quad N_3=\tfrac{1}{2}(\varepsilon_{34}-\varepsilon_{43}) \\ &[J_i, J_j]=\varepsilon_{ijk}J_k, \quad [K_i, K_j]=\varepsilon_{ijk}K_k, \quad [J_i, K_j]=0. \end{aligned} \tag{B.4}$$

We shall also need to specify the unitary transformation V:

$$V[\tau_i \otimes I]V^+=J_i, \quad V[I\otimes \tau_i]V^+=K_i, \tag{B.5}$$

where $\otimes$ defines the usual tensor product of matrices, I is the two-dimensional unit matrix and τ_i are the two-dimensional generators of SU(2) defined earlier. It turns out that

$$V=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & -1 \\ -i & 0 & 0 & -i \\ 0 & -1 & -1 & 0 \\ 0 & -i & i & 0 \end{bmatrix}. \tag{B.6}$$

After these preliminaries, we proceed with the actual construction of the solution vector q. Clearly, the integrability condition for (B.2) reads:

$$\partial_\xi \bar{\omega}_{1,(\pm)}-\partial_\eta \bar{\omega}_{2,(\pm)}+\bar{\omega}_{1,(\pm)}\times \bar{\omega}_{2,(\pm)}=0. \tag{B.7}$$

It should be noted that $\bar{\omega}_{(-)}$ is obtained from $\bar{\omega}_{(+)}$, by simply changing the sign of the eigenvalue parameter γ, see Equations (B.3). Both choices lead to the same system of equations for the invariants α, u, and v, already derived in [6]:

$$\begin{aligned} &\alpha_{,\eta\xi}+\sin\alpha+\frac{uv}{\sin\alpha}=0, \\ &u_{,\eta}=\frac{\alpha_{,\xi}v}{\sin\alpha}, \quad v_{,\xi}=\frac{\alpha_{,\eta}u}{\sin\alpha}. \end{aligned} \tag{B.8}$$

The inverse scattering problem is formulated in the spinor representation of the SU(2) group. Let ϕ be a solution of

$$\phi_{,\eta}=\omega^i_{1,(+)}\tau_i\phi, \quad \phi_{,\xi}=\omega^i_{2,(+)}\tau_i\phi. \tag{B.9}$$

In complete analogy with the 0(3)-case, ϕ can be normalized according to

$$\phi^+\phi\equiv\phi_1^*\phi_1+\phi_2^*\phi_2=1, \tag{B.10}$$

and the SU(2) matrix:

$$r_\gamma\equiv\begin{bmatrix}\phi_1 & -\phi_2^* \\ \phi_2 & \phi_1^*\end{bmatrix}, \tag{B.11}$$

obeys the equations

$$\partial_\eta r_\gamma=\omega^i_{1,(+)}\tau_i r_\gamma, \quad \partial_\xi r_\gamma=\omega^i_{2,(+)}\tau_i r_\gamma. \tag{B.12}$$

r_γ is parametrized by three arbitrary real numbers, after imposing the normalization (B.10). Let now $\phi' = \phi'(\ ;\gamma)$ be a second solution of (B.9), corresponding in general to a different choice of those three parameters, but the same invariants α, u, and v. Define:

$$\psi \equiv \phi'(\ ;-\gamma), \tag{B.13a}$$

$$\psi_{,\eta} = \omega^i_{1,(-)}\tau_i\psi, \qquad \psi_{,\xi} = \omega^i_{2,(-)}\tau_i\psi. \tag{B.13b}$$

The associated SU(2) matrix $r_{-\gamma}$:

$$r_{-\gamma} \equiv \begin{bmatrix} \psi_1 & -\psi_2^* \\ \psi_2 & \psi_1^* \end{bmatrix}, \tag{B.14}$$

satisfies the same system (B.13b). The 4×4 matrix $r_\gamma \otimes r_{-\gamma}$ depends on six arbitrary real parameters and obeys the equations:

$$\begin{aligned} \partial_\eta[r_\gamma \otimes r_{-\gamma}] &= \{\omega^i_{1,(+)}[\tau_i \otimes I] + \omega^i_{1,(-)}[I \otimes \tau_i]\}\,[r_\gamma \otimes r_{-\gamma}] \\ \partial_\xi[r_\gamma \otimes r_{-\gamma}] &= \{\omega^i_{2,(+)}[\tau_i \otimes I] + \omega^i_{2,(-)}[I \otimes \tau_i]\}\,[r_\gamma \otimes r_{-\gamma}] \end{aligned} \tag{B.15}$$

obtained by using the equations for r_γ and $r_{-\gamma}$, and elementary properties of the tensor product such as

$$[A_1 \otimes A_2]\,[B_1 \otimes B_2] = [A_1B_1] \otimes [A_2B_2].$$

It is now clear that the rows of the 4×4 real orthogonal matrix:

$$R \equiv V[r_\gamma \otimes r_{-\gamma}]V^+, \tag{B.16}$$

with V constructed in (B.6), specify the orthonormal tetrahedral Z in terms of the wave-function. It is sufficient to consider $\hat{z}_4 = q$, whose components are given explicitly as follows:

$$\begin{aligned} q^1 &= \frac{i}{2}(\phi_2\psi_1 - \phi_1\psi_2) + \text{c.c.}, \qquad q^2 = \frac{1}{2}(\phi_1\psi_2 - \phi_2\psi_1) + \text{c.c.} \\ q^3 &= \frac{i}{2}(\varphi_1\psi_1^* + \phi_2\psi_2^*) + \text{c.c.}, \qquad q^4 = \frac{1}{2}(\phi_1\psi_1^* + \phi_2\psi_2^*) + \text{c.c.} \end{aligned} \tag{B.17}$$

where ϕ_i, ψ_i, $i = 1, 2$ are the upper and lower components of the spinors ϕ and ψ. (B.17) generalizes Equation (A.11) to the present 0(4)-case.

So far, our construction made little reference to the details of the inverse problem. Since the latter has already been considered in the context of the $U(2)$ chiral theory in Section VI, and for the present theory in [9], we shall only include here some remarks concerning the practical use of Equation (B.17) and illustrate it in a simple example. In general, representations other than (B.9) are more convenient for the actual solution of the inverse problem. They are obtained by performing suitable space-time dependent transformations to spinors $\phi \to \chi$, $\chi = C\phi$, where C is a 2×2 matrix depending on the invariant functions. The latter are "independently" constructed in the course of inverse scattering. Therefore the inverse transformation $\phi = C^{-1}\chi$ is always possible before attempting the construction summarized in Equation (B.17).

An example of such a transformation is

$$C=\frac{i}{\sqrt{2}}\begin{bmatrix} e^{-\frac{i\alpha}{4}} & e^{\frac{i\alpha}{4}} \\ e^{-\frac{i\alpha}{4}} & -e^{\frac{i\alpha}{4}} \end{bmatrix}. \tag{B.18}$$

Then $\chi=C\phi$ satisfies the system of equations

$$\chi_{,\eta}=\frac{1}{2i}\begin{bmatrix} \gamma+v\,\mathrm{ctg}\,\alpha & \alpha_{,\eta}+iv \\ \alpha_{,\eta}-iv & -\gamma-v\,\mathrm{ctg}\,\alpha \end{bmatrix}\chi$$

$$\chi_{,\xi}=\frac{1}{2i\gamma}\begin{bmatrix} \cos\alpha-\gamma u/\sin\alpha & i\sin\alpha \\ -i\sin\alpha & -\cos\alpha+\gamma u/\sin\alpha \end{bmatrix}\chi. \tag{B.19}$$

In this form, the inverse problem follows essentially the analysis of Zakharov and Shabat [8]. Introducing the integrating factor β, [6]:

$$u=\beta_{,\xi}\mathrm{tg}\left(\frac{\alpha}{2}\right), \qquad v=-\beta_{,\eta}\mathrm{tg}\left(\frac{\alpha}{2}\right) \tag{B.20}$$

the system for the invariants α and β becomes

$$\alpha_{,\eta\xi}+\sin\alpha=\frac{\mathrm{tg}^2\left(\frac{\alpha}{2}\right)}{\sin\alpha}\beta_{,\eta}\beta_{,\xi},$$

$$\sin\alpha\beta_{,\eta\xi}+\alpha_{,\xi}\beta_{,\eta}+\alpha_{,\eta}\beta_{,\xi}=0 \tag{B.21}$$

which should be compared with Equations (6.17) obtained in the context of the $U(2)$ chiral theory (a sign difference is trivially compensated by the reflection $\eta\to-\eta$). (B.19) and (B.20) is the form under which the inverse problem was discussed in [9]. Further, by introducing a function ω through the compatible equations

$$\omega_{,\xi}=\frac{\beta_{,\xi}}{2\cos^2\left(\frac{\alpha}{2}\right)}, \qquad \omega_{,\eta}=\frac{\beta_{,\eta}\cos\alpha}{2\cos^2\left(\frac{\alpha}{2}\right)} \tag{B.22}$$

the system (B.19) transforms, under the change of variables:

$$\chi_1'=e^{-\frac{i}{2}\omega}\chi_1, \qquad \chi_2'=e^{\frac{i}{2}\omega}\chi_2, \tag{B.23}$$

into

$$\chi'_{,\eta}=\frac{1}{2i}\begin{bmatrix} \gamma & \left[\alpha_{,\eta}-i\beta_{,\eta}\mathrm{tg}\left(\frac{\alpha}{2}\right)\right]e^{-i\omega} \\ \left[\alpha_{,\eta}+i\beta_{,\eta}\mathrm{tg}\left(\frac{\alpha}{2}\right)\right]e^{i\omega} & -\gamma \end{bmatrix}\chi'$$

$$\chi'_{,\xi}=\frac{1}{2i\gamma}\begin{bmatrix} \cos\alpha & i\sin\alpha\, e^{-i\omega} \\ -i\sin\alpha\, e^{i\omega} & -\cos\alpha \end{bmatrix}\chi' \tag{B.24}$$

which is essentially the representation given in [6] and was also used in Section VI.

We close with a simple example illustrating Equation (B.17). We consider the special case $u=0=v$, so that the inverse problem reduces to the sine-Gordon theory. Since the matrix C in (B.18) is unitary, $\chi^+\chi=\phi^+\phi$, therefore $\chi^+\chi$ is space-time independent and must be normalized as $\chi^+\chi=1$. This is true for the solution:

$$\chi_1=\frac{-\varepsilon}{\gamma+i\varepsilon}\frac{e^{i\theta/2}}{\cosh\varrho}, \qquad \chi_2=\frac{[\gamma-i\varepsilon\tanh\varrho]e^{i\theta/2}}{\gamma+i\varepsilon}$$
$$\varrho\equiv\varepsilon^{-1}\xi-\varepsilon\eta, \qquad \theta=\gamma^{-1}\xi+\gamma\eta \tag{B.25a}$$

$$\operatorname{tg}\left(\frac{\alpha}{4}\right)=e^{-\varrho}. \tag{B.25b}$$

The wave function ϕ is then given by:

$$\phi_1=\frac{i}{\sqrt{2}(\gamma+i\varepsilon)}\left[\gamma-\frac{\varepsilon}{\cosh\varrho}-i\varepsilon\tanh\varrho\right]e^{\frac{i}{2}\left(\theta+\frac{\alpha}{2}\right)}$$

$$\phi_2=\frac{-i}{\sqrt{2}(\gamma+i\varepsilon)}\left[\gamma+\frac{\varepsilon}{\cosh\varrho}-i\varepsilon\tanh\varrho\right]e^{\frac{i}{2}\left(\theta-\frac{\alpha}{2}\right)}. \tag{B.26}$$

The wave-function ψ entering Equation (B.17) is taken here to be $\psi=\phi(\ ;-\gamma)$ and a direct application of (B.17) yields the solution vector:

$$q_1=0, \qquad q^2=\frac{-2\varepsilon\gamma}{\gamma^2+\varepsilon^2}\frac{1}{\cosh\varrho}$$

$$\begin{aligned} q^3=&\frac{1}{(\gamma^2+\varepsilon^2)^2}\{2\varepsilon\gamma(\gamma^2-\varepsilon^2)(1+\tanh\varrho)\cos\theta\\ &+[4\varepsilon^2\gamma^2\tanh\varrho-(\gamma^2-\varepsilon^2)^2]\sin\theta\}\\ q^4=&\frac{1}{(\gamma^2+\varepsilon^2)^2}\{[(\gamma^2-\varepsilon^2)^2-4\varepsilon^2\gamma^2\tanh\varrho]\cos\theta\\ &+2\varepsilon\gamma(\gamma^2-\varepsilon^2)(1+\tanh\varrho)\sin\theta\}.\end{aligned} \tag{B.27}$$

It is interesting to verify explicitly that $(q_{,\eta}q_{,\xi})=\cos\alpha$, obtained from (B.27), is compatible with the value of α given directly from the inverse scattering technique, Equation (B.25b). This is a special case of the important fact that working with the orthonormal basis and imposing the constraint $\phi^+\phi=1$, as explained above in detail, suffices to guarantee the correct identification of the fundamental invariants calculated directly in the inverse scattering procedure and ultimately in terms of the solution vector q. The remaining invariants u and v have been assumed to be equal to zero in our special example, a fact correctly reproduced in (B.27), since one of the components vanishes identically. In effect, (B.27) is a solution of the 0(3)-theory.

Finally, it seems possible to establish a more explicit connection of scale invariant Fermi interactions and appropriately generalized σ-models. Direct comparison of the content of Section IV and Section A of this Appendix suggests that it should be possible to formulate the symplectic theory in the three dimensional representation of $\mathrm{Sp}(2,\mathbb{R})$, in terms of three real variables q^i, $i=1,2,3$, which are constructed as suitable bilinear functionals of the two-dimensional spinors, and interact through the quadratic constraint

$$k_{ij}q^i q^j = \text{const}. \tag{B.28}$$

In (B.28), k_{ij} is the Killing form of the regular representation of the $\mathrm{Sp}(2,\mathbb{R}) \sim \mathrm{SO}(2,1)$ Lie algebra and can be brought into the diagonal form $(1,1,-1)$. In conclusion, a universal approach emerges for the solution of non-linear two-dimensional scale invariant classical field theories.

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