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KMS Conditions and Local Thermodynamical Stability of Quantum Lattice Systems. II

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Abstract. We prove that local thermodynamical stability (LTS), as defined in [1], implies the KMS conditions in quantum lattice systems, without any assumption of translational invariance. This result, together with those of [1], establishes the equivalence between the LTS and the KMS conditions for such systems.

Section 1

In an article by Araki and the author [1], the concept of local thermodynamical stability (LTS) was defined for quantum lattice systems; and it was shown that, if the forces were suitably tempered, the LTS conditions were implied by, and in the case of translationally invariant states equivalent to, those of Kubo-Martin-Schwinger (KMS). In the present article, we prove that the LTS conditions imply those of KMS, without any assumption of translational invariance, and thus establish the following theorem.

Theorem 1. *The* LTS *and* KMS *conditions are mutually equivalent for quantum lattice systems, subject to the same assumptions on the interactions as in* [1].

Comment. It has already been observed (cf. Note following Definition 2.2 in [2]) that, for classical lattice and hard-core continuous systems, the LTS conditions are equivalent to those of Dobrushin-Lanford-Ruelle (DLR). Thus, in view of the above theorem, we now conclude that LTS $\equiv$ KMS for quantum lattice systems, and $\equiv$ DLR for classical lattice and hard-core continuous ones.

Our notation will be based on that of [1]. We take Γ to be the lattice on which the system is situated; here it suffices to consider Γ as a denumerably infinite point set. The family $\{\Lambda\}$ of finite point subsets of Γ will be denoted by L. The algebra of observables, $\mathscr{A}(\Lambda)$, for the region $\Lambda(\in L)$ will be assumed to be a finite-dimensional, type-I factor, that is isotonic with respect to Λ and commutes with $\mathscr{A}(\Lambda')$ if $\Lambda \cap \Lambda' = \emptyset$; and the C^*-algebra of observables, $\mathscr{A}$, for the system will be taken to be the norm completion of $\mathscr{A}_L \equiv \bigcup_{\Lambda \in L} \mathscr{A}(\Lambda)$. The state space, Ω, of the system will be assumed to be the set of positive, normalised, linear functionals on $\mathscr{A}$. For $\Lambda \in L$, we define the density matrix $\varrho_\Lambda^\omega(\in \mathscr{A}(\Lambda))$, induced by the state ω, according to the formula

$$\omega(A) = \tau(\varrho_\Lambda^\omega A), \forall A \in \mathscr{A}(\Lambda), \tag{1.1}$$

where τ is the central tracial state. The local entropy and conditional entropy functionals, S_Λ and $\tilde{S}_\Lambda$, respectively, are defined by the formulae

$$S_\Lambda(\omega) = -\tau(\varrho_\Lambda^\omega \ln \varrho_\Lambda^\omega), \forall \omega \in \Omega, \tag{1.2}$$

and

$$\tilde{S}_\Lambda(\omega) = \lim_{\Lambda' \uparrow} [S_{\Lambda'}(\omega) - S_{\Lambda' \setminus \Lambda}(\omega)], \forall \omega \in \Omega. \tag{1.3}$$

As in [1], it is assumed that the forces in the system are sufficiently tempered to permit the definition of an energy observable $\tilde{H}_\Lambda(\in \mathscr{A})$, for each $\Lambda \in L$, that corresponds to the total interaction energy of the particles in Λ both with one another and with those in $\Lambda^c(\equiv \Gamma \setminus \Lambda)$. We say that $\omega(\in \Omega)$ satisfies the LTS conditions for temperature $\beta^{-1}(>0)$ if, for each $\Lambda \in L$,

$$\beta\omega(\tilde{H}_\Lambda) - \tilde{S}_\Lambda(\omega) \leqq \beta\omega'(\tilde{H}_\Lambda) - \tilde{S}_\Lambda(\omega') \quad \text{for} \quad \omega'_{\Lambda^c} = \omega_{\Lambda^c}. \tag{1.4}$$

The dynamics of the system is assumed to correspond to a one-parameter group $\{\sigma_t | t \in \boldsymbol{R}\}$ of automorphisms of $\mathscr{A}$, whose infinitesimal generator δ has $\mathscr{A}_L$ as a core and is given by the formula

$$\delta_{|\mathscr{A}(\Lambda)} = i[\tilde{H}_\Lambda, \cdot] \; \forall \Lambda \in L. \tag{1.5}$$

Theorem 1 is an immediate consequence of the following two theorems, that will be proved in Sections 2 and 3, together with parts (*b*) and (*c*) of the theorem of [1].

Theorem 2. *If the state ω satisfies the* LTS *conditions, then*

$$\frac{i\beta}{2}\omega(\delta(A^*)A - A^*\delta(A)) \geqq \Phi(\omega(A^*A), \omega(AA^*)), \forall A \in D(\delta), \tag{1.6}$$

where $D(\delta)$ is the domain of δ and Φ is the function from $[0, \infty)^2$ to $[-0, \infty]$ given by:

$$\Phi(u, v) = \left.\begin{matrix} u \ln u - u \ln v & \text{if} & u + v > 0 \\ 0 & \text{if} & u = v = 0. \end{matrix}\right\} \tag{1.7}$$

Theorem 3. *If $\omega(\in \Omega)$ is stationery in time and satisfies the inequality (1.6), then it also satisfies the* KMS *conditions.*

The author is indebted to M. Fannes and A. Verbeure for discussions of their current work [3] on the derivation of the KMS conditions from the Roepstorff inequalities for states on W^*-algebras. Indeed, it was these discussions that prompted him to adopt the strategy of proceeding from LTS to KMS via appropriate correlation inequalities. The author is also grateful to H. Araki for suggesting some improvements to the first draft of the article.

Section 2

This Section will be devoted to the proof of four Lemmas and thence to that of Theorem 2.

Definition 4. (i) We define $\{\gamma_s | s \in \boldsymbol{R}_+ \equiv [0, \infty)\}$ to be the one-parameter semigroup of transformations of $\mathscr{A}$, whose generator L is given by:

$$LA = x^* A x - \tfrac{1}{2}\{x^* x, A\}_+ \ \forall A \in \mathscr{A}, \tag{2.1}$$

with $x \in \mathscr{A}(\Lambda)$ and $\Lambda \in L$. Thus, by [4; Theorem 2], γ_s is a completely positive map; and it follows from (2.1) that the restriction of γ_s to $\mathscr{A}(\Lambda^c)$ is the identity.

(ii) For $\omega \in \Omega$ and $s \in \boldsymbol{R}_+$, we define $\omega_s = \omega \circ \gamma_s (\in \Omega)$. Thus, by (1.1) and (2.1),

$$\frac{d}{ds}\omega_s(A) = \tau\left(\frac{d\varrho_{\Lambda'}^{\omega_s}}{ds} A\right) \forall A \in \mathscr{A}(\Lambda'), \Lambda' \supset \Lambda, \tag{2.2}$$

where

$$\frac{d\varrho_{\Lambda'}^{\omega_s}}{ds} = x\varrho_{\Lambda'}^{\omega_s} x^* - \tfrac{1}{2}\{x^* x, \varrho_{\Lambda'}^{\omega_s}\}_+ .$$

Definition 5. We define Ω_1 to be $\{\omega \in \Omega | (\omega(y^* y) = 0) \wedge (y \in \mathscr{A}_L) \Rightarrow y = 0\}$. Thus, if $\omega \in \Omega_1$, then $\varrho_{\Lambda'}^{\omega}$ is strictly positive, and hence has a logarithm, for $\Lambda' \in L$.

Lemma 6. $-\omega(L \ln \varrho_{\Lambda'}^{\omega}) \geqq \Phi(\omega(x^* x), \omega(x x^*)), \forall \omega \in \Omega_1, \Lambda \supset \Lambda.$ (2.3)

Proof. Let $\omega \in \Omega_1$ and $\Lambda' \supset \Lambda$. As noted above, $\varrho_{\Lambda'}^{\omega}$ is strictly positive, and may therefore be expressed in the form

$$\varrho_{\Lambda'}^{\omega} = \sum_{j=1}^{m} c_j E_j, \tag{2.4}$$

where the c_j's are positive numbers and the E_j's are a maximal set of orthogonal projectors in $\mathscr{A}(\Lambda')$, i.e.

$$E_j E_k = E_j \delta_{jk} \quad \text{and} \quad \sum_{j=1}^{m} E_j = I. \tag{2.5}$$

It follows that

$$\ln \varrho_{\Lambda'}^{\omega} = \sum_{j=1}^{m} (\ln c_j) E_j, \tag{2.6}$$

and hence, by Equations (1.1), (1.7), (2.1), (2.4)–(2.6) and the tracial property of τ,

$$-\omega(L \ln \varrho_{\Lambda'}^{\omega}) = \sum_{j,k=1}^{m} g_{jk} \Phi(c_j, c_k) \tag{2.7}$$

where

$$g_{ik} = \tau((E_k x E_j)(E_k x E_j)^*) \geqq 0. \tag{2.8}$$

Now it follows from Equation (1.7) that Φ is jointly convex in its two arguments, i.e.

$$\sum_{n=1}^{N} \alpha_n \Phi(u_n, v_n) \geqq \Phi\left(\sum_{n=1}^{N} \alpha_n u_n, \sum_{n=1}^{N} \alpha_n v_n\right)$$

$$\text{if} \quad \alpha_1, \ldots, \alpha_N \geqq 0 \quad \text{and} \quad \sum_{n=1}^{N} \alpha_n = 1. \tag{2.9}$$

Since, by (1.7), Φ is also a homogeneous function of the first order, it follows that the inequality (2.9) remains valid even without the restriction that $\sum_{n=1}^{N} \alpha_n = 1$. Thus, as $g_{jk} \geqq 0$, by (2.8), we may apply (2.9) to the R.H.S. of (2.7), thereby obtaining the following inequality:

$$-\omega(L \ln \varrho_{A'}^{\omega}) \geqq \Phi\left(\sum_{j,k=1}^{m} g_{jk} c_j, \sum_{j,k=1}^{m} g_{jk} c_k \right). \tag{2.10}$$

Further, it follows from (1.1), (2.4) and (2.8) that

$$\sum_{j,k=1}^{m} g_{jk} c_j = \omega(x^*x), \quad \text{and} \quad \sum_{j,k=1}^{m} g_{jk} c_k = \omega(xx^*). \tag{2.11}$$

The required inequality (2.3) follows immediately from (2.10) and (2.11). Q.E.D.

Lemma 7. $S_{A'}(\omega_s) - S_{A'}(\omega) \geqq \int_0^s dr \Phi(\omega_r(x^*x), \omega_r(xx^*)) \forall \omega \in \Omega_1,$

$$\Lambda' \supset \Lambda, \quad s > 0. \tag{2.12}$$

Proof. For $\omega \in \Omega_1$. it follows from Definitions 4(ii), 5 and the positivity of γ_s that $\omega_s \in \Omega_1 \, \forall s > 0$. Hence, by Lemma 6, the inequality (2.12) will be established if we prove that

$$S_{A'}(\omega_s) - S_{A'}(\omega) = -\int_0^s dr \omega_r(L \ln \varrho_{A'}^{\omega_r}). \tag{2.13}$$

In order to prove this latter formula, we note again that, as $\omega \in \Omega_1$, $\varrho_{A'}^{\omega}$ is strictly positive, i.e. $\varrho_{A'}^{\omega} > 2b_{A'}I$ for some positive number $b_{A'}$. Thus, if A is any positive element of $\mathscr{A}(\Lambda')$, then $\tau(\varrho_{A'}^{\omega_s}A) \equiv \tau(\varrho_{A'}^{\omega}\gamma_s A) \geqq 2b_{A'}\tau(\gamma_s A)$, in view of Definition 4(ii) and the positivity of γ_s. Further, as Definition 4(i) implies that $\frac{d}{ds}\tau(\gamma_s A)$ $= \tau((xx^* - x^*x)\gamma_s A) \geqq -\|xx^* - x^*x\| \, \tau(\gamma_s A)$, it follows that $\tau(\gamma_s A)$ $\geqq e^{-\|xx^*-x^*x\|s}\tau(A)$; and hence $\tau(\varrho_{A'}^{\omega_s}A) \geqq 2b_{A'}e^{-\|xx^*-x^*x\|s}\tau(A)$. Therefore $\varrho_{A'}^{\omega_s}$ $\geqq b_{A'}I \forall s < s_1 \equiv (\ln 2)/\|xx^* - x^*x\|$. On the other hand, (1.1) implies that, for any state ω', $\varrho_{A'}^{\omega'} \leqq c_{A'}I$ with $c_{A'} = (\dim \mathscr{A}(\Lambda'))^{1/2}$. Thus,

$$b_{A'}I \leqq \varrho_{A'}^{\omega_s} \leqq c_{A'}I, \quad \forall s < s_1, \tag{2.14}$$

where $b_{A'}$ and $c_{A'}$ are positive, finite numbers.

Since $LI = 0$, by (2.1), it follows from (1.1) and (2.2) that the R.H.S. of (2.13) is equal to

$$-\int_0^s dr \tau\left(\frac{d\varrho_r}{dr}(I + \ln \varrho_r)\right), \tag{2.15}$$

where $\varrho_r \equiv \varrho_{A'}^{\omega_r}$. In view of the bounds (2.14) on ϱ_r, we may represent $I + \ln \varrho_r$ by the following formula:

$$I + \ln \varrho_r = \int_0^\infty da \left(\frac{1}{a+I} - \frac{1}{a+\varrho_r} + \frac{1}{a+\varrho_r}\varrho_r \frac{1}{a+\varrho_r} \right). \tag{2.16}$$

Hence, as $\tau(AB)\equiv\tau(BA)$, it follows from (2.16) that the expression (2.15) is equal to

$$-\int_0^s dr\int_0^\infty da\,\tau\left(\frac{d\varrho_r}{dr}\left(\frac{1}{a+I}-\frac{1}{a+\varrho_r}+\frac{1}{a+\varrho_r}\varrho_r\frac{1}{a+\varrho_r}\right)\right). \tag{2.17}$$

The bounds on ϱ_r, given by (2.14), ensure that the order of integration w.r.t. a and r may be exchanged in (2.17). Hence, as the integrand in that expression is equal to $\frac{d}{dr}\tau\left(\varrho_r\left(\frac{1}{a+I}-\frac{1}{a+\varrho_r}\right)\right)$, it follows that (2.17), and thus the R.H.S. of (2.13), is equal to

$$\left[-\int_0^\infty da\,\tau\left(\varrho_r\left(\frac{1}{a+I}-\frac{1}{a+\varrho_r}\right)\right)\right]_{r=0}^{r=s}\equiv-\tau(\varrho_{A'}^{\omega_s}\ln\varrho_{A'}^{\omega_s})+\tau(\varrho_{A'}^{\omega}\ln\varrho_{A'}^{\omega})\equiv S_{A'}(\omega_s)-S_{A'}(\omega).$$

Q.E.D.

Lemma 8. $\liminf_{s\to+0} s^{-1}(\tilde{S}_A(\omega_s)-\tilde{S}_A(\omega))\geqq\Phi(\omega(x^*x),\omega(xx^*)),\quad\forall\omega\in\Omega\,.$ (2.18)

Proof. We deal separately with the following three cases: $(a)\omega(x^*x)=0$; $(b)\omega(x^*x)$ and $\omega(xx^*)$ both >0; $(c)\omega(x^*x)>0$ and $\omega(xx^*)=0$.

Case a). The assumption that $\omega(x^*x)=0$ implies, by (1.7), that the R.H.S. of (2.18) is zero. This assumption also implies, by the Schwartz inequality, that $\omega(x^*y)=\omega(y^*x)=0\forall y\in\mathscr{A}$; and thus, by Definition 4, that $\omega(L(\cdot))=0$, i.e. $\omega_s=\omega\forall s\in R_+$. Thus the L.H.S. of (2.18) is also zero, and therefore that formula is satisfied.

Case b). $\omega(x^*x)$ and $\omega(xx^*)$ both >0. Let $\omega^\varepsilon=(1-\varepsilon)\omega+\varepsilon\tau$, with $0<\varepsilon<1$. Then $\omega^\varepsilon\in\Omega_1$, and hence, by Lemma 7,

$$s^{-1}(S_{A'}(\omega_s^\varepsilon)-S_{A'}(\omega^\varepsilon))\geqq s^{-1}\int_0^s dr\,\Phi(\omega_r^\varepsilon(x^*x),\omega_r^\varepsilon(xx^*))\quad\forall A'\supset A\,, \tag{2.19}$$

with $\omega_s^\varepsilon\equiv(\omega^\varepsilon)_s$. By (1.7), Φ is continuous in each of its arguments when they are both non-zero; while, by Definition 4(ii) and our definition of ω_s^ε, this state is w^*-continous in ε and s. Thus, for suitable positive numbers ε_0 and s_0, the integrand in (2.19) is continuous in both ε and r, provided that $\varepsilon<\varepsilon_0$ and $s<s_0$, which we shall henceforth assume to be the case. Further, the functional $S_{A'}$ is w^*-continuous on Ω, and hence the L.H.S. of (2.19) is continuous in ε. Consequently, we may pass to the limit as $\varepsilon\to0$ of the inequality (2.19), thereby obtaining the result that

$$s^{-1}(S_{A'}(\omega_s)-S_{A'}(\omega))\geqq s^{-1}\int_0^s dr\,\Phi(\omega_r(x^*x),\omega_r(xx^*))\,. \tag{2.20}$$

Since, by Definition 4(ii), ω_s coincides with ω on $\mathscr{A}(A^c)$, it follows from (1.3) and (2.20) that

$$s^{-1}(\tilde{S}_A(\omega_s)-\tilde{S}_A(\omega))\geqq s^{-1}\int_0^s dr\,\Phi(\omega_r(x^*x),\omega_r(xx^*))\,. \tag{2.21}$$

The R.H.S. of this inequality tends to $\Phi(\omega(x^*x),\omega(xx^*))$, as $s\to+0$, since the integrand is continuous in r; and therefore

$$\liminf_{s\to+0} s^{-1}(\tilde{S}_A(\omega_s)-\tilde{S}_A(\omega))\geqq\Phi(\omega(x^*x),\omega(xx^*))\,,$$

as required.

Case c). $\omega(x^*x)>0$ and $\omega(xx^*)=0$. Thus, by (1.7), $\Phi(\omega(x^*x),\omega(xx^*))=\infty$. Thus, it is necessary and sufficient to prove that the L.H.S. of (2.18) is also ∞. For this purpose we again use Equation (2.19), with the same notation as before.

By Equation (1.7), $\Phi(u_\alpha, v_\alpha)\to\infty$ as $u_\alpha\to u>0$ and $v_\alpha\to 0$. Hence, as ω_s^ε is w^*-continuous in both ε and s, it follows that for each $N\in\boldsymbol{R}_+$, $\exists$ positive numbers $s_0(N)$, $\varepsilon_0(N)$ such that the integrand in (2.19) exceeds N whenever $s<s_0(N)$ and $\varepsilon<\varepsilon_0(N)$. Thus, by (2.19),

$$s^{-1}(S_{\Lambda'}(\omega_s^\varepsilon)-S_{\Lambda'}(\omega^\varepsilon))>N\,\forall s<s_0(N),\qquad \varepsilon<\varepsilon_0(N).$$

Since the L.H.S. is continuous in ε, we may pass to the limit as $\varepsilon\to 0$, thereby obtaining the inequality

$$s^{-1}(S_{\Lambda'}(\omega_s)-S_{\Lambda'}(\omega))>N\,\forall s<s_0(N).$$

As this formula is valid for all $\Lambda'\supset\Lambda$, it follows from (1.3) that

$$s^{-1}(\tilde{S}_\Lambda(\omega_s)-\tilde{S}_\Lambda(\omega))>N\,\forall s<s_0(N)$$

and therefore $\liminf_{s\to+0} s^{-1}(\tilde{S}_\Lambda(\omega_s)-\tilde{S}_\Lambda(\omega))=\infty$, as required Q.E.D.

Lemma 9. *If* ω *satisfies the* LTS *conditions, then*

$$\frac{i\beta}{2}\omega(\delta(x^*)x-x^*\delta(x))\geqq\Phi(\omega(x^*x),\omega(xx^*)).\tag{2.22}$$

Proof. Since, by Definition 4(ii), ω_s coincides with ω on $\mathscr{A}(\Lambda^c)$, it follows from (1.4) that, if ω is LTS then

$$\beta s^{-1}(\omega_s(\tilde{H}_\Lambda)-\omega(\tilde{H}_\Lambda))\geqq s^{-1}(\tilde{S}_\Lambda(\omega_s)-\tilde{S}_\Lambda(\omega))\,\forall s>0.\tag{2.23}$$

Further, it follows from Equation (1.5) and Definition 4(i), (ii) that

$$\lim_{s\to+0} s^{-1}(\omega_s(\tilde{H}_\Lambda)-\omega(\tilde{H}_\Lambda))=\frac{i}{2}\omega(\delta(x^*)x-x^*\delta(x)).\tag{2.24}$$

This equation, together with Lemma 8, implies the required inequality (2.22).

Q.E.D.

Proof of Theorem 2. Since x can be chosen to be any element of $\mathscr{A}(\Lambda)$, and Λ any bounded region of Γ, it follows from Lemma 9 that (1.6) is valid for all $A\in\mathscr{A}_L$. Thus, it remains for us to extend that inequality to all $A\in D(\delta)$. This we shall do firstly for $\{A\in D(\delta)\,|\,\omega(A^*A)=0\}$ and then for the remaining elements of $D(\delta)$.

If $A\in D(\delta)$ and $\omega(A^*A)=0$, it follows from the Schwartz inequality that $\omega(A^*\delta(A))=\omega(\delta(A^*)A)=0$, and thus that the L.H.S. of (1.6) is zero; while it follows from (1.7) that the R.H.S. of (1.6) is also zero. Hence (1.6) is satisfied.

Suppose now that $\omega(A^*A)>0$ and $A\in D(\delta)$. Since $\mathscr{A}_L$ is a core for δ, there exists a net $\{A_\alpha\}$ in $\mathscr{A}_L$ such that the norm limits of A_α, $\delta(A_\alpha)$ are A, $\delta(A)$, respectively. As (1.6) is satisfied if A is replaced by any element of $\mathscr{A}_L$, it follows that

$$\frac{i\beta}{2}\omega(\delta(A_\alpha^*)A_\alpha-A_\alpha^*\delta(A_\alpha))\geqq\Phi(\omega(A_\alpha^*A_\alpha),\omega(A_\alpha A_\alpha^*)).\tag{2.25}$$

The application of $\lim_\alpha$ to this formula yields the required inequality (1.6), since by (1.7), $\lim_\alpha\Phi(u_\alpha,v_\alpha)=\Phi(u,v)$ if $\lim_\alpha u_\alpha=u>0$ and $\lim_\alpha v_\alpha=v\geqq 0$. Q.E.D.

We observe now that the L.H.S. of (1.6) is finite for all $A \in D(\delta)$, and that the R.H.S. would be infinite if $\omega(AA^*)$ were zero and $\omega(A^*A)$ positive. Hence we arrive at

Corollary 10. *If ω is* LTS *and $A \in D(\delta)$, then $\omega(AA^*)=0$ implies that $\omega(A^*A)=0$.*

Section 3

We shall devote this Section to proving Theorem 3.

Definition 11. (i) Let $(\mathscr{H}, \pi, \Psi)$ be the GNS triple induced by a state ω, that is stationary in time. We define the Hamiltonian H by the standard formula, i.e.

$$H\pi(A)\Psi = \pi(\delta(A))\Psi \forall A \in D(\delta) \tag{3.1}$$

Thus,

$$e^{iHt}\pi(A)\Psi = \pi(\sigma_t(A))\Psi \forall A \in \mathscr{A}, \qquad t \in \boldsymbol{R}. \tag{3.2}$$

(ii) For $A \in \mathscr{A}$, we define the positive-valued Radon measures μ_A, ν_A on $\boldsymbol{R}$ by the equations

$$\mu_A(f) = \int f(s) d\|E_s \pi(A)\Psi\|^2 \tag{3.3}$$

and

$$\nu_A(f) = \int f(-s) d\|E_s \pi(A^*)\Psi\|^2, \tag{3.4}$$

for all continuous functions f with compact support, where $\{E_s\}$ is the spectral family of projectors for H, i.e. $H = \int s dE_s$.

Lemma 12. *Let ω be a stationary state such that, for all A in a norm-dense linear, involutive subset Δ of $\mathscr{A}$, ν_A is absolutely continuous with respect to μ_A and $d\nu_A/d\mu_A = e^{-\beta s}$. Then ω satisfies the* KMS *conditions.*

Proof. Let f be a function on $\boldsymbol{R}$ whose Fourier transform $\hat{f}$ is $\mathscr{D}$-class; and let f_β be the function on $\boldsymbol{R}$ whose Fourier transform is given by

$$\hat{f}_\beta(s) = e^{-\beta s}\hat{f}(s). \tag{3.5}$$

Then it follows from Equations (3.2), (3.3) and (3.5) that, if $d\nu_A/d\mu_A = e^{-\beta s}$, then

$$\nu_A(\hat{f}) = \mu_A(\hat{f}_\beta) = \int dt\, f_\beta(t)\omega((\sigma_t A^*)A); \tag{3.6}$$

while it follows from (3.2), (3.4) that

$$\nu_A(\hat{f}) = \int dt\, f(t)\omega(A(\sigma_t A^*)). \tag{3.7}$$

Hence, by (3.6) and (3.7)

$$\int dt[f_\beta(t)\omega((\sigma_t A^*)A) - f(t)\omega(A(\sigma_t A^*))] = 0 \forall A \in \Delta, \hat{f} \in \mathscr{D}(\boldsymbol{R}). \tag{3.8}$$

This equation may readily be extended by linearity and continuity to the form

$$\int dt[f_\beta(t)\omega((\sigma_t A^*)B) - f(t)\omega(B(\sigma_t A^*))] = 0, \forall A, B \in \Delta, \hat{f} \in \mathscr{D}(\boldsymbol{R}), \tag{3.9}$$

which constitutes the KMS conditions [5]. Q.E.D.

Proof of Theorem 3. Assuming that ω is a stationary state that satisfies (1.6), we shall prove that, for all $A \in D(\delta)$, ν_A is absolutely continuous w.r.t. μ_A and that $d\nu_A/d\mu_A = e^{-\beta s}$: by Lemma 12, this will suffice to prove the theorem.

Let f be a function on $\boldsymbol{R}$, whose Fourier transform $\hat{f}$ (with $\hat{f}(s) \equiv \int dt f(t) e^{-ist}$) is $\mathscr{D}$-class, and let $\bar{s}(\hat{f})$ (resp. $\underline{s}(\hat{f})) = \sup(\text{resp.} \inf)\{s \in \operatorname{supp} \hat{f}\}$. For $A \in \mathscr{A}$, we define

$$A(f) = \int dt f(-t) \sigma_t A. \tag{3.10}$$

Hence it follows from Equations (3.1)–(3.4) and (3.10) that

$$\omega(A(f)^* A(f)) = \int |\hat{f}(s)|^2 d\mu_A(s) \equiv \mu_A(|\hat{f}|^2) \tag{3.11}$$

$$\omega(A(f) A(f)^*) = \int |\hat{f}(s)|^2 d\nu_A(s) \equiv \nu_A(|\hat{f}|^2) \tag{3.12}$$

$$\begin{aligned} &\frac{i}{2}\omega(\delta(A(f)^*)A(f) - A(f)^* \delta(A(f))) \\ &\quad = \int s |\hat{f}(s)|^2 d\mu_A(s) \leqq \bar{s}(\hat{f}) \mu_A(|\hat{f}|^2) \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &\frac{i}{2}\omega(\delta(A(f)) A(f)^* - A(f) \delta(A(f)^*)) \\ &\quad = -\int s |\hat{f}(s)|^2 d\nu_A(s) \leqq -\underline{s}(\hat{f}) \nu_A(|\hat{f}|^2). \end{aligned} \tag{3.14}$$

On replacing A by $A(f)$ in (1.6) and using Equations (3.11)–(3.13), we obtain the inequality

$$\beta \bar{s}(\hat{f}) \mu_A(|\hat{f}|^2) \geqq \Phi(\mu_A(|\hat{f}|^2), \nu_A(|\hat{f}|^2)). \tag{3.15}$$

Likewise, on replacing A by $A(f)^*$ in (1.6) and using equations (3.11), (3.12) and (3.14), we see that

$$-\beta \underline{s}(\hat{f}) \nu_A(|\hat{f}|^2) \geqq \Phi(\nu_A(|\hat{f}|^2), \mu_A(|\hat{f}|^2)). \tag{3.16}$$

By Corollary 10 and Equations (3.11) and (3.12), $\mu_A(|\hat{f}|^2)$ and $\nu_A(|\hat{f}|^2)$ are either both zero or both positive. In the latter case, it follows from the formulae (1.7), (3.15) and (3.16) that

$$\mu_A(|\hat{f}|^2) \exp(-\beta \underline{s}(\hat{f})) \geqq \nu_A(|\hat{f}|^2) \geqq \mu_A(|\hat{f}|^2) \exp(-\beta \bar{s}(\hat{f})); \tag{3.17}$$

while, in the former case ($\mu_A(|\hat{f}|^2) = \nu_A(|\hat{f}|^2) = 0$), these relations are trivially satisfied. Thus, the inequalities (3.17) are applicable in all cases.

For given $\varepsilon > 0$, let $\{h_n\}$ be a sequence in $\mathscr{D}(\boldsymbol{R})$ forming a partition of unity, such that $\exp(-\beta \underline{s}(h_n)) - \exp(-\beta \bar{s}(h_n)) < \varepsilon$ for every n. Thus, on putting $|\hat{f}|^2 = h_n g$ in (3.17), where g is an arbitrary positive element of $\mathscr{D}(\boldsymbol{R})$, it follows that

$$\begin{aligned} &\int h_n(s) g(s) e^{-\beta s} d\mu_A(s) + \varepsilon \mu_A(h_n g) \\ &\quad \geqq \mu_A(h_n g) e^{-\beta \underline{s}(h_n g)} \geqq \nu_A(h_n g) \\ &\quad \geqq \mu_A(h_n g) e^{-\beta \bar{s}(h_n g)} \\ &\quad \geqq \int h_n(s) g(s) e^{-\beta s} d\mu_A(s) - \varepsilon \mu_A(h_n g), \end{aligned}$$

for every n. By summing these inequalities over n, we obtain the formula

$$\int g(s)\,e^{-\beta s}\,d\mu_A(s)+\varepsilon\mu_A(g)\geqq \nu_A(g)\geqq \int g(s)\,e^{-\beta s}\,d\mu_A(s)-\varepsilon\mu_A(g).$$

Therefore, as ε may be chosen to be arbitrarily small, it follows that

$$\int g(s)\,e^{-\beta s}\,d\mu_A(s)=\nu_A(g)$$

for every positive g in $\mathscr{D}(\boldsymbol{R})$, and hence, by linearity and continuity, for every continuous function g on $\boldsymbol{R}$ with compact support. This implies the required result.
Q.E.D.

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