Commun. math. Phys. 53, 285—294 (1977)

Communications in Mathematical Physics

Spectral Theory of the Operator $(p^2+m^2)^{1/2}$—Ze^2/r

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Abstract. Using dilation invariance and dilation analytic techniques, and with the help of a new virial theorem, we give a detailed description of the spectral properties of the operator $(\boldsymbol{p}^2+m^2)^{1/2}-Ze^2/r$. In the process the norm of the operator $|\boldsymbol{x}|^{-\alpha}|\boldsymbol{p}|^{-\alpha}$ is calculated explicitly in $L^p(\mathbb{R}^N)$.

I. Introduction

The *classical* Hamiltonian describing the interaction of a relativistic particle of charge e and mass m with an electromagnetic field [vector potential $\boldsymbol{A}(\boldsymbol{x})$ and scalar potential $\phi(\boldsymbol{x})$] is given by [1]

$$[(\boldsymbol{p}-e\boldsymbol{A}(\boldsymbol{x}))^2+m^2]^{1/2}+e\phi(\boldsymbol{x}). \tag{1.1}$$

To make the transition to quantum mechanics, the usual procedure (which is of course fraught with ambiguities) is to change the classical Hamiltonian into an operator on the Hilbert space $L^2(\mathbb{R}^3)$ by replacing $\boldsymbol{p}$ by $-i\nabla$. Because of the troublesome square root in (1.1), the standard procedure just described has received very little attention in treating a relativistic particle in an electromagnetic field. Historically, an alternative procedure was followed resulting in the Klein-Gordon (K.G.) equation [2]. Calling the energy function of (1.1) E, one finds

$$(E-e\phi(\boldsymbol{x}))^2-(\boldsymbol{p}-e\boldsymbol{A}(\boldsymbol{x}))^2-m^2=0.$$

One now makes the Ansatz $\boldsymbol{p}=-\mathrm{i}\nabla$ and tries to solve the implicit eigenvalue problem

$$\{(E-e\phi(\boldsymbol{x}))^2-(\boldsymbol{p}-e\boldsymbol{A}(\boldsymbol{x}))^2-m^2\}\psi(\boldsymbol{x})=0 \tag{1.2}$$

subject to "appropriate" boundary conditions. The K.G. equation has a definite virtue when the interaction is the Coulomb potential ($\boldsymbol{A}\equiv 0, \phi(\boldsymbol{x})=-Ze/|\boldsymbol{x}|$): The equation can be solved explicitly. It seems to us that this explicit solvability is the

* Supported in part by NSF Grant MPS 74 22844

main reason for the comparative neglect of the more difficult operator

$$H=(p^2+m^2)^{1/2}-Ze^2/|x|. \tag{1.3}$$

This operator describes the same system as the K.G. equation, namely a spin zero particle in the Coulomb field of an infinitely heavy nucleus of charge Z. However, the theory of operator (1.3) does not suffer from the difficulties of interpretation of the K.G. theory [2] which are connected with the fact that the latter is not really a Hamiltonian theory. The operator (1.3) also has the virtue that it is stable over a larger range of Z's than the K.G. theory: The operator H is non-negative if $Ze^2 \leqq 2/\pi$ (see Theorem 2.1 below) while the energy of the ground state in the K.G. theory becomes complex when $Ze^2 > \frac{1}{2}$ [2]. (This is to be compared with the Dirac equation for a spin $\frac{1}{2}$ particle in a Coulomb field which is unstable if $Ze^2 > 1$ [2, 3].)

It is thus clear that there is a range of atomic number over which the K.G. energies will differ appreciably from the eigenvalues of the operator in (1.3). It would be very interesting if Nature's preference could be seen experimentally. Unfortunately, this question is clouded by other effects which play an important role in π and K mesic atoms [4] and it may be on the borderline of being untestable.

In this paper we examine the spectral properties of the operator of Equation (1.3) from an abstract point of view. Our results are summarized in Theorems 2.1 through 2.5.

II. Spectral Properties

In this section we first state and then prove Theorems 2.1 through 2.5. We use the notation $Q(A)=\mathscr{D}(|A|^{1/2})$ for any self-adjoint operator A. We work on $L^2(\mathbb{R}^3)$ unless otherwise stated.

Theorem 2.1. *Let* $H_0=(p^2+m^2)^{1/2}$, $m \geqq 0$.

a) *If* $Ze^2 \leqq 2/\pi$, *then as a form on* $Q(H_0)$

$$H_0 - Ze^2/|x| \geqq 0. \tag{2.1}$$

b) *If* $Ze^2 > 2/\pi$, *then* $H_0-Ze^2/|x|$ *is unbounded below as a form on* $Q(H_0)$.

c) $\| |x|^{-1}(H_0+1)^{-1}\| = 2$ *and thus in particular* $\mathscr{D}(|x|^{-1}) \supseteqq \mathscr{D}(H_0)$ *and* $H_0-Ze^2/|x|$ *is essentially self-adjoint on* $\mathscr{D}(H_0)$ *if* $Ze^2 \leqq \frac{1}{2}$.

Remark. a) Of Theorem 2.1 is stated in Kato [5] (without proof); c) is a well known result [5, 6]. Theorem 2.1 will follow from a more general result proved in Theorem 2.5 below.

Definition. We define the operator $H=H_0-Ze^2/|x|$ for $Ze^2 \leqq 2/\pi$ to be the Friedrichs extension of $(H_0-Ze^2/|x|)\restriction_{\mathscr{D}(H_0)}$. We remark that because of Theorem 2.1c), if $Ze^2 < 2/\pi$, the Friedrichs extension coincides with the form sum of H_0 and $-Ze^2/|x|$.

Theorem 2.2. *Suppose* $Ze^2 < 2/\pi$. *Then the spectrum of* H *in* $[0, m)$ *is discrete (consisting of eigenvalues of finite multiplicity with no points of accumulation). We have the lower bound*

$$H \geqq m(1-(\tfrac{1}{2}\pi Ze^2)^2)^{1/2}. \tag{2.2}$$

Remark. The ground state energy for the Klein-Gordon equation is $E_0=(\frac{1}{2}+\sqrt{\frac{1}{4}-(Ze^2)^2})^{1/2}m$ and for the Dirac equation $E_0=m(1-(Ze^2)^2)^{1/2}$.

Theorem 2.3. *Suppose $Ze^2<2/\pi$. Then $\sigma(H)\supseteq[m,\infty)$ and the spectrum of H in $[m,\infty)$ is purely absolutely continuous.*

Remark. To prove Theorem 2.3 we will use dilation analytic techniques developed by Aguilar and Combes [7] and applied by them to handle certain perturbations of the Laplacian. Their techniques have also been applied to operators of the form

$$(\boldsymbol{p}^2+m^2)^{1/2}+\lambda|\boldsymbol{x}|^{-\beta}$$

for $\beta\in(0,1)$ by Weder [8]. Evidently Weder could not handle the case $\beta=1$ because of the more severe singularity. We fill this gap.

Theorem 2.4 (Virial theorem). *Suppose $m>0$, $Ze^2<2/\pi$ and ψ is an eigenstate of H with eigenvalue E and norm 1. Then if*

$$\boldsymbol{v}=\boldsymbol{p}(\boldsymbol{p}^2+m^2)^{-1/2}$$

we have

$$E=m^2(\psi,H_0^{-1}\psi), \tag{2.3}$$

$$(\psi,Ze^2/|\boldsymbol{x}|\psi)=(\psi,\boldsymbol{v}\cdot\boldsymbol{p}\psi). \tag{2.4}$$

Remark. Equation (2.4) is also true when $(\boldsymbol{p}^2+m^2)$ is replaced by $\boldsymbol{p}^2/2m$ if in addition $\boldsymbol{v}$ is replaced by $\boldsymbol{p}/m$. In this case it is the usual non-relativistic virial theorem.

We begin the proof of Theorem 2.1 with a calculation of the norm of a certain operator:

Theorem 2.5. *Define the operator C_α on $\mathscr{S}(\mathbb{R}^N)$ by*

$$C_\alpha\equiv|\boldsymbol{x}|^{-\alpha}|\boldsymbol{p}|^{-\alpha},\ \boldsymbol{p}=-i\nabla \tag{2.5}$$

and let $p^{-1}+q^{-1}=1$. Suppose $\alpha>0$ and $N\alpha^{-1}>p>1$. Then C_α extends to a bounded operator on $L^p(\mathbb{R}^N)$ with

$$\|C_\alpha\|_{L^p(\mathbb{R}^N)\to L^p(\mathbb{R}^N)}=2^{-\alpha}\frac{\Gamma(\frac{1}{2}(Np^{-1}-\alpha))\Gamma(\frac{1}{2}Nq^{-1})}{\Gamma(\frac{1}{2}(Nq^{-1}+\alpha))\Gamma(\frac{1}{2}Np^{-1})}. \tag{2.6}$$

If $p\geqq N\alpha^{-1}$ or $p=1$, then C_α is unbounded.

Before embarking on a proof of Theorem 2.5 we make a few remarks. We restrict ourselves to the case $p=2$. First note that the operator

$$C_\alpha^m\equiv|\boldsymbol{x}|^{-\alpha}(\boldsymbol{p}^2+m^2)^{-\alpha/2}\qquad m\geqq0$$

has the same norm as C_α. To prove this we can do a unitary scale transformation (dilation) of C_α^m to get the operator $(\lambda>0)$ $C_\alpha^{m\lambda}$, so it follows that $\|C_\alpha^m\|\equiv\gamma$ is independent of m for $m>0$. We clearly have $\gamma\leqq\|C_\alpha\|$ and in addition since $C_\alpha^{m\lambda}\xrightarrow{s}C_\alpha$ as $\lambda\to0$

$$\|C_\alpha\|\leqq\lim_{\lambda\to0}\|C_\alpha^{m\lambda}\|=\gamma.$$

Thus

$$\|C_\alpha\| = \|C_\alpha^m\| . \tag{2.7}$$

We mention the special case $\alpha=1$: From Equation (2.7) we have for $N \geqq 3$ and any $m \geqq 0$

$$\| |\boldsymbol{x}|^{-1}(\boldsymbol{p}^2+m^2)^{-1/2}\| = 2(N-2)^{-1} . \tag{2.8}$$

We also derive Hardy's inequality. For $N \geqq 3$

$$((N-2)/2)^2|\boldsymbol{x}|^{-2} \leqq |\boldsymbol{p}|^2 . \tag{2.9}$$

Another application of Equation (2.6) is given in [9] where the local singularities of the eigenfunctions of

$$-\Delta + V(\boldsymbol{x})$$

are discussed for a potential $V(\boldsymbol{x})$ which behaves like $-\lambda^2/|\boldsymbol{x}|^2$ near $\boldsymbol{x}=0$.

Proof of Theorem 2.5. Define the isometric dilation on $L^p(\mathbb{R}^N)$:

$$(U_\theta f)(\boldsymbol{x}) = e^{N\theta p^{-1}} f(e^\theta \boldsymbol{x}) ; \qquad \theta \in (-\infty, \infty) . \tag{2.10}$$

The idea of the computation is that C_α commutes with dilations and thus in a representation where dilations are simply translations (in the variable θ), C_α is a convolution operator (at least in one of the variables). It will turn out that it is convolution by a *positive* function. The norm of such an operator is just the integral of the function. (The author thanks Barry Simon for pointing out the simple fact that this holds on *all* L^p spaces.) This idea needs only minor modifications due to the presence of other variables.

Thus let $U : L^p(\mathbb{R}^N, d^N x) \to L^p(\mathbb{R} \times S_{N-1}, dx d\omega)$ be given by

$$(Uf)(x, \boldsymbol{\omega}) = e^{Np^{-1}x} f(e^x \boldsymbol{\omega}) , \tag{2.11}$$

where $\boldsymbol{\omega}$ runs over S_{N-1}, the surface of the unit ball in $\mathbb{R}^N$ and $d\omega$ is the invariant surface measure ($d^N x = |\boldsymbol{x}|^{N-1} dx d\omega$) U is an isometry and satisfies

$$(UU_\theta U^{-1} f)(x, \boldsymbol{\omega}) = f(x+\theta, \boldsymbol{\omega}) . \tag{2.12}$$

We now transform the operator C_α. First note that $|\boldsymbol{p}|^{-\alpha}$ is convolution with the function

$$(2\pi)^{-N} \int |\boldsymbol{p}|^{-\alpha} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} d^N p = \gamma |\boldsymbol{x}|^{-(N-\alpha)} , \tag{2.13}$$

where $\gamma = \gamma(N, \alpha) = (2^\alpha \pi^{N/2} \Gamma(\alpha/2))^{-1} \Gamma(\frac{1}{2}(N-\alpha))$ [10]. Thus C_α is an integral operator with kernel

$$\gamma |\boldsymbol{x}_1|^{-\alpha} |\boldsymbol{x}_1 - \boldsymbol{x}_2|^{-(N-\alpha)} . \tag{2.14}$$

The kernel of $UC_\alpha U^{-1}$ is then easily computed. It is

$$G(x_1 - x_2 ; \boldsymbol{\omega}_1, \boldsymbol{\omega}_2) = \gamma e^{\beta(x_1 - x_2)} (2\cosh(x_1 - x_2) - 2\boldsymbol{\omega}_1 \cdot \boldsymbol{\omega}_2)^{-(N-\alpha)/2} , \tag{2.15}$$

where $\beta = N(p^{-1} - \frac{1}{2}) - \frac{1}{2}\alpha$.

Let $\hat{C} = UC_\alpha U^{-1}$ and $Q = L^p(\mathbb{R} \times S_{n-1}, dx d\omega)$. We claim that

$$\|\hat{C}\|_{Q \to Q} = \int dx d\omega G(x ; \boldsymbol{\omega}, \boldsymbol{e}) . \tag{2.16}$$

Denote the right-hand side of (2.16) by Λ. (Note that Λ is independent of $\boldsymbol{e}$ because of rotation invariance.) It is easy to see that $\|\hat{C}\|_{Q\to Q}\leqq\Lambda$: An easy estimate shows that $\hat{C}$ is bounded on $L^\infty(\mathbb{R}\times S_{N-1};dxd\omega)$ and on $L^1(\mathbb{R}\times S_{n-1};dxd\omega)$ by Λ, and the Riesz-Thorin interpolation theorem then gives $\|\hat{C}\|_{Q\to Q}\leqq\Lambda$. To show the opposite inequality, let $\hat{C}_0=\hat{C}\restriction_{Q_0}$ where Q_0 is the set of rotation invariant vectors in $L^p(\mathbb{R}\times S_{n-1};dxd\omega)$ $Q_0\cong L^p(\mathbb{R};dx)$. Then

$$(\hat{C}_0 f)(x)=\int g(x-x')f(x')dx', \tag{2.17}$$

where

$$g(x)\equiv\int G(x;\boldsymbol{\omega},\boldsymbol{e})d\omega.$$

We have

$$\begin{aligned}\|\hat{C}_0\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})}&=\sup|(f,\hat{C}_0h)|/\|f\|_q\|h\|_p\\&\geqq\int_{-\lambda}^{\lambda}dx\int_{-\lambda}^{\lambda}dy\,g(x-y)(2\lambda)^{-1}\\&=\int_{-2\lambda}^{2\lambda}dx(1-|x|(2\lambda)^{-1})g(x).\end{aligned}\tag{2.18}$$

The first inequality follows by taking $f=h=\chi_{[-\lambda,\lambda]}$ and (2.18) by integrating over $x+y$. As $\lambda\to\infty$, (2.18) converges to $\|g\|_1$ by the Lebesgue dominated convergence theorem. This gives

$$\Lambda\geqq\|\hat{C}\|_{Q\to Q}\geqq\|\hat{C}_0\|_{L^p\to L^p}\geqq\int g(x)dx=\Lambda$$

which is (2.16). It follows from the fact that U maps $L^p(\mathbb{R}^N;d^Nx)$ onto $L^p(\mathbb{R}\times S_{n-1};dxd\omega)$ isometrically that

$$\|C_\alpha\|_{L^p(\mathbb{R}^N)\to L^p(\mathbb{R}^N)}=\|\hat{C}\|_{Q\to Q}.$$

In an appendix we compute $\int dxd\omega G(x;\boldsymbol{\omega},\boldsymbol{e})$ and show that it is the right side of Equation (2.6). The fact that C_α is unbounded if $p\geqq N\alpha^{-1}$ or $p=1$ follows by inspection.

Proof of Theorem 2.1. This theorem is essentially a corollary of Theorem 2.5. From Equations (2.6) and (2.7) it follows that in $L^2(\mathbb{R}^3)$

$$\|H_0^{-1/2}|\boldsymbol{x}|^{-1}H_0^{-1/2}\|=\|C_{1/2}\|^2=\pi/2. \tag{2.19}$$

Thus $Q(H_0)\subseteqq Q(|\boldsymbol{x}|^{-1})$ and if $Ze^2\leqq 2/\pi$, $V(\boldsymbol{x})=-Ze^2/|\boldsymbol{x}|$ satisfies

$$\|H_0^{-1/2}VH_0^{-1/2}\|\leqq 1$$

which implies $H_0+V=H_0^{1/2}(1+H_0^{-1/2}VH_0^{-1/2})H_0^{1/2}\geqq 0$. Conversely if

$$H_0+V\geqq -E$$

then by performing a dilation we find for any $\lambda>0$

$$(\boldsymbol{p}^2+(m\lambda)^2)^{1/2}+V\geqq -E\lambda$$

so that taking $\lambda\to 0$ gives

$$H_0+V\geqq 0$$

which gives

$$-H_0^{-1/2}VH_0^{-1/2}\leqq 1\,.$$

By Equation (2.19) this can only be true if $Ze^2\leqq 2/\pi$.

This completes the proof of Parts a) and b). Part c) follows from Equation (2.8) and the scaling arguments used in proving Equation (2.7).

Proof of Theorem 2.4. We follow a method developed by Weidmann [11] which skirts possible domain problems.

Temporarily we write $H_m=(\boldsymbol{p}^2+m^2)^{1/2}-Ze^2/|\boldsymbol{x}|$ indicating the dependence on m explicitly. It is easy to see that

$$\mathscr{D}(H_{m_1})=\mathscr{D}(H_{m_2}) \tag{2.20}$$

and in fact

$$H_{m_1}=H_{m_2}+B_{m_1,m_2}\,, \tag{2.21}$$

where B_{m_1,m_2} is a bounded operator:

$$B_{m_1,m_2}=(m_1^2-m_2^2)\left((\boldsymbol{p}^2+m_1^2)^{1/2}+(\boldsymbol{p}^2+m_2^2)^{1/2}\right)^{-1}. \tag{2.22}$$

Let $U(a)\psi(\boldsymbol{x})=a^{3/2}\psi(a\boldsymbol{x})$. $U(a)$ is unitary and satisfies

$$U(a)\mathscr{D}(H_m)=\mathscr{D}(H_m)$$

and in fact

$$U(a)H_mU^{-1}(a)=a^{-1}H_{ma}\,. \tag{2.23}$$

Suppose $H_m\psi=E\psi$ with $\|\Psi\|^2=1$. Denote $\psi_a=U(a)\psi$. We have by Equation (2.23)

$$H_{ma}\psi_a=aE\psi_a \tag{2.24}$$

and thus taking the complex conjugate of the inner product of (2.24) with ψ and subtracting from the equation $(\psi_a,H_m\psi)=E(\psi_a,\psi)$ results in

$$(1-a)^{-1}(\psi_a,(H_m-H_{ma})\psi)=E(\psi,\psi_a)\,.$$

Since $(1-a)^{-1}(H_m-H_{ma})=(1+a)m^2\left((\boldsymbol{p}^2+m^2)^{1/2}+(\boldsymbol{p}^2+(ma)^2)^{1/2}\right)^{-1}$ converges strongly on ψ to $m^2H_0^{-1}\psi$ as $a\to 1$, we have

$$m^2(\psi,H_0^{-1}\psi)=E\,. \tag{2.25}$$

This is Equation (2.3). Equation (2.4) follows from

$$(\psi,V\psi)=(\psi,(E-H_0)\psi)=(\psi,(m^2H_0^{-1}-H_0)\psi)\,.$$

Lemma 2.6. *Suppose $Ze^2<2/\pi$. Then $(H+1)^{-1}-(H_0+1)^{-1}$ is compact.*

Remark. Ideas very similar to those used in the proof of Lemma 2.6 can be found in [12] which also contains references to original contributions.

Proof. Let $R=(H+1)^{-1}$, $R_0=(H_0+1)^{-1}$ and $V=-Ze^2/|\boldsymbol{x}|$. We have

$$R-R_0=R_0|V|^{1/2}|V|^{1/2}R.$$

R maps $L^2(\mathbb{R}^3)$ into $\mathscr{D}(H)\subseteqq\mathscr{D}(H^{1/2})=\mathscr{D}(H_0^{1/2})\subseteqq\mathscr{D}(|V|^{1/2})$ so that by the closed graph theorem, $|V|^{1/2}R$ is bounded. Thus we need only show that $R_0|V|^{1/2}$ is compact. We give an elementary argument: Let χ be multiplication by the characteristic function of the unit ball. Then $R_0|V|^{1/2}$ is compact because it is the norm convergent limit of the Hilbert-Schmidt operators

$$e^{-\varepsilon p^2}R_0|V|^{1/2}(1-\chi)e^{-\delta x^2}+e^{-\varepsilon p^2}R_0|V|^{1/2}\chi$$

as $\delta,\varepsilon\downarrow 0$. [Note $\|e^{-\varepsilon p^2}R_0^{1/2}-R_0^{1/2}\|\to 0$, $R_0^{1/2}|V|^{1/2}$ is bounded, and $\|(1-\chi)(1-e^{-\delta x^2})\|\to 0$.]

Proof of Theorem 2.2. Let $\varrho=Ze^2\pi/2<1$, and suppose $0\leqq\lambda<m$. Then if $V=-Ze^2/r$ we have that

$$B=(H_0-\lambda)^{-1/2}V(H_0-\lambda)^{-1/2}$$

satisfies

$$\begin{aligned}\|B\| &\leqq \|\,|\boldsymbol{p}|^{1/2}(H_0-\lambda)^{-1/2}\|^2\,\|\,|\boldsymbol{p}|^{-1/2}V|\boldsymbol{p}|^{-1/2}\| \\ &=\left(\sup_{x\geqq 0} x((x^2+m^2)^{1/2}-\lambda)^{-1}\right)\varrho \\ &=m(m^2-\lambda^2)^{-1/2}\varrho\,.\end{aligned}$$

Thus if

$$m(m^2-\lambda^2)^{-1/2}\varrho<1 \tag{2.26}$$

$$(H-\lambda)^{-1}=(H_0-\lambda)^{-1/2}(1+B)^{-1}(H_0-\lambda)^{-1/2}$$

so that λ is not in the spectrum of H. Condition (2.26) is however the same as the condition

$$\lambda<m(1-\varrho^2)^{1/2}\,.$$

Thus $H\geqq m(1-\varrho^2)^{1/2}$. The proof is completed after noting that Lemma 2.6 guarantees the equality of the essential spectra of H and H_0. Thus $\sigma_{\text{essential}}(H)=[m,\infty)$.

Proof of Theorem 2.3. The virial theorem guarantees the absence of discrete spectrum in $[m,\infty)$ while Lemma 2.6 shows that $\sigma(H)\supseteqq[m,\infty)$ so that it only remains to show the absence of singular continuous spectrum. This we do using a technique invented by Aguilar and Combes [7] for perturbations of the Laplacian and applied by Weder [8] to operators of the form $(\boldsymbol{p}^2+m^2)^{1/2}+\lambda|\boldsymbol{x}|^{-\beta}$ for $\beta<1$.

Define $H_0'(z)=(\boldsymbol{p}^2+(me^{-z})^2)^{1/2}$ where $z\in\mathbb{C}$, $|\mathrm{Im}\,z|<\pi/2$ and $\mathrm{Re}(\boldsymbol{p}^2+(me^{-z})^2)^{1/2}\geqq 0$. Note that $H_0'(z)$ is a normal operator whose spectrum in the complex λ plane is the curve

$$\{\lambda=\lambda_1+\lambda_2:\lambda_1\geqq\mathrm{Re}(me^{-z}),\lambda_1\lambda_2=\tfrac{1}{2}m^2\,\mathrm{Im}\,e^{-2z}\}\,.$$

Consider the family of operators

$$H_0'(z) + V = H'(z), \tag{2.27}$$

where since $H_0'(z) - H_0$ is bounded, $H'(z) = H + (H_0'(z) - H_0)$ is a holomorphic family of type A [5], analytic in the region $|\mathrm{Im}\, z| < \frac{1}{2}\pi$.

We define $\sigma_{\mathrm{discrete}}(A)$ for a closed operator A to be the set of all $\lambda \in \sigma(A)$ which are isolated and have the property that for all small enough $\varepsilon > 0$, the projection

$$P_\lambda \equiv (2\pi i)^{-1} \int_{|\lambda - \mu| = \varepsilon} (\mu - A)^{-1} d\mu$$

is finite dimensional. We define $\sigma_{\mathrm{essential}}(A) \equiv \sigma(A) \setminus \sigma_{\mathrm{discrete}}(A)$.

By methods similar to those used in proving Lemma 2.6, it is easy to see that for $\mu \in \varrho(H'(z)) \cap \varrho(H_0'(z))$,

$$(H_0'(z) - \mu)^{-1} - (H'(z) - \mu)^{-1}$$

is compact. Thus by the theorems of Section XIII.3 of [12] (see references given there for original contributions) $H_0'(z)$ and $H'(z)$ have the same essential spectrum. Consider now the operator family $\{H(z) : |\mathrm{Im}\, z| < \frac{1}{2}\pi\}$, given by

$$H(z) = e^z H'(z). \tag{2.28}$$

Note that for z real, $H(z) = U_z H U_z^{-1}$ and that Equation (2.28) thus provides an analytic continuation of $H(\theta)$ from real θ to the strip indicated. The essential spectrum is a curve in the complex λ plane given by

$$\{\lambda = (te^{2z} + m^2)^{1/2} : \mathrm{Re}\, \lambda > 0, t \geqq 0\}. \tag{2.29}$$

It is shown in [7] that under these circumstances, the eigenvalues of $H(z)$ all lie between the real axis and the curve (2.29) and are independent of z. Suppose $\mathrm{Im}\, z > 0$. Let S be the set of eigenvalues of $H(z)$ on the real axis. S is a discrete set. (Presumably $S = \emptyset$ [7], but we do not need this result.) Let $\mathscr{D} = \{\psi : U_z \psi \text{ is entire}\}$. Suppose $\varepsilon > 0$, $\lambda \in \mathbb{R} \setminus S$. Then for $\psi \in \mathscr{D}$

$$\begin{aligned}(\psi, (H - \lambda + i\varepsilon)^{-1} \psi) &= (U_\theta \psi, (H(\theta) - \lambda + i\varepsilon)^{-1} U_\theta \psi) \\ &= (U_{\bar{z}} \psi, (H(z) - \lambda + i\varepsilon)^{-1} U_z \psi)\end{aligned}$$

and this is uniformly bounded as $\varepsilon \downarrow 0$ uniformly in λ for λ in compacts of $\mathbb{R} \setminus S$. This proves $\sigma_{\mathrm{sing.cont}}(H) = \emptyset$ (by standard arguments), and thus the proof of Theorem 2.3 is complete.

Acknowledgements. It is a pleasure to thank H. Brascamp and B. Simon for useful conversations.

Appendix. The Integral $\int dxd\omega G(x;\omega,e)$

We begin by noting the equality

$$y^{-\lambda}=(\Gamma(\lambda))^{-1}\int_0^\infty dte^{-ty}t^{\lambda-1}. \tag{A1}$$

Substituting $y=\cosh x-\boldsymbol{\omega}\cdot\boldsymbol{e}$ and abbreviating $\lambda=\frac{1}{2}(N-\alpha)$, $\beta=N(p^{-1}-\frac{1}{2})-\frac{1}{2}\alpha$, we can rewrite the integral in Equation (2.16) as

$$(2^{\frac{1}{2}\alpha}\Gamma(\tfrac{1}{2}\alpha)(2\pi)^{N/2})^{-1}\int_0^\infty dtt^{\lambda-1}\left\{\int_{-\infty}^{\infty} dxe^{\beta x}e^{-t\cosh x}\right\}\{\int d\omega e^{t\boldsymbol{\omega}\cdot\boldsymbol{e}}\}. \tag{A2}$$

The integrals in curly brackets can be recognized as Bessel functions [13]:

$$\int_{-\infty}^{\infty} dxe^{\beta x}e^{-t\cosh x}=2K_\beta(t), \tag{A3}$$

$$\int d\omega e^{t\boldsymbol{\omega}\cdot\boldsymbol{e}}=(2\pi)^{N/2}t^{-\frac{1}{2}N+1}I_{\frac{1}{2}N-1}(t). \tag{A4}$$

Substituting in (A2) we find

$$\int dxd\omega G(x;\boldsymbol{\omega},\boldsymbol{e})=2(2^{\frac{1}{2}\alpha}\Gamma(\tfrac{1}{2}\alpha))^{-1}\int_0^\infty dtt^{-\alpha/2}K_\beta(t)I_{\frac{1}{2}N-1}(t). \tag{A5}$$

The integral above is expressible in terms of Γ functions [13]

$$\int_0^\infty dtt^{-\alpha/2}K_\beta(t)I_{\frac{1}{2}N-1}(t)=2^{-\frac{1}{2}\alpha-1}\frac{\Gamma(\frac{1}{2}(Np^{-1}-\alpha))\Gamma(\frac{1}{2}Nq^{-1})\Gamma(\alpha/2)}{\Gamma(\frac{1}{2}(Nq^{-1}+\alpha))\Gamma(\frac{1}{2}Np^{-1})}, \tag{A6}$$

where we have used $\beta=N(p^{-1}-\frac{1}{2})-\frac{1}{2}\alpha$ and set $q=p/(p-1)$. Finally

$$\int dxd\omega G(x;\boldsymbol{\omega},\boldsymbol{e})=2^{-\alpha}\frac{\Gamma(\frac{1}{2}(Np^{-1}-\alpha))\Gamma(\frac{1}{2}Nq^{-1})}{\Gamma(\frac{1}{2}(Nq^{-1}+\alpha))\Gamma(\frac{1}{2}Np^{-1})}. \tag{A7}$$

References

1. Bergmann, P.: Introduction to the theory of relativity, p. 118. Englewood Cliffs: Prentice-Hall, Inc. 1942
2. Bethe, H.: Intermediate quantum mechanics. New York: Benjamin 1964
3. Schmincke, U.-W.: Math. Z. **124**, 47—50 (1972)
4. Kim, Y.N.: Mesic atoms and nuclear structure. Amsterdam: North-Holland (1971)
5. Kato, T.: Perturbation theory for linear operators. Berlin-Heidelberg-New York: Springer 1966
6. Kalf, H., Schmincke, U.-W., Walter, J., Wüst, R.: Spectral theory and differential equations. Lecture Notes in Mathematics, Vol. 448. Berlin-Heidelberg-New York: Springer 1975
7. Aguilar, J., Combes, J.M.: Commun. math. Phys. **22**, 269—279 (1971)
8. Weder, R.: Ann. Inst. H. Poincaré **20**, 211—220 (1974)

9. Herbst, I., Sloan, A.: Perturbation of translation invariant positivity preserving semigroups in $L^2(\mathbb{R}^N)$. Preprint
10. Gel'fand, I. M., Shilov, G. E.: Generalized functions, Vol. I, p. 194. New York: Academic Press 1964
11. Weidmann, J.: Bull. AMS **73**, 452—456 (1967)
12. Reed, M., Simon, B.: Methods of modern mathematical physics, Vol. III. New York: Academic Press (to be published)
13. Gradshteyn, I. S., Ryzhik, I. M.: Table of integrals, series, and products, pp. 321, 358, 693, 1042. New York: Academic Press 1965

Communicated by W. Hunziker

Received October 25, 1976