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Scattering Theory and Quadratic Forms: On a Theorem of Schechter*

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Abstract. We show how to extend Cook's method to a class of pairs whose difference is only a quadratic form.

This note is connected with the existence question for generalized wave operators

$$\Omega^{\pm}(A,B) = \underset{t\to\mp\infty}{s\text{-lim}}\, e^{itA}e^{-itB}P_{ac}(B),$$

where A and B are self-adjoint operators on a Hilbert space, $\mathscr{H}$, and $P_{ac}(B)$ is the projection onto $\mathscr{H}_{ac}(B)$, the set of vectorswhose spectral measure relative to B is absolutely continuous with respect to Lebesgue measure. Nearly twenty years ago, Cook [2] proved a result whose abstract form says

Theorem 1. *Let A, B be self-adjoint operators and suppose that there is a subset $\mathscr{D}$ of $D(B)$ dense in $\mathscr{H}_{ac}(B)$ so that for any $\varphi\in\mathscr{D}$, there is a T_0 with $e^{itB}\varphi\in D(A)$ for $|t|>T_0$ and*

$$\int_{|t|\geqq T_0} \|(B-A)e^{itB}\|\,dt<\infty\,. \tag{1}$$

Then $\Omega^{\pm}(A,B)$ exist.

Cook's theorem is widely applicible (see e.g. [6]) and it has the advantage of having an extremely simple proof. As regards existence alone, its main disadvantage is that it does not accomodate operators defined as sum of forms, e.g. $B=-\Delta$, $A=-\Delta+V$. If $(1+|x|)^{+1+\varepsilon}V\in L^{3/2}(R^3)+L^{\infty}(R^3)$, one can define A as a form sum and expects $\Omega^{\pm}(A,B)$ to exist. Indeed, this has been proven [5], but only by developing the rather elaborate machinery of smoothness [4] and weighted L^2 estimates [1, 3] *and the proof doesn't work in the multiparticle case.* For many years, there have been no improvement in results based on an estimate like (1), but Schechter [7] has recently proven:

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Theorem 2. *Let A and B be self-adjoint operators on a Hilbert space $\mathscr{H}$. Let C, D, be operators from $\mathscr{H}$ to another Hilbert space $\mathscr{K}$ so that C is relatively A bounded and so that $A=B+C^*D$ in the sense that*

$$(A\varphi,\psi)=(\varphi,B\psi)+(C\varphi,D\psi)$$

for all $\varphi\in D(A)$, $\psi\in D(B)$. Suppose that there is a dense set $\mathscr{D}$ of $\mathscr{H}_{ac}(B)$ so that for any $\varphi\in\mathscr{D}$, there is T_0 with $e^{-itB}\varphi\in D(D)$ for $|t|>T_0$ and

$$\int_{|t|>T_0}\|De^{-iBt}\varphi\|\,dt<\infty\,.$$

Then $\Omega\pm(A,B)$ exist.

While Schechter's result is quite pleasing, it has the disadvantage of having a proof without the simplicity of Cook's method. It is our purpose here to show how to prove a result closely related to Schechter's using a small modification of Cook's original proof.

Theorem 3. *Let A and B be self-adjoint operators on a Hilbert space $\mathscr{H}$ with $D(|A|^{1/2})=D(|B|^{1/2})$. Suppose that C and D are operators from $\mathscr{H}$ to a Hilbert space $\mathscr{K}$ and that there is a dense set $\mathscr{D}$ in $\mathscr{H}_{ac}(B)\cap D(B)^{1/2}$ so that*

(i) *C is $|B|^{1/2}$ relatively bounded.*

(ii) *For each $\varphi\in\mathscr{D}$, there is a T_0 so that for $|t|>T_0$, $e^{-iBt}\varphi\in D(D)$ and for any $\psi\in D(|B|^{1/2})$:*

$$(\psi,Ae^{-iBt}\varphi)=(\psi,Be^{-iBt}\varphi)+(C\psi,De^{-iBt}\varphi)\,.$$

(iii) $$\int_{|t|>T_0}\|De^{-iBt}\varphi\|\,dt<\infty\,.$$

Then $\Omega^{\pm}(A,B)$ exist.

Before proving this theorem, we note that it is "essentially" a specialization of Schechter's theorem although it will accommodate most practical situations where Theorem 2 is applicable. We mention:

Example. If $V=V_1+V_2$ where $V_1\in L^{3/2}\cap L^{3/2-\varepsilon}(\mathbb{R}^3)$, and $W=V_2\,(1+|x|)^{1+\varepsilon}\in L^{3/2}+L^{\infty}$, then Theorem 3 shows that $\Omega^{\pm}\,(-\Delta+V-\Delta)$ exist and generally is applicable to the multiparticle situation with these potentials. For let $C_1,C_2,D_1,D_2:\mathscr{H}\to\mathscr{H}$ by $C_1=|V_1|^{1/2}$, $D_1=V_1/|V_1|^{1/2}$, $C_2=W/|W|^{1/2}$, $D_2|W|^{1/2}(1+|x|)^{-1-\varepsilon}$. Since V is $-\Delta$ form-bounded with relative form bound zero, $Q(-\Delta+V)=Q(-\Delta)$. Letting $\mathscr{D}=\mathscr{H}(\mathbb{R}^3)$ $\mathscr{K}=\mathscr{H}\oplus\mathscr{H}$ and $C=C_1\oplus C_2$; $D=D_1\oplus D_2$, (i), (ii), (iii) hold. To check (iii) we note that $\int_{|t|\ge 1}\|D_1e^{-iBt}\varphi\|\,dt<\infty$ by a standard argument ([6]) since D_1 is multiplication by a function in $L^{3-\delta}$. Moreover, since $|W|^{1/2}(-\Delta+1)^{-1}$ is bounded and $\varrho=(1+|x|)^{-1-\varepsilon}$ has ϱ, $\nabla\varrho$, $\Delta\varrho\in L^{3-\delta}$

$$\begin{aligned}\|D_2e^{it\Delta}\varphi\|&\le\text{const}\,\|(-\Delta+1)\varrho e^{it\Delta}\varphi\|\\&\le\text{const}(\|\varrho e^{it\Delta}(-\Delta+1)\varphi\|+\|(\Delta\varrho)e^{it\Delta}\varphi\|+2\|\nabla\varrho\cdot e^{it\Delta}\nabla\varphi\|\end{aligned}$$

is in L^1.

Proof of Theorem 3. Let $W(t)=e^{iAt}e^{-iBt}$. As usual [2], it suffices to prove that $\|(W(t) - W(s))\varphi\|^2 \to 0$ as $t, s\to\infty$ (or $-\infty$), or equivalently that $(W(t)\varphi, (W(t)-W(s)\varphi)\to 0$ as $s,t\to\infty$. Now norm $D(|B|^{1/2})$ with the norm $(\| |B|^{1/2}\varphi\| + \|\varphi\|)\equiv |||\varphi|||$. Clearly e^{-itB} is an isometry in $|||\varphi|||$. Since $|||\cdot|||$ is equivalent to the norm $\| |A|^{1/2}\varphi\| + \|\varphi\|$ (by the closed graph theorem) e^{-itA} is bounded in $|||\cdot|||$ *uniformly in t.*

$$(W(t)\varphi, (W(t)-W(s))\varphi) = \int_s^t (W(t)\varphi, e^{iuA}i(A-B)e^{-iuB}\varphi)$$

so by (ii) for $s, t > T_0$ and say $t > s$

$$|W(t)\varphi, (W(t)-W(s))\varphi| \leqq \left[\sup_{\substack{-\infty \leqq u \leqq \infty \\ -\infty \leqq v \leqq \infty}} \|Ce^{-iuA}W(v)\varphi\|\right] \int_s^t \|De^{-iuB}\varphi\|\, du .$$

Now, since the maps $e^{-iuA}W(v)$ are uniformly bounded from $D(|B|^{1/2})$ to itself $\varphi\in D(|B|^{1/2})$ and C is bounded from $D(|B|^{1/2})$ to $\mathscr{K}$, the sup is finite. By (iii), the integral goes to zero if $s, t\to\infty$.

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