Commun. math. Phys. 52, 267—294 (1977)

Communications in Mathematical Physics

Charged Particles in External Fields

II. The Quantized Dirac and Klein-Gordon Theories

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Abstract. The second-quantized Dirac and Klein-Gordon equations with external fields are solved. It is shown that the interpolating field is local and satisfies the Yang-Feldman equations. The Capri-Wightman approach and the Friedrichs-Segal approach are shown to lead to the same unitary S-operator. The evolution operator and S-operator are studied. A divergence-free perturbation expansion of the S-operator is derived and the connection with the Feynman-Dyson series is established.

1. Introduction

In a previous paper [1] (which we shall refer to as I) a number of perturbation-theoretic results were obtained which were applied to the classical Dirac and Klein-Gordon equations with external fields. The quantized theories will now be considered.

As detailed in [2] (referred to as B), one can treat the external field problem in a rigorous way by using the operators from the classical theory to generate transformations of the field operators on Fock space which amount to Bogoliubov transformations. If such a transformation is implementable the resulting Fock space operator is regarded as the physical operator corresponding to the unphysical operator on the classical Hilbert space. This approach, which goes back to Friedrichs [3], was further developed in [4–8, 2]. A closely related but more algebraic approach, inspired by the ideas of Segal [9], was used in [10–17].

A rather different strategy, based on the Yang-Feldman equations, was initiated by Capri [18] and further developed by Wightman [19]; it can be used for any generalized Dirac equation. Yet another treatment, using ideas from renormalization theory, was recently given by Bellissard [20, 21].

On a formal level the scattering of (especially spin-$\frac{1}{2}$) particles at external fields has been considered some time ago. Some references are [22–24]. Detailed accounts of the formal theory can be found in the books by Schweber [25] and Thirring [26].

One of the main results of this paper is that for (massive, relativistic, charged) spin-0 and spin-$\frac{1}{2}$ particles in external fields which are test functions on space-time the Friedrichs-Segal and Capri-Wightman approaches lead to the same S-operator, and that the divergence-free perturbation expansion of this unitary S-operator corresponds in a quite natural way to the formal Feynman-Dyson series. (Earlier results going in this direction can be found in [20].) In particular, the relative (cf. [22, 25]) S-matrix elements are the same in the rigorous and the formal theory, while the vacuum-to-vacuum transition amplitude resp. its modulus are equal in a formal sense to be specified below.

In Section 2 the Dirac theory is treated. Subsection A contains definitions of various field operators, the equations they satisfy and their interrelationship. It is proved that the interpolating field is local and satisfies the Yang-Feldman equations, and the equivalence of the Capri-Wightman and Friedrichs-Segal approaches is established. In Subsection B the evolution operator and S-operator are studied. Explicit expressions are derived and various continuity and analyticity properties are proved. The S-operator is shown to be Lorentz covariant and causal up to a phase factor. In Subsection C perturbation expansions are derived, the connection with the Feyman-Dyson series is established, and the analogue of Furry's theorem is proved. In Section 3 the Klein-Gordon theory is treated along the same lines as the Dirac theory. The paper ends with Section 4, which contains concluding remarks.

In Sections 2 and 3 it is assumed that the external fields are real-valued test functions on space time. As in I, several results could easily be extended to more general functions, but we shall not consider this.

2. The Quantized Dirac Theory

A. Field Operators

In this section and the next one we shall make extensive use of the notation and results of *I* and *B*. Thus [cf. B(2.9)], we have field operators on $\mathscr{F}_a(\mathscr{H})$, defined by

$$\Phi(v) = a(P_+ v) + b^*(\overline{P_- v}) \qquad \forall v \in \mathscr{H} \tag{2.1}$$

where $P_+(P_-)$ is the projection corresponding to the positive (negative) part of the Dirac Hamiltonian H_0 acting on the classical Hilbert space $\mathscr{H}$ (cf. I § 3A). Clearly,

$$[\Phi(u), \Phi^*(v)]_+ = (u, v) \qquad \forall u, v \in \mathscr{H} \tag{2.2}$$

where

$$\Phi^*(v) \equiv \Phi(v)^*. \tag{2.3}$$

Defining

$$\psi_t^0(f) = \Phi(\exp(iH_0 t) W^{-1} f) \qquad \forall f \in \check{\mathscr{H}} \tag{2.4}$$

one has the formal relation

$$\psi_t^0(f) = \int dx \bar{f}(x) \cdot \psi^0(t, x) \tag{2.5}$$

where $\psi^0(x)$ is the usual formal free Dirac field

$$\psi^0(x)=(2\pi)^{-3/2}\sum_i \int d\boldsymbol{p}(m/E_p)^{1/2}\,(a_i(\boldsymbol{p})u_i(\boldsymbol{p})\exp(-ipx)$$
$$+b_i^*(\boldsymbol{p})v_i(\boldsymbol{p})\exp(ipx)) \tag{2.6}$$

with our conventions for the u_i and v_i [cf. I (3.6–7)]. To see this, use I (3.4), (3.10) and set, e.g. [cf. B (4.6)],

$$\sum_i \int d\boldsymbol{p}\, a_i(\boldsymbol{p})\bar{g}_i(\boldsymbol{p})\equiv a(g) \qquad g\in\mathscr{H}_+ . \tag{2.7}$$

If $f\in D(\check{H}_0)$, then

$$d\psi_t^0(f)/dt=\psi_t^0(i\check{H}_0 f) \tag{2.8}$$

where the differentiation is in the norm topology. Note that if one smears the formal relation

$$\beta(-i\partial\!\!\!/+m)\psi^0(x)=0 \tag{2.9}$$

with $\bar{f}(\boldsymbol{x})$ and uses partial integration and (2.5) one obtains (2.8). One can therefore regard (2.8) as a rigorous analogue of (2.9). Similarly, the relation

$$[\psi_t^0(f),\psi_t^0(g)^*]_+=(f,g) \qquad \forall f,g\in\check{\mathscr{H}} \tag{2.10}$$

is the analogue of

$$[\psi_\alpha^0(t,\boldsymbol{x}),\psi_\beta^{0*}(t,\boldsymbol{x}')]_+=\delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{2.11}$$

Since $\exp(iH_0t)$ acts as multiplication by $\exp(iE_pt)$ on $\mathscr{H}_+$ and by $\exp(-iE_pt)$ on $\mathscr{H}_-$ it follows from (2.1) and (2.4) that

$$\psi_t^0(f)=\exp(iB_0t)\psi_0^0(f)\exp(-iB_0t) \tag{2.12}$$

where

$$B_0\equiv\Omega(qH_0). \tag{2.13}$$

B_0 is by definition the free Fock space Hamiltonian. It is the sum operator derived from the operator qH_0, which acts as multiplication by E_p both on the one-particle space $\mathscr{H}_+$ and on the one-antiparticle space $\mathscr{H}_-$. Thus, B_0 is a positive self-adjoint operator on $\mathscr{F}_a(\mathscr{H})$ with continuous spectrum in $[m,\infty)$ and eigenvalue 0 on Ω, the vacuum. On physical vectors (cf. B § 2) built up from vectors in $D(H_0)$ one has (cf. B § 4)

$$B_0=\int d\boldsymbol{p}E_p(a^*(\boldsymbol{p})a(\boldsymbol{p})+b^*(\boldsymbol{p})b(\boldsymbol{p})) \tag{2.14}$$

which is of course the usual expression (the spin indices are suppressed).

More generally, the transformation

$$\Phi(v)\to\Phi(U^*(a,\Lambda)v) \tag{2.15}$$

where $U(a,\Lambda)$ is the representation of $i\mathrm{SL}(2,C)$ in $\mathscr{H}$ [see I (3.44)] is implemented by unitary operators

$$\mathscr{U}(a,\Lambda)=\Gamma(\tilde{U}(a,\Lambda)) \tag{2.16}$$

in which

$$(\tilde{U}(a,\Lambda)v)^i_\varepsilon(\boldsymbol{p})\equiv\exp(ipa)\,((\Lambda^{-1}p)_0/p_0)^{1/2} \cdot\sum_j[(\tilde{p}/m)^{1/2}A(\underbrace{\Lambda^{-1}p/m})^{1/2}]_{ij}v^j_\varepsilon(\overrightarrow{\Lambda^{-1}p})\qquad\forall v\in\mathscr{H}\,. \tag{2.17}$$

Thus, $\mathscr{U}(a,\Lambda)$ acts in the same fashion on particle and antiparticle states.

One also verifies that the gauge transformation

$$\Phi(v)\to\Phi(\exp(i\alpha)v) \tag{2.18}$$

is implemented by the unitary operator $\exp(iQ\alpha)$, where Q is the charge operator

$$Q\equiv\Omega(q)\,. \tag{2.19}$$

Moreover [cf. I (3.11)],

$$\Phi^*(Cv)=\mathscr{C}\Phi(v)\mathscr{C}^*\,, \tag{2.20}$$

where $\mathscr{C}$ is the Fock space charge conjugation operator, given by

$$\mathscr{C}=\Gamma(C') \tag{2.21}$$

where

$$(C'v)^i_\varepsilon(\boldsymbol{p})\equiv v^i_{-\varepsilon}(\boldsymbol{p})\qquad\forall v\in\mathscr{H}\,. \tag{2.22}$$

It should be noted that $\mathscr{C}$ is unitary, whereas C is anti-unitary.

The formal perturbed Dirac field should satisfy

$$\beta(-i\partial\!\!\!/+m-B(x))\psi(x)=0\,, \tag{2.23}$$

$$[\psi_\alpha(t,\boldsymbol{x}),\psi^*_\beta(t,\boldsymbol{x}')]_+=\delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{x}')\,. \tag{2.24}$$

In (2.23) $V(x)\equiv\beta B(x)$ is by definition a function from R^4 to the Hermitean 4×4 matrices, the matrix elements of which belong to $S(R^4)$. Smearing with $\bar{f}(\boldsymbol{x})\in\check{\mathscr{H}}$ and partially integrating one obtains as rigorous analogues:

$$d\psi_t(f)/dt=\psi_t(i\check{H}(t)f)\qquad\forall f\in D(\check{H}_0)\,, \tag{2.25}$$

$$[\psi_t(f),\psi_t(g)^*]_+=(f,g)\qquad\forall f,g\in\check{\mathscr{H}}\,. \tag{2.26}$$

We assert that for any $a\in\tilde{R}$

$$\psi_{t,a}(f)\equiv\Phi(U^*(t,a)\exp(iH_0t)W^{-1}f) \tag{2.27}$$

is a solution to (2.25) and (2.26), where U is the interaction picture evolution operator corresponding to $V(x)$ (cf. I § 3B). Indeed, the verification of (2.26) is trivial and (2.25) follows from the relation

$$U^*(t,a)=U(a,t) \tag{2.28}$$

and I (2.23).

Evidently,

$$\lim_{t\to-\infty} \psi_{t,-\infty}(\exp(-i\check{H}_0t)f)=\Phi(W^{-1}f), \tag{2.29}$$

$$\lim_{t\to\infty} \psi_{t,-\infty}(\exp(-i\check{H}_0t)f)=\Phi(S^*W^{-1}f) \tag{2.30}$$

where the limits are norm limits and S is the classical S-operator on $\mathscr{H}$. Hence, we set

$$\psi_{\mathrm{in}}(f)\equiv\Phi(W^{-1}f), \tag{2.31}$$

$$\psi_{\mathrm{int},t}(f)\equiv\Phi(U^*(t,-\infty)W^{-1}f)=\psi_{t,-\infty}(\exp(-i\check{H}_0t)f), \tag{2.32}$$

$$\psi_{\mathrm{out}}(f)\equiv\Phi(S^*W^{-1}f), \tag{2.33}$$

where int stands for interpolating. In a formal sense, $\psi_{\mathrm{int},t}$ is the interaction picture analogue of the Heisenberg picture field $\psi_{t,-\infty}$.

We now define field operators, smeared with test functions $F\in S(R^4)^4$ by

$$\psi^{\mathrm{ex}}(F)=\int dt\,\psi_{\mathrm{ex}}(\exp(i\check{H}_0t)\check{\bar{F}}(t,\cdot)), \tag{2.34}$$

$$\psi^{\mathrm{int}}(F)=\int dt\,\psi_{t,-\infty}(\check{\bar{F}}(t,\cdot)), \tag{2.35}$$

where the integrals are Riemann integrals in $\mathscr{L}(\mathscr{F}_a)$ and ex = in, out. To conform to common usage these field operators depend linearly on F. Clearly, we can also write,

$$\psi^{\mathrm{ex}}(F)=\psi_{\mathrm{ex}}(\int dt\exp(i\check{H}_0t)\check{\bar{F}}(t,\cdot)), \tag{2.36}$$

$$\psi^{\mathrm{int}}(F)=\Phi(\int dt\,U^*(t,-\infty)\exp(iH_0t)W^{-1}\check{\bar{F}}(t,\cdot)), \tag{2.37}$$

where the integrals are strong Riemann integrals in $\check{\mathscr{H}}$ resp. $\mathscr{H}$. The relation

$$\psi^{\mathrm{in}}(F)=\int dxF(x)\psi^0(x) \tag{2.38}$$

now follows in the same way as (2.5). One also verifies that

$$\psi^{\mathrm{ex}}((i\partial\!\!\!/^T+m)F)=0, \tag{2.39}$$

$$\psi^{\mathrm{int}}((i\partial\!\!\!/^T+m-B^T)F)=0. \tag{2.40}$$

Thus, $\psi^{\mathrm{ex}}(x)$and $\psi^{\mathrm{int}}(x)$ satisfy the free resp. perturbed Dirac equation in the sense of operator-valued distributions. Moreover, we have

Theorem 2.1. *The interpolating field ψ^{int} is local and satisfies the Yang-Feldman equations:*

$$\psi^{\mathrm{int}}(T_RF)=\psi^{\mathrm{in}}(F), \tag{2.41}$$

$$\psi^{\mathrm{int}}(T_AF)=\psi^{\mathrm{out}}(F), \tag{2.42}$$

where

$$(T_I F)(x) \equiv F(x) - \int dy F(y) S_I(y-x) B(x) \qquad I = R, A\,. \tag{2.43}$$

Proof. If supp$F$ and supp$G$ are spacelike separated:

$$\begin{aligned}
[\psi^{\text{int}}(F), \psi^{\text{int}}(G)^*]_+ &= (\int dt U^*(t, -\infty) \exp(iH_0 t) W^{-1} \check{\bar{F}}(t, \cdot), \int dt' U^*(t', -\infty) \\
&\qquad \exp(iH_0 t') W^{-1} \check{\bar{G}}(t', \cdot)) \\
&= \int dt dt' (\check{\bar{F}}(t, \cdot), \check{U}^s(t, t') \check{\bar{G}}(t', \cdot)) \\
&= (-i[G_R] + i[G_A])(F, \gamma^0 \bar{G}) = 0
\end{aligned} \tag{2.44}$$

where we used (2.28) and I (2.21), (2.27), Theorem 3.2.

To prove (2.41) we note that

$$(\overline{T_R F})(x) = \bar{F}(x) - \gamma^0 B(x) \int dy S_A(x-y) \gamma^0 \bar{F}(y) \tag{2.45}$$

$$= \bar{F}(x) + iV(x) \left[\int_t^\infty dt' \exp(-i\check{H}_0(t-t')) \check{\bar{F}}(t', \cdot) \right](x) \tag{2.46}$$

where we used the well-known relation

$$S_R^*(y-x) = \gamma^0 S_A(x-y) \gamma^0 \tag{2.47}$$

and I Theorem 3.2 [for $V(x)=0$]. Thus,

$$\begin{aligned}
&\int dt \check{U}^*(t, -\infty) \exp(i\check{H}_0 t) (\check{\overline{T_R F}})(t, \cdot) = \int dt \check{U}^*(t, -\infty) \exp(i\check{H}_0 t) \check{\bar{F}}(t, \cdot) \\
&\quad + i \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^{\infty} dt \int_{-\infty}^{t} dt_1 \ldots \int_{-\infty}^{t_{n-1}} dt_n \check{O}(t_n) \ldots \check{O}(t_1) \check{O}(t) \\
&\quad \cdot \int_t^\infty dt' \exp(i\check{H}_0 t') \check{\bar{F}}(t', \cdot) \\
&= \int dt' \check{U}^*(t', -\infty) \exp(i\check{H}_0 t') \check{\bar{F}}(t', \cdot) + i \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^{\infty} dt' \int_{-\infty}^{t'} dt \int_{-\infty}^{t} dt_1 \ldots \\
&\quad \cdot \int_{-\infty}^{t_{n-1}} dt_n \check{O}(t_n) \ldots \check{O}(t_1) \check{O}(t) \exp(i\check{H}_0 t') \check{\bar{F}}(t', \cdot) \\
&= \int dt \exp(i\check{H}_0 t) \check{\bar{F}}(t, \cdot)\,.
\end{aligned} \tag{2.48}$$

Hence, (2.41) holds. Similarly,

$$\begin{aligned}
&\int dt \check{U}^*(t, -\infty) \exp(i\check{H}_0 t) (\check{\overline{T_A F}})(t, \cdot) = \int dt' \check{U}^*(t', -\infty) \exp(i\check{H}_0 t') \check{\bar{F}}(t', \cdot) \\
&\quad - i \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^{\infty} dt' \left(\int_{-\infty}^{\infty} - \int_{-\infty}^{t'} \right) dt \int_{-\infty}^{t} dt_1 \ldots \\
&\quad \cdot \int_{-\infty}^{t_{n-1}} dt_n \check{O}(t_n) \ldots \check{O}(t_1) \check{O}(t) \exp(i\check{H}_0 t') \check{\bar{F}}(t', \cdot) \\
&= \int dt \check{U}^*(\infty, -\infty) \exp(i\check{H}_0 t) \check{\bar{F}}(t, \cdot)\,.
\end{aligned} \tag{2.49}$$

Thus, (2.42) holds. ■

We note that, if $V(x) \neq 0$, ψ^{in} and ψ^{int} cannot be unitarily equivalent. (If they were, one would have

$$[\psi^{\text{int}}(F), \psi^{\text{int}}(G)^*]_+ = [\psi^{\text{in}}(F), \psi^{\text{in}}(G)^*]_+ \qquad \forall F, G \in S(R^4)^4 \tag{2.50}$$

so

$$U(t, t') = 1 \qquad \forall t, t' \in R^2 \tag{2.51}$$

from which it follows by I (2.22) that $V(x) = 0$.) However, this does not imply that ψ_{in} and $\psi_{\text{int},t}$ are unitarily inequivalent.

In contrast, if $\mathscr{S}$ is a unitary operator on $\mathscr{F}_a$ such that

$$\psi^{\text{out}}(F) = \mathscr{S}^* \psi^{\text{in}}(F) \mathscr{S} \qquad \forall F \in S(R^4)^4 \tag{2.52}$$

then also

$$\psi_{\text{out}}(f) = \mathscr{S}^* \psi_{\text{in}}(f) \mathscr{S} \qquad \forall f \in \check{\mathscr{H}} \tag{2.53}$$

and vice versa. This is an easy consequence of (2.36), (2.31), and (2.33). Evidently, existence of the Fock space S-operator is in turn equivalent to the implementability of the field operator transformation

$$\Phi(v) \to \Phi(S^* v). \tag{2.54}$$

Following Capri [18] and Wightman [19] one can define, provided that T_R, T_A are bijections of $S(R^4)^4$, an out field by

$$\psi^{\text{out}}_{cw}(F) = \psi^{\text{in}}(T_R{}^{-1} T_A F) \qquad \forall F \in S(R^4)^4. \tag{2.55}$$

It easily follows from I Theorem A.1 that this condition is satisfied (use (2.45) and its analogue for T_A and observe that $\bar{\Delta}_I = \Delta_I$, $I = R, A$). Using (2.41–42) we now conclude

$$\psi^{\text{out}}_{cw}(F) = \psi^{\text{out}}(F). \tag{2.56}$$

Thus,

Theorem 2.2. *The Capri-Wightman approach and the Friedrichs-Segal approach lead to the same S-operator.* ■

Using (2.16) and I (3.46) one verifies that

$$\mathscr{U}(a, \Lambda) \psi^{\text{in}}(F) \mathscr{U}^*(a, \Lambda) = \psi^{\text{in}}(F^{a,\Lambda}) \tag{2.57}$$

where

$$F^{a,\Lambda}(x) \equiv S(\Lambda^{-1})^T F(\Lambda^{-1}(x - a)). \tag{2.58}$$

Hence, ψ^{in} satisfies the Wightman axioms [27].

We finally observe that if one chooses a different representation of the γ-algebra, $\{\gamma^{\mu'}\}$, and proceeds in the same way as we have done (defining, e.g.,

$$W' = MW \tag{2.59}$$

where M is the unitary matrix which connects the representations), then the resulting field $\psi^{\mathrm{in}'}$ is not unitarily equivalent to ψ^{in}. Indeed, if it were, (2.50) should hold with $\psi^{\mathrm{int}} \to \psi^{\mathrm{in}'}$. However, $S(x-y)\gamma^0$ and $S'(x-y)\gamma^{0'}$ are different distributions. Even the Wightman axioms do not determine the free field up to unitary equivalence. Indeed, there exist different representations of the γ-algebra having the same $\{S(A)\}$, since the $\{S(A)\}$ have a non-trivial commutant.

B. The Evolution Operator and S-Operator in $\mathscr{F}_a(\mathscr{H})$

We now assume that only the timelike component of the vector field (i.e. the electric field) and/or the pseudovector field are non-zero. It then follows from I Theorem 3.3 that the hypothesis of I Theorem 2.8 is satisfied, and from I Corollary 3.4 that $U_\lambda(T_2, T_1)$ is implementable in $\mathscr{F}_a$ for any $(\lambda, T_2, T_1) \in R \times \tilde{R}^2$. Denoting the resulting three-parameter family of unitary operators by $\mathscr{U}_\lambda(T_2, T_1)$, it follows in particular that

$$\psi_{\mathrm{int},t}(f) = \mathscr{U}_1^*(t, -\infty)\psi_{\mathrm{in}}(f)\mathscr{U}_1(t, -\infty) \quad \forall f \in \check{\mathscr{H}} \quad \forall t \in \tilde{R}. \tag{2.60}$$

If (λ, T_2, T_1) is such that

$$(\Omega, \mathscr{U}_\lambda(T_2, T_1)\Omega) \neq 0 \tag{2.61}$$

then we normalize $\mathscr{U}$ by requiring

$$(\Omega, \mathscr{U}\Omega) > 0. \tag{2.62}$$

In the next theorem the set $E(T_2, T_1)$ is defined in I Theorem 2.8, and the operator Λ by I (2.57).

Theorem 2.3. (i) *For any $(\lambda, T_2, T_1) \in R \setminus E(T_2, T_1) \times \tilde{R}^2$ (2.61) holds true. For these values of the arguments one has for any $\phi \in D$:*

$$\mathscr{U}\phi = \det(1_{--} + \Lambda_{+-}{}^*\Lambda_{+-})^{-\frac{1}{2}} : \exp(\Lambda_{+-}a^*b^* + \Lambda_{++}a^*a + \Lambda_{--}bb^* + \Lambda_{-+}ba) : \phi. \tag{2.63}$$

(ii) *$\mathscr{U}_\lambda(T_2, T_1)$ is strongly continuous on $\tilde{R}^2$ for any $\lambda \in (-l, l)$, and on $R \setminus E(T_2, T_1)$ for any $(T_2, T_1) \in \tilde{R}^2$.*

(iii) *On $(-l, l) \times \tilde{R}^2$:*

$$\begin{aligned} &\mathscr{U}_\lambda(T, T) = 1 \\ &\mathscr{U}_\lambda(T_3, T_2)\mathscr{U}_\lambda(T_2, T_1) = \exp(i\chi(\lambda, T_3, T_2, T_1))\mathscr{U}_\lambda(T_3, T_1) \end{aligned} \tag{2.64}$$

where χ is a real-valued function.

(iv) *For any $(T_2, T_1) \in \tilde{R}^2$ and $\phi \in D$ the vector-valued function $\mathscr{U}_\lambda(T_2, T_1)\phi$ on $(-l, l)$ has a unique analytic continuation to $D_{l_{E(T_2, T_1)}}$ where*

$$l_E \equiv \mathrm{dist}(E, \{0\}) \tag{2.65}$$

and a, possibly two-valued, analytic continuation to $C \setminus E(T_2, T_1)$.

Proof. It follows from B that (2.61) is equivalent to existence of a uniquely determined bounded operator A on $\mathscr{H}$, satisfying

$$(U-1)-A-(U-1)P_-A=(U-1)-A-AP_-(U-1)=0. \tag{2.66}$$

Comparing (2.66) and I (2.60) we conclude that the first statement of (i) holds true. The second one then follows from (2.62) and B (4.18).

Since $\mathscr{U}_\lambda(T_2, T_1)$ is unitary its strong continuity will follow if we prove that the function $M(\lambda, T_2, T_1)$, defined by

$$M(\lambda, T_2, T_1)=\left(\prod_{i=1}^{n} a^*(f_i) \prod_{j=1}^{r} b^*(\bar{g}_j)\Omega, \mathscr{U}_\lambda(T_2, T_1) \prod_{i=1}^{n'} a^*(f'_i) \prod_{j=1}^{r'} b^*(\bar{g}'_j)\Omega\right)$$
$$n, r, n', r' \in N \qquad f_i, f'_i \in \mathscr{H}_+ \qquad g_j, g'_j \in \mathscr{H}_- \tag{2.67}$$

is continuous on $\tilde{R}^2$ for any $\lambda \in (-l, l)$ and on $R \setminus E(T_2, T_1)$ for any $(T_2, T_1) \in \tilde{R}^2$. Using (2.63) and the CAR one easily sees that $M(\lambda, T_2, T_1)$ is equal to a finite sum of terms, each of which is the product of $(+$ or $-)\det(\ldots)^{-\frac{1}{2}}$ and a finite number of terms of the form $(f_i, A_{+-}g_j)$, $(f_i, A_{++}f'_j)$, $(g'_i, A_{--}g_j)$, $(g'_i, A_{-+}f'_j)$, (f_i, f'_j) or (g'_i, g_j). Since det $(1+\cdot)$ is a continuous function on the trace class [28] we conclude from I Theorem 2.8 that $\det(\ldots)^{-\frac{1}{2}}$ has the required continuity properties. The same conclusion for the remaining terms follows from the norm continuity of $A_\lambda(T_2, T_1)$ in λ and (T_2, T_1). Thus, M has the required properties.

(2.64) is an easy consequence of I (2.21) and the irreducibility of the field operators.

To prove (iv) we first observe that

$$E(T_2, T_1)=\bar{E}(T_2, T_1) \qquad \forall (T_2, T_1) \in \tilde{R}^2. \tag{2.68}$$

Indeed, $U_{\lambda--}$ is singular if and only if $V_{\bar{\lambda}--}$ is, since

$$V_{\bar{\lambda}--}=U^*_{\lambda--} \qquad \forall \lambda \in C. \tag{2.69}$$

[To see this, use I (2.39) and the unitarity of U_λ for $\lambda \in R$.] We then note that on $(-l, l) \times \tilde{R}^2$ [cf. B (3.2)]

$$1_{--}+A_{+-}{}^*A_{+-}=U^*_{--}{}^{-1}U_{--}{}^{-1} \tag{2.70}$$

so

$$(1_{--}+A_{+-}{}^*A_{+-})(1_{--}-U_{-+}U_{-+}{}^*)=1_{--}. \tag{2.71}$$

It follows from I Theorem 2.8 that (2.70–71) can be continued to $C \setminus E$ and that $A_{\bar{\lambda}+-}{}^*A_{\lambda+-}$ and $U_{\lambda-+}U_{\bar{\lambda}-+}{}^*$ are $\|\cdot\|_1$-analytic functions in $C \setminus E$ resp. C. Hence (28),

$$g(\lambda) \equiv \det(1_{--}+A_{\bar{\lambda}+-}{}^*A_{\lambda+-})^{-1}=\det(1_{--}-U_{\lambda-+}U_{\bar{\lambda}-+}{}^*) \tag{2.72}$$

is an entire function which only vanishes on E. Denoting its positive square root on $(-l, l)$ by $v(\lambda)$ it follows from the monodromy theorem that $v(\lambda)$ has a unique analytic continuation to D_{l_E}. Clearly, $v(\lambda)$ can be analytically continued to an, in general two-valued, function on $C \setminus E$. We assume first that $v(\lambda)$ can be continued to an entire function.

We define for any $(T_2, T_1)\in\tilde{R}^2$, $\lambda\in C\setminus E(T_2, T_1)$ and ϕ of the form

$$\phi=\prod_{i=1}^{n} a^*(f_i)\prod_{j=1}^{r} b^*(\bar{g}_j)\Omega \qquad n, r\in N \quad f_i\in\mathscr{H}_+ \quad g_j\in\mathscr{H}_-, \tag{2.73}$$

$$\mathscr{U}_\lambda\phi\equiv v(\lambda)\prod_{i=1}^{n}(a^*(U_{\lambda++}f_i)+b(\overline{U_{\lambda-+}f_i}))\prod_{j=1}^{r}(b^*(\overline{U_{\lambda--}g_j})+a(U_{\lambda+-}g_j))$$
$$\cdot\exp(A_{\lambda+-}a^*b^*)\Omega. \tag{2.74}$$

The products in (2.73–74) are by definition in the natural order of the indices. It follows from B that the r.h.s. of (2.74) belongs to $\mathscr{F}_a$ and that it is equal to $\mathscr{U}_\lambda\phi$ if $\lambda\in(-l, l)$. Choosing $\lambda_0\in C\setminus E$ and an open ball O_{λ_0} with center λ_0 such that $\overline{O_{\lambda_0}}\cap E=\emptyset$, it follows from I (2.13) and B (3.47) that $\|\mathscr{U}_\lambda\phi\|$ is bounded on $\overline{O_{\lambda_0}}$. Hence, $\mathscr{U}_\lambda\phi$ is analytic in λ_0 if $(\psi, \mathscr{U}_\lambda\phi)$ is analytic in O_{λ_0} for any $\psi\in D$. However, this follows from the analyticity of $v(\lambda)$, U_λ and $A_{\lambda+-}$ in $C\setminus E$ by the same argument that we used to prove the continuity properties of $\mathscr{U}$.

It is clear from the above that (iv) holds as well if some points of E are branch points of $v(\lambda)$. In this case the continuation to $C\setminus E$ is two-valued. ■

Using results recently obtained by Bellissard [20, 21] and Palmer [17] we shall now study the Fock space S-operator, which corresponds to a function $V(x)$ as considered in Subsection A, multiplied by a real coupling constant λ. It is by definition the unitary operator $\mathscr{S}_\lambda$ satisfying

$$\Phi(S_\lambda^* v)=\mathscr{S}_\lambda^*\Phi(v)\mathscr{S}_\lambda \qquad \forall v\in\mathscr{H} \tag{2.75}$$

(if such an operator exists, cf. B). We denote by l_s the supremum of the numbers $\alpha>0$ such that $S_{\lambda\pm\mp}$ are $\|\cdot\|_2$-analytic in D_α. It follows from (20, Lemmas A 5.4, A 5.6), using the Neumann series argument of [21, Theorem II 1.1], that $l_s>0$. Since each term of their perturbation series is H.S. and the series converges in norm, $S_{\lambda\pm\mp}$ are compact for any $\lambda\in C$. Thus, arguing as in the proof of I Theorem 2.8, it follows that $E\equiv E(\infty, -\infty)$ is a discrete set outside D_l, and that $A_\lambda\equiv F_\lambda(\infty, -\infty)$ can be continued to $C\setminus E$. It moreover follows from [17] that $S_{\lambda\pm\mp}$ are H.S. for any $\lambda\in R$, so $\mathscr{S}_\lambda$ exists for any $\lambda\in R$ and $A_{\lambda\pm\mp}$ are H.S. for any $\lambda\in R\setminus E$. If $\lambda\in R$ is such that

$$(\Omega, \mathscr{S}_\lambda\Omega)\neq 0 \tag{2.76}$$

then we require

$$(\Omega, \mathscr{S}_\lambda\Omega)>0. \tag{2.77}$$

Setting

$$l_c\equiv\min(l_E, l_s) \tag{2.78}$$

we have

Theorem 2.4. (i) *For any $\lambda\in R\setminus E$ (2.76) holds true. For these λ one has for any $\phi\in D$:*

$$\mathscr{S}\phi=\det(1_{--}+A_{+-}{}^*A_{+-})^{-\frac{1}{2}}:\exp(A_{+-}a^*b^*+A_{++}a^*a+A_{--}bb^*$$
$$+A_{-+}ba):\phi. \tag{2.79}$$

(ii) *$\mathcal{S}_\lambda$ is strongly continuous on $R\setminus E$.*

(iii) *For any $\phi\in D$ the vector-valued function $\mathcal{S}_\lambda\phi$ on $(-l_c, l_c)$ has a unique analytic continuation to D_{l_c}; if $l_s > l_E$ it has a, possibly two-valued, analytic continuation to $D_{l_s}\setminus E$.*

(iv) *For any $\lambda\in R\setminus E$ $\mathcal{S}_\lambda$ is causal, up to a phase factor, and Lorentz covariant; for any $\lambda\in R\cap E$ $\mathcal{S}_\lambda$ is causal and Lorentz covariant, up to a phase factor.*

Proof. In view of the above it suffices to prove (iv). However, this statement is an easy consequence of I Theorem 3.1. ■

We remark that, by I Theorem 3.3, 3.1, $l_s=\infty$ for any $V(x)$ which, in some inertial frame, is equal to the sum of an electric and a "pseudo-electric" field. Thus, $l_c=l_E$ for these V. We further note that if the vacuum-to-vacuum transition amplitude $(\Omega, \mathcal{S}_\lambda\Omega)=0$, i.e. if $\lambda\in R\cap E$, then $\mathcal{S}_\lambda\phi$ is given by the r.h.s. of B (5.15), with $U\to S_\lambda$ (up to a phase factor). Finally, we mention that Schwinger [24] formally obtained the expression (2.79) for $\mathcal{S}_\lambda$ in the case of an electromagnetic field.

C. *The Connection with the Feynman-Dyson Series*

According to Theorem 2.4 $\mathcal{S}_\lambda\phi$ can, for any $\phi\in D$, be expanded in a power series, the convergence radius of which is greater than or equal to l_c. In this subsection we will derive explicit expressions for the expansions of

$$v(\lambda)\equiv(\Omega, \mathcal{S}_\lambda\Omega) \qquad \lambda\in D_{l_c} \tag{2.80}$$

and of

$$\mathcal{R}_\lambda\phi\equiv\mathcal{S}_\lambda\phi/v(\lambda) \qquad \phi\in D \quad \lambda\in D_{l_c} \tag{2.81}$$

and compare the result with the expressions which one obtains from the formal Feynman-Dyson (F.D.) series for the Fock space S-operator [25, 26].

We set for any $\lambda\in D_{l_c}$ (cf. B §4)

$$M_\lambda\equiv\Lambda_{\lambda+-}a^*b^*+\Lambda_{\lambda++}a^*a+\Lambda_{\lambda--}bb^*+\Lambda_{\lambda-+}ba. \tag{2.82}$$

From I §3B we have, explicitly exhibiting the spin indices,

$$\Lambda^{ii'}_{\lambda\varepsilon\varepsilon'}(\boldsymbol{p},\boldsymbol{q})=\sum_{n=1}^{\infty}\lambda^n\Lambda^{(n)ii'}_{\varepsilon\varepsilon'}(\boldsymbol{p},\boldsymbol{q}) \tag{2.83}$$

where

$$\begin{aligned}\Lambda^{(n)ii'}_{\varepsilon\varepsilon'}(\boldsymbol{p},\boldsymbol{q})=2\pi i\int dk_1\dots dk_{n-1}\left(\frac{m}{E_p}\right)^{\frac{1}{2}}\tilde{w}^i_\varepsilon(\boldsymbol{p})\tilde{B}(\varepsilon p-k_1)\tilde{S}_F(k_1)\tilde{B}(k_1-k_2)\\ \dots\tilde{S}_F(k_{n-1})\tilde{B}(k_{n-1}-\varepsilon' q)w^{i'}_{\varepsilon'}(\boldsymbol{q})\left(\frac{m}{E_q}\right)^{\frac{1}{2}}.\end{aligned} \tag{2.84}$$

Clearly,

$$:M_\lambda:\phi=s\cdot\lim_{N\to\infty}\sum_{n=1}^{N}\lambda^n:M^{(n)}:\phi \qquad \forall\phi\in D \tag{2.85}$$

where

$$M^{(n)} \equiv \Lambda^{(n)}{}_{+-}a^*b^* + \Lambda^{(n)}{}_{++}a^*a + \Lambda^{(n)}{}_{--}bb^* + \Lambda^{(n)}{}_{-+}ba. \tag{2.86}$$

In (2.86) we again suppressed the spin indices. It follows from (2.79) (cf. B § 4) that

$$\mathscr{R}_\lambda\phi = s\cdot\lim_{N\to\infty}\sum_{L=0}^{N}(L!)^{-1}{:}M_\lambda^L{:}\phi \quad \forall\phi\in D \tag{2.87}$$

where, explicitly,

$$\begin{aligned}{:}M_\lambda{}^L{:} = &\sum_{\substack{i,j,k,l=0\\ i+j+k+l=L}}^{L}(L!/i!j!k!l!)\Lambda_{\lambda+-}^{i}\Lambda_{\lambda++}^{j}(-\Lambda_{\lambda--}^{T})^k\Lambda_{\lambda-+}^{l}\\ &\cdot a^{*i}b^{*i}a^{*j}b^{**k}b^kb^la^la^j.\end{aligned} \tag{2.88}$$

Using arguments familiar by now we conclude that ${:}M_\lambda{}^L{:}\phi$ is analytic in D_{l_c} for any $\phi\in D$. In fact, one easily sees that

$${:}M_\lambda{}^L{:}\,\phi = s\cdot\lim_{N\to\infty}\sum_{n=L}^{N}\lambda^n{:}M^{(n,L)}{:}\,\phi \qquad L\geqq 1 \tag{2.89}$$

where

$$M^{(n,L)} \equiv \sum_{\substack{j_1,\ldots,j_L=1\\ j_1+\ldots+j_L=n}}^{n-L+1}\prod_{l=1}^{L}M^{(j_l)} \qquad n\geqq L\geqq 1. \tag{2.90}$$

Thus,

$$\mathscr{R}_\lambda\phi = \phi + \sum_{L=1}^{\infty}(L!)^{-1}\sum_{n=L}^{\infty}\lambda^n{:}M^{(n,L)}{:}\,\phi \qquad \forall\phi\in D. \tag{2.91}$$

We now have

Theorem 2.5. *For any $\phi\in D$ and $\lambda\in D_{l_c}$:*

$$\mathscr{R}_\lambda\phi = s\cdot\lim_{N\to\infty}\sum_{n=0}^{N}\lambda^n\mathscr{R}^{(n)}\phi \tag{2.92}$$

where

$$\mathscr{R}^{(0)} \equiv 1$$

$$\mathscr{R}^{(n)} \equiv \sum_{L=1}^{n}(L!)^{-1}{:}M^{(n,L)}{:} \qquad n\geqq 1. \tag{2.93}$$

Proof. Since $\mathscr{R}_\lambda\phi$ is analytic it suffices to show that

$$\sum_{n=0}^{\infty}\lambda^n(\psi,\mathscr{R}^{(n)}\phi) = (\psi,\mathscr{R}_\lambda\phi) \qquad \forall\psi\in D. \tag{2.94}$$

However, as $\phi,\psi\in D$, there exists a $K<\infty$ such that

$$(\psi,\mathscr{R}_\lambda\phi) = (\psi,\phi) + \sum_{L=1}^{K}(L!)^{-1}\sum_{n=L}^{\infty}\lambda^n(\psi,{:}M^{(n,L)}{:}\,\phi) \tag{2.95}$$

where we used (2.91). Hence, (2.94) follows. ■

With our conventions for the Dirac equation and the field operators [cf. (2.23) and (2.6)] the F.D. S-operator is given by

$$\mathscr{S}_{\lambda}^{\mathrm{F.D.}} = T(\exp(i \int dx \mathscr{L}_I(x))) \tag{2.96}$$

where

$$\mathscr{L}_I(x) \equiv \lambda : \tilde{\psi}^0(x) B(x) \psi^0(x) : . \tag{2.97}$$

Expanding the exponential and using Wick's theorem the fully contracted factors sum up to the well-known multiplicative divergent factor $(\Omega, \mathscr{S}_{\lambda}^{\mathrm{F.D.}}\Omega)$ by the usual arguments [25]; omitting this factor one obtains $\mathscr{R}_{\lambda}^{\mathrm{F.D.}}$. We denote the term of the coefficient of λ^n in its expansion ($n \geqq 1$) which has L uncontracted ψ and $\tilde{\psi}$ ($1 \leqq L \leqq n$) by $(L!)^{-1}\ M_{\mathrm{F.D.}}^{(n,L)}$. Using the relation [cf. (2.6) and I § 3A]

$$(\Omega, T(\psi_{\alpha}^0(x) \tilde{\psi}_{\beta}^0(y))\Omega) = -iS_F(x-y)_{\alpha\beta} \tag{2.98}$$

and a combinatorial argument it then follows that

$$\begin{aligned} M_{\mathrm{F.D.}}^{(n,L)} = i^L \sum_{\substack{j_1, \ldots, j_L = 1 \\ j_1 + \ldots + j_L = n}}^{n-L+1} \int dx_1 \ldots dx_n \\ : [\tilde{\psi}^0(x_1) B(x_1) S_F(x_1 - x_2) B(x_2) \ldots S_F(x_{j_1 - 1} - x_{j_1}) B(x_{j_1}) \psi^0(x_{j_1})] \\ \ldots [\tilde{\psi}^0(x_{j_1 + \ldots + j_{L-1} + 1}) B(x_{j_1 + \ldots + j_{L-1} + 1}) \ldots B(x_n) \psi^0(x_n)] : . \end{aligned} \tag{2.99}$$

However, from (2.6), (2.84) and (2.86) it easily follows that

$$i \int dx_1 \ldots dx_l \tilde{\psi}^0(x_1) B(x_1) S_F(x_1 - x_2) \ldots B(x_l) \psi^0(x_l) = M^{(l)} . \tag{2.100}$$

Thus, using (2.90),

$$M_{\mathrm{F.D.}}^{(n,L)} = : M^{(n,L)} : \tag{2.101}$$

so

$$\mathscr{R}_{\lambda}^{\mathrm{F.D.}} = \mathscr{R}_{\lambda} . \tag{2.102}$$

Of course, the r.h.s. of (2.99) is a priori not defined in a rigorous mathematical sense. What we have shown by the formal calculations leading from (2.96) to (2.102) is that expressions like (2.99) can be associated in a quite natural way with well-defined operators mapping the subspace of physical vectors D into D_f and that, with this association, the relative F.D. S-operator acts in the same way on physical vectors as $\mathscr{R}_{\lambda}$. Thus, the formal relative S-matrix elements, given by

$$\begin{aligned} &\mathscr{R}_{\lambda}^{\mathrm{F.D.}}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_r; \boldsymbol{p}'_1, \ldots, \boldsymbol{p}'_{n'}, \boldsymbol{q}'_1, \ldots, \boldsymbol{q}'_{r'}) \\ &= \left(\prod_{i=1}^{n} a^*(\boldsymbol{p}_i) \prod_{j=1}^{r} b^*(\boldsymbol{q}_j)\Omega, \mathscr{R}_{\lambda}^{\mathrm{F.D.}} \prod_{i=1}^{n'} a^*(\boldsymbol{p}'_i) \prod_{j=1}^{r'} b^*(\boldsymbol{q}'_j)\Omega \right) \end{aligned} \tag{2.103}$$

$(n, r, n', r' = 0, 1, \ldots)$, are equal to their rigorous counterparts, viz. the tempered distributions $\mathscr{R}_{\lambda}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_r; \boldsymbol{p}'_1, \ldots, \boldsymbol{p}'_{n'}, \boldsymbol{q}'_1, \ldots, \boldsymbol{q}'_{r'})$ which are defined by the requirement

$$\mathscr{R}_\lambda(f_1,\ldots,f_n,g_1,\ldots,g_r;f'_1,\ldots,f'_{n'},g'_1,\ldots,g'_{r'})$$
$$=\left(\prod_{i=1}^{n} a^*(\bar{f_i})\prod_{j=1}^{r} b^*(\bar{g_j})\Omega,\mathscr{R}_\lambda\prod_{i=1}^{n'} a^*(f'_i)\prod_{j=1}^{r'} b^*(g'_j)\Omega\right)$$
$$\forall f_i,g_j,f'_i,g'_j\in S(R^3)^2\,. \tag{2.104}$$

We further note that the $A^{(n)ii'}_{\varepsilon\varepsilon'}(\boldsymbol{p},\boldsymbol{q})$ (which together with the δ-function are the constituents of the terms of the perturbation series of any relative S-matrix element) are functions in $S(R^3)$ in $\boldsymbol{p}$ and $\boldsymbol{q}$ separately (cf. I § 3B) whereas the complete amplitudes $A_{\lambda\varepsilon\varepsilon'}^{ii'}(\boldsymbol{p},\boldsymbol{q})$ are tempered distributions which are not necessarily functions.

We shall now show that the coefficients of the expansion of $v(\lambda)$ easily follow from the $L^2(R^6)$ functions $A^{(n)ii'}_{+-}(\boldsymbol{p},\boldsymbol{q})$. Indeed, denoting by $[x]$ the greatest integer less than or equal to x, we have

Theorem 2.6. *For any* $\lambda\in D_{l_c}$*:*

$$v(\lambda)=\sum_{n=0}^{\infty} d_n\lambda^n\,, \tag{2.105}$$

where

$$d_0\equiv 1\qquad d_1\equiv 0$$
$$nd_n\equiv\sum_{k=2}^{n} ka_k d_{n-k}\qquad n\geqq 2 \tag{2.106}$$

and

$$a_k\equiv\sum_{n=1}^{[\frac{1}{2}k]}\frac{(-)^n}{2n}\sum_{\substack{i_1,\ldots,j_n=1\\ i_1+\ldots+j_n=k}}^{k-2n+1}\operatorname{Tr}A^{(i_1)}{}_{+-}{}^*A^{(j_1)}{}_{+-}\ldots A^{(i_n)}{}_{+-}{}^*A^{(j_n)}{}_{+-}\qquad k\geqq 2\,. \tag{2.107}$$

Proof. Since $A_{\lambda+-}$ is $\|\cdot\|_2$-analytic in D_{l_c} and $A_{0+-}=0$ we have, if $|\lambda|$ is small enough [28],

$$v(\lambda)=\det(1_{--}+A_{\bar\lambda+-}{}^*A_{\lambda+-})^{-\frac{1}{2}}$$
$$=\exp\left(\tfrac{1}{2}\sum_{n=1}^{\infty}(-)^n n^{-1}\sigma_n(\lambda)\right)\equiv\exp(f(\lambda)) \tag{2.108}$$

where

$$\sigma_n(\lambda)\equiv\operatorname{Tr}(A_{\bar\lambda+-}{}^*A_{\lambda+-})^n\,. \tag{2.109}$$

Clearly, $f(\lambda)$ is analytic in D_{l_c} and

$$f(\lambda)=\sum_{k=2}^{\infty} a_k\lambda^k\,. \tag{2.110}$$

Thus, differentiating the identity

$$v(\lambda)=\exp\left(\sum_{k=2}^{\infty} a_k\lambda^k\right) \tag{2.111}$$

at both sides and equating coefficients, (2.106) follows. ■

The F.D. analogue of a_k is given by

$$a_k^{\mathrm{F.D.}} = -k^{-1} \int dx_1 \dots dx_k \operatorname{Tr} B(x_1) S_F(x_1 - x_2) B(x_2) \dots S_F(x_k - x_1) \qquad k \geqq 2. \tag{2.112}$$

Formally Fourier transforming to time-momentum variables the integrand becomes a measurable function, since $\hat{S}_F(t, \boldsymbol{p})$ is. However, one easily sees that the integral is not absolutely convergent for $k=2$. For higher k it presumably does not converge either, but this is difficult to prove. Transforming to energy-momentum variables and replacing $\tilde{S}_F$ by S_F^δ with $\delta > 0$ [cf. I (3.19)] it might be convergent for $k \geqq 5$, but it is not clear whether the limit $\delta \downarrow 0$ then exists. If it does, one could probably obtain any other number by choosing a different sequence of functions approximating $\tilde{S}_F$ (in the sense of distributions). We also note that two renormalizations (in the sense of Hepp [29]) of the undefined product of the S_F in (2.112) differ in general by a finite renormalization which gives a non-zero contribution.

The same remarks apply to the real part of $a_k^{\mathrm{F.D.}}$, given by

$$\begin{aligned} \operatorname{Re} a_k^{\mathrm{F.D.}} = &-(2k)^{-1} \int dx_1 \dots dx_k \operatorname{Tr} [B(x_1) S_F(x_1 - x_2) \\ &\dots S_F(x_k - x_1) + B(x_1) S_{\bar{F}}(x_1 - x_2) \dots S_{\bar{F}}(x_k - x_1)], \end{aligned} \tag{2.113}$$

which of course is the "observable" part, since the imaginary part only gives rise to a phase factor. In particular, we have not been able to show that one obtains a_k if one evaluates $\operatorname{Re} a_k^{\mathrm{F.D.}}$ by the usual Feynman techniques.

However, it should be realized that in view of (2.102) one ought to require

$$|(\Omega, \mathscr{S}_\lambda^{\mathrm{F.D.}} \Omega)| = v(\lambda) \qquad \forall \lambda \in (-l_c, l_c) \tag{2.114}$$

if $\mathscr{S}_\lambda^{\mathrm{F.D.}}$ is to correspond to a unitary operator on Fock space. Since $v(\lambda)$ is analytic this requirement can only be satisfied if

$$\operatorname{Re} a_k^{\mathrm{F.D.}} = a_k \qquad \forall k \geqq 2. \tag{2.115}$$

As noted above, we could not obtain a satisfactory definition of the r.h.s. of (2.113) which implies (2.115) or, equivalently, (2.114). We shall now show that, nevertheless, (2.114) can be formally derived.

Substituting

$$S_F = S_R + S_- \tag{2.116}$$

in (2.112) and multiplying through, the term without S_- drops out since the integrand is zero a.e. The sum of the remaining terms can be written as

$$\begin{aligned} a_k^{\mathrm{F.D.}} = -i \sum_{n=1}^{k} n^{-1} \sum_{\substack{i_1, \dots, i_n = 1 \\ i_1 + \dots + i_n = k}}^{k-n+1} &\int d\boldsymbol{x}\, dx_1 \dots dx_k \operatorname{Tr} S_-(\boldsymbol{x} - x_1) B(x_1) S_R(x_1 - x_2) \\ &\dots B(x_{i_1}) S_-(x_{i_1} - x_{i_1+1}) B(x_{i_1+1}) S_R(x_{i_1+1} - x_{i_1+2}) \\ &\dots B(x_{k-i_n}) S_-(x_{k-i_n} - x_{k-i_n+1}) B(x_{k-i_n+1}) S_R(x_{k-i_n+1} - x_{k-i_n+2}) \\ &\dots B(x_k) S_-(x_k - \boldsymbol{x}) \gamma^0 . \end{aligned} \tag{2.117}$$

(To see this, first use [cf. I (3.28)]

$$i \int d\boldsymbol{x} S_-(x_k - \boldsymbol{x}) \gamma^0 S_-(\boldsymbol{x} - x_1) \doteq S_-(x_k - x_1). \tag{2.118}$$

Observe then that two terms with the same n but with $\{i_1, \ldots, i_n\}$ which differ by a cyclic permutation give the same contribution. A moment's reflection now shows that the number of such terms, multiplied by n^{-1}, equals the number of such terms in the expansion of (2.112), multiplied by k^{-1}. Thus, (2.117) follows.) We now note (cf. I § 3B) that the r.h.s. of (2.117) is formally equal to the coefficient of λ^k in the expansion of

$$\sum_{n=1}^{\infty} (-)^{n+1} n^{-1} \operatorname{Tr} \check{R}_{\lambda--}{}^{n}. \tag{2.119}$$

(This would be rigorously true if $R_{\lambda--}$ were $\|\cdot\|_1$-analytic in a neighbourhood of the origin, which it is not in the present case.) Thus,

$$(\Omega, \mathscr{S}_{\lambda}^{\mathrm{F.D.}} \Omega) = \det(1_{--} + \check{R}_{\lambda--}) \tag{2.120}$$

so

$$\begin{aligned} |(\Omega, \mathscr{S}_{\lambda}^{\mathrm{F.D.}} \Omega)| &= (\det(1_{--} + R_{\lambda--}) \det(1_{--} - A_{\lambda--}))^{\frac{1}{2}} \\ &= \det(U_{\lambda--} U_{\bar{\lambda}--}^{*})^{\frac{1}{2}} = v(\lambda) \end{aligned} \tag{2.121}$$

where we used I (2.55), some properties of infinite determinants [28] and (2.70).

Of course, this derivation is purely formal. Nevertheless, we have the following analogue of Furry's theorem, which closes this section.

Theorem 2.7. *Let $V(t)$ be such that there exists a conjugation C, satisfying*

$$CV(t) = V(t)C \qquad \forall t \in R, \tag{2.122}$$

$$CH_0 = -H_0 C. \tag{2.123}$$

Then

$$a_{2n+1} = 0 \qquad \forall n \in N^+. \tag{2.124}$$

Proof. It follows from (2.123) by the functional calculus that

$$CP_{\varepsilon} = P_{-\varepsilon} C. \tag{2.125}$$

Thus, using I (2.8),

$$C\Lambda_{\lambda} C = -\Lambda_{-\lambda}^{*} \qquad \forall \lambda \in (-l_c, l_c). \tag{2.126}$$

Hence [cf. (2.109)]

$$\begin{aligned} \sigma_n(\lambda) &= \operatorname{Tr}(C(\Lambda_{\lambda+-}{}^{*} \Lambda_{\lambda+-})^n C) = \operatorname{Tr}(\Lambda_{-\lambda+-} \Lambda_{-\lambda+-}{}^{*})^n = \sigma_n(-\lambda) \\ &\qquad \forall \lambda \in (-l_c, l_c). \end{aligned} \tag{2.127}$$

In the last step we used the fact that $\Lambda_{+-} \Lambda_{+-}{}^{*}$ has the same eigenvalues, including multiplicities, as $\Lambda_{+-}{}^{*} \Lambda_{+-}$. From (2.108) and (2.127) it clearly follows that $f(\lambda)$ is even, so (2.124) holds true. ■

The theorem holds in particular for electromagnetic fields, since the charge conjugation operator then satisfies (2.122–123).

3. The Quantized Klein-Gordon Theory

A. Field Operators

The basic field operators on $\mathscr{F}_s(\mathscr{H})$ are defined on $\tilde{D}$ by (2.1) (cf. B § 2). The analogue of (2.2) is

$$[\Phi(u), \Phi^*(v)]_- = (u, qv) \qquad \forall u, v \in \mathscr{H} \tag{3.1}$$

where $\Phi^*(v)$ is the restriction of $\Phi(v)^*$ to $\tilde{D}$; (3.1) holds on D_∞.

One could now, in analogy to the spin-$\frac{1}{2}$ case, introduce a formal two-component Klein-Gordon field

$$\psi^0(x) \equiv (2\pi)^{-3/2} \int d\boldsymbol{p}(a(\boldsymbol{p}) w_+(\boldsymbol{p}) \exp(-ipx) + b^*(\boldsymbol{p}) w_-(\boldsymbol{p}) \exp(ipx)) \tag{3.2}$$

and an adjoint field

$$\tilde{\psi}^0(x) \equiv (2\pi)^{-3/2} \int d\boldsymbol{p}(a^*(\boldsymbol{p}) w_+^*(\boldsymbol{p}) \exp(ipx) - b(\boldsymbol{p}) w_-^*(\boldsymbol{p}) \exp(-ipx)) \tag{3.3}$$

(cf. I § 4A). It can be seen that the interaction Lagrangean

$$\mathscr{L}_I(x) \equiv :\tilde{\psi}^0(x) B(x) \psi^0(x): \tag{3.4}$$

[cf. I (4.44)] leads to the same Feynman-Dyson S-matrix as the one obtained from the usual theory. However, since one of our main goals is to establish the relation of the rigorous theory with the customary formal theory, we shall not consider these fields any further.

The usual free fields are [25]:

$$\begin{cases} \phi^0(x) \equiv (2\pi)^{-3/2} \int d\boldsymbol{p}(2E_p)^{-\frac{1}{2}}(a(\boldsymbol{p}) \exp(-ipx) + b^*(\boldsymbol{p}) \exp(ipx)) \\ \phi^{0*}(x) \equiv \phi^0(x)^*, \end{cases} \tag{3.5}$$

$$\pi^0(x) \equiv \partial_t \phi^{0*}(x) \qquad \pi^{0*}(x) \equiv \partial_t \phi^0(x). \tag{3.6}$$

They satisfy the relations

$$(\Box + m^2)\phi^0(x) = 0, \tag{3.7}$$

$$\begin{cases} [\phi^0(t, \boldsymbol{x}), \pi^0(t, \boldsymbol{x}')]_- = i\delta(\boldsymbol{x} - \boldsymbol{x}') \\ [\phi^0(t, \boldsymbol{x}), \phi^{0*}(t, \boldsymbol{x}')]_- = [\phi^0(t, \boldsymbol{x}), \pi^{0*}(t, \boldsymbol{x}')]_- = [\pi^0(t, \boldsymbol{x}), \pi^{0*}(t, \boldsymbol{x}')]_- = 0 \end{cases} \tag{3.8}$$

and the adjoint relations (i.e. the relations obtained by taking formal adjoints).

We define for any $f \in W_{1/2}(R^3)$

$$\phi_t^0(f) = \Phi(\exp(iH_0 t)|H_0|^{-1} W^{-1} f), \tag{3.9}$$

$$\pi_t^0(f) = \Phi^*(i \exp(iH_0 t) q W^{-1} f) \tag{3.10}$$

where, at the r.h.s.,

$$f \equiv \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{3.11}$$

The operators $\phi_t^{0*}(f)$, $\pi_t^{0*}(f)$ are defined in the obvious way. Clearly,

$$\phi_t^{0(*)}(f) = (\int d\mathbf{x} \bar{f}(\mathbf{x})\phi^0(t,\mathbf{x}))^{(*)}, \tag{3.12}$$

$$\pi_t^{0(*)}(f) = (\int d\mathbf{x} f(\mathbf{x})\pi^0(t,\mathbf{x}))^{(*)}. \tag{3.13}$$

Moreover, for any $\psi \in D_f$,

$$\pi_t^0(f)\psi = \frac{d}{dt}\phi_t^{0*}(f)\psi \qquad \forall f \in W_{1/2}(R^3), \tag{3.14}$$

$$\frac{d^2}{dt^2}\phi_t^0(f)\psi = \phi_t^0((\Delta - m^2)f)\psi \qquad \forall f \in W_{5/2}(R^3). \tag{3.15}$$

The time differentiations are in the strong sense, and Δ acts in the sense of distributions. One also verifies, using (3.1), that on D_∞

$$\begin{aligned} &[\phi_t^0(f), \pi_t^0(g)]_- = i\int d\mathbf{x}\bar{f}(\mathbf{x})g(\mathbf{x}) \\ &[\phi_t^0(f), \phi_t^{0*}(g)]_- = [\phi_t^0(f), \pi_t^{0*}(g)]_- = [\pi_t^0(f), \pi_t^{0*}(g)]_- = 0 \end{aligned} \tag{3.16}$$

for any $f, g \in W_{1/2}(R^3)$. The relations (3.14–16) and their adjoints can be regarded as rigorous analogues of the relations (3.6–8) and their adjoints.

The analogue of (2.12) is

$$\psi_t^0(f) = \exp(iB_0 t)\psi_0^0(f)\exp(-iB_0 t) \qquad \psi = \pi, \phi \tag{3.17}$$

where B_0 is the free Fock space Hamiltonian, defined by (2.13). It clearly has the same properties as in the Dirac case [in particular (2.14) holds true in the same sense]. Similarly, the field operator transformation (2.15), where $U(a, \Lambda)$ is the representation of the Poincaré group in $\mathscr{H}$, given by I (4.55), is implemented by unitary operators $\mathscr{U}(a, \Lambda)$ given by (2.16), in which

$$(\tilde{U}(a, \Lambda)v)_\varepsilon(\mathbf{p}) \equiv \exp(ipa)((\Lambda^{-1}p)_0/p_0)^{1/2} v_\varepsilon(\overrightarrow{\Lambda^{-1}p}) \qquad \forall v \in \mathscr{H}. \tag{3.18}$$

Also, the gauge transformation (2.18) is implemented by $\exp(iQ\alpha)$, where Q is the charge operator (2.19). It follows from I (4.17) that (2.20–21) hold true as well, with

$$(C'v)_\varepsilon(\mathbf{p}) \equiv v_{-\varepsilon}(\mathbf{p}) \qquad \forall v \in \mathscr{H}. \tag{3.19}$$

Again, $\mathscr{C}$ is unitary while C is anti-unitary.

The perturbed Klein-Gordon fields should satisfy

$$(\Box + m^2 - K(x))\phi(x) = 0, \tag{3.20}$$

$$\pi(x) \equiv \partial_t \phi^*(x) + iA_0(x)\phi^*(x) \tag{3.21}$$

and (3.8) with the $_0$'s omitted, and the adjoint relations. [In (3.20)

$$K \equiv iA_\mu \partial^\mu + i\partial_\mu A^\mu + A_\mu A^\mu + A_4 \tag{3.22}$$

where A_l ($l = 0, \ldots, 4$) are real-valued functions in $S(R^4)$.] Smearing with $\bar{f}(\mathbf{x})$ resp. $f(\mathbf{x})$ and partially integrating one obtains

$$d^2\phi_t(f)/dt^2 + d\phi_t(2iA_0 f)/dt + \phi_t((-\Delta + m^2 - K)f) = 0 \tag{3.23}$$

$$\pi_t(f) = d\phi_t^*(f)/dt + \phi_t^*(iA_0 f) \tag{3.24}$$

and (3.16) with the $_0$'s omitted, and the adjoint relations. We assert that for any $a\in\tilde{R}$

$$\phi_{t,a}(f)\equiv\Phi(U^*(t,a)\exp(iH_0t)|H_0|^{-1}W^{-1}f), \tag{3.25}$$
$$\pi_{t,a}(f)\equiv\Phi^*(iU^*(t,a)\exp(iH_0t)qW^{-1}f) \tag{3.26}$$

and their adjoints are solutions to (3.23–24), commutation relations and adjoint relations in the sense specified before [cf. (3.14–16)]. [In (3.25–26) U is the interaction picture evolution operator corresponding to $\{A_l\}_{l=0}^4$ (cf. I § 4B).] Indeed, the commutation relations follow from (3.1) and the pseudo-unitarity of U while on $D(H_0)$, by I (2.23),

$$\begin{aligned} d/dt\,U^*(t,a)\exp(iH_0t) &= d/dt\,qU(a,t)q\exp(iH_0t)\\ &= iU^*(t,a)\exp(iH_0t)qH(t)q\,. \end{aligned} \tag{3.27}$$

It therefore remains to show

$$\begin{aligned} &(-\check{q}\check{H}(t)^2\check{q}-i\check{q}\check{V}(t)\check{q})|\check{H}_0|^{-1}\begin{pmatrix}f\\0\end{pmatrix}+2i|\check{H}_0|^{-1}\begin{pmatrix}\dot{A}_0f\\0\end{pmatrix}-2\check{q}\check{H}(t)\check{q}|\check{H}_0|^{-1}\begin{pmatrix}A_0f\\0\end{pmatrix}\\ &+|\check{H}_0|^{-1}\begin{pmatrix}(-\Delta+m^2-K)f\\0\end{pmatrix}=0, \end{aligned} \tag{3.28}$$

$$i\check{q}\begin{pmatrix}f\\0\end{pmatrix}=i\check{q}\check{H}(t)\check{q}|\check{H}_0|^{-1}\begin{pmatrix}f\\0\end{pmatrix}+i|\check{H}_0|^{-1}\begin{pmatrix}A_0f\\0\end{pmatrix}. \tag{3.29}$$

The verification of (3.28–29) is straightforward.

Clearly, on D_f,

$$s\cdot\lim_{t\to-\infty}\phi_{t,-\infty}(\exp(-i\check{H}_0t)f)=\Phi(|H_0|^{-1}W^{-1}f), \tag{3.30}$$

$$s\cdot\lim_{t\to\infty}\phi_{t,-\infty}(\exp(-i\check{H}_0t)f)=\Phi(S^*|H_0|^{-1}W^{-1}f). \tag{3.31}$$

Similar relations hold for $\pi_{t,-\infty}(f)$. Therefore we set

$$\begin{cases}\phi_{\text{in}}(f)\equiv\Phi(|H_0|^{-1}W^{-1}f)\\ \pi_{\text{in}}(f)\equiv\Phi^*(iqW^{-1}f),\end{cases} \tag{3.32}$$

$$\begin{cases}\phi_{\text{int},t}(f)\equiv\Phi(U^*(t,-\infty)|H_0|^{-1}W^{-1}f)\\ \pi_{\text{int},t}(f)\equiv\Phi^*(iU^*(t,-\infty)qW^{-1}f),\end{cases} \tag{3.33}$$

$$\begin{cases}\phi_{\text{out}}(f)\equiv\Phi(S^*|H_0|^{-1}W^{-1}f)\\ \pi_{\text{out}}(f)\equiv\Phi^*(iS^*qW^{-1}f).\end{cases} \tag{3.34}$$

The adjoints are defined in the obvious way. We note that, if f ranges over $W_{1/2}(R^3)$, the arguments of the Φ and Φ^* in e.g. (3.32) and its adjoint range over a dense subspace of $\mathscr{H}$, which ensures the irreducibility of the field operators. [Properly speaking, of e.g. the set of unitary operators $\exp(i(\overline{\phi_{\text{in}}(f)+\phi^*_{\text{in}}(f)}))$, $\exp(i(\overline{\pi_{\text{in}}(f)+\pi^*_{\text{in}}(f)}))$.]

We now define field operators smeared with test functions $F\in S(R^4)$ by, e.g. [cf. (3.32–34)],

$$\begin{cases}\phi^{\rm in}(F)=\Phi(\int dt\exp(iH_0t)|H_0|^{-1}W^{-1}\check{\bar{F}}(t,\cdot))\\ \pi^{\rm in}(F)=\Phi^*(i\int dt\exp(iH_0t)qW^{-1}\check{\bar{F}}(t,\cdot)),\end{cases}\tag{3.35}$$

$$\phi^{\rm int}(F)=\Phi(\int dtU^*(t,-\infty)\exp(iH_0t)|H_0|^{-1}W^{-1}\check{\bar{F}}(t,\cdot))\tag{3.36}$$

etc.; adjoints are defined by, e.g.,

$$\phi^{\rm in*}(F)=\phi^{\rm in}(\bar{F})^*\upharpoonright\tilde{D}.\tag{3.37}$$

Thus these field operators depend linearly on F. Clearly, (2.34–35) hold with $\psi=\pi,\phi$ if the integral is interpreted as a strong Riemann integral on D_f. One easily verifies the relations

$$\phi^{\rm in}(F)=\int dxF(x)\phi^0(x),\tag{3.38}$$

$$\phi^{\rm ex}((\square+m^2)F)=0,\tag{3.39}$$

$$\pi^{\rm ex}(F)=\phi^{\rm ex*}(-\partial_tF),\tag{3.40}$$

$$\phi^{\rm int}((\square+m^2-\bar{K})F)=0,\tag{3.41}$$

$$\pi^{\rm int}(F)=\phi^{\rm int*}((-\partial_t+iA_0)F).\tag{3.42}$$

[Use (3.27–29) to obtain (3.41–42).] Thus $\phi^{\rm ex}$, $\pi^{\rm ex}$, $\phi^{\rm int}$, $\pi^{\rm int}$ satisfy (3.7), (3.6), (3.20), (3.21) in the sense of operator-valued distributions. Notice that the $\phi^{\rm ex(*)}(F)$, $\phi^{\rm int(*)}(F)$ form an irreducible set of operators, in contrast to the sharp time fields $\phi^{(*)}_{\rm ex}(f)$, $\phi^{(*)}_{{\rm int},t}(f)$. Furthermore:

Theorem 3.1. *The interpolating field $\phi^{\rm int}$ is local and satisfies the Yang-Feldman equations:*

$$\phi^{\rm int}(T_RF)=\phi^{\rm in}(F),\tag{3.43}$$

$$\phi^{\rm int}(T_AF)=\phi^{\rm out}(F)\tag{3.44}$$

where

$$(T_IF)(x)\equiv F(x)-\int dyF(y)\Delta_I(y-x)\overleftarrow{\bar{K}}(x)\qquad I=R,A.\tag{3.45}$$

Proof. If supp$F$ and supp$G$ are spacelike separated one has on D_∞:

$$\begin{aligned}&[\phi^{\rm int}(F),\phi^{\rm int*}(G)]_-=(\int dtU^*(t,-\infty)\exp(iH_0t)|H_0|^{-1}W^{-1}\check{\bar{F}}(t,\cdot),\\ &q\int dt'U^*(t',-\infty)\exp(iH_0t)|H_0|^{-1}W^{-1}\check{G}(t',\cdot))\\ &=\int dtdt'\left(\check{L}\begin{pmatrix}\bar{F}(t,\cdot)\\ 0\end{pmatrix},\check{q}\check{U}^s(t,t')\begin{pmatrix}0\\ G(t',\cdot)\end{pmatrix}\right)\\ &=(-i\overset{0}{G}_R+i\overset{0}{G}_A)(F,G)=0\end{aligned}\tag{3.46}$$

where we used I (2.21), (2.27), Theorem 4.3.

To prove (3.43) we observe that

$$(\overline{T_R F})(x) = \bar{F}(x) - K(x) \int dy \Delta_A(x-y) \bar{F}(y). \tag{3.47}$$

Using (3.27) and I Theorem 4.2 [for $K(x)=0$] it then follows that

$$\begin{aligned}
&\int dt \check{U}^*(t,-\infty) \exp(i\check{H}_0 t) |\check{H}_0|^{-1} (\check{\overline{T_R F}})(t,\cdot) \\
&= \int dt \check{U}^*(t,-\infty) \exp(i\check{H}_0 t) |\check{H}_0|^{-1} \check{\bar{F}}(t,\cdot) \\
&\quad - \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^{\infty} dt \int_{-\infty}^{t} dt_1 \ldots \int_{-\infty}^{t_{n-1}} dt_n \check{q} \check{O}(t_n) \ldots \check{O}(t_1) \exp(i\check{H}_0 t) \check{H}_0^{-1} \\
&\quad \cdot \left[\begin{pmatrix} (A_\mu A^\mu(t,\cdot) + i\nabla \cdot A(t,\cdot) + iA(t,\cdot)\cdot \nabla + A_4(t,\cdot)) \int dx' \Delta_A(t-t',\cdot - \mathbf{x}') \bar{F}(x') \\ 0 \end{pmatrix} \right. \\
&\quad + \begin{pmatrix} iA_0(t,\cdot) \int dx' \dot{\Delta}_A(t-t',\cdot-\mathbf{x}') \bar{F}(x') \\ 0 \end{pmatrix} \\
&\quad \left. + \check{H}_0 \check{H}(t) \check{H}_0^{-1} \begin{pmatrix} A_0(t,\cdot) \int dx' \Delta_A(t-t',\cdot-\mathbf{x}') \bar{F}(x') \\ 0 \end{pmatrix} \right] \\
&= \ldots - \Sigma \ldots \check{O}(t_1) \exp(i\check{H}_0 t) \\
&\quad \cdot \begin{pmatrix} A_0(t,\cdot) \int \ldots \Delta_A \ldots \\ iA_0(t,\cdot) \int \ldots \dot{\Delta}_A \ldots + (-A^2(t,\cdot) + \ldots + A_4(t,\cdot)) \int \ldots \Delta_A \ldots \end{pmatrix} \\
&= \ldots + i\Sigma \ldots \check{O}(t_1) \exp(i\check{H}_0 t) \check{V}(t) \check{H}_0^{-1} \int_t^{\infty} dt' \exp(-i\check{H}_0(t-t')) \check{\bar{F}}(t',\cdot) \\
&= \int dt \exp(i\check{H}_0 t) |\check{H}_0|^{-1} \check{\bar{F}}(t,\cdot).
\end{aligned} \tag{3.48}$$

The proof of (3.44) is similar [cf. the proof of (2.42)]. ■

It is easily seen that ϕ^{in} and ϕ^{int} cannot be unitarily equivalent if $K(x) \neq 0$, and that ϕ^{out} and ϕ^{in} are unitarily equivalent if and only if ψ_{out} and ψ_{in} are ($\psi = \pi, \phi$), and if and only if the transformation (2.54) is unitarily implementable.

Since, by I Theorem A.1, $T_{R,A}$ are bijections of $S(R^4)$, one can define an out field by

$$\phi_{cw}^{\text{out}}(F) = \phi^{\text{in}}(T_R^{-1} T_A F) \qquad \forall F \in S(R^4). \tag{3.49}$$

From (3.43–44) it then follows that

$$\phi_{cw}^{\text{out}}(F) = \phi^{\text{out}}(F). \tag{3.50}$$

Hence,

Theorem 3.2. *The Capri-Wightman approach and the Friedrichs-Segal approach lead to the same S-operator.* ■

We finally note that on $\tilde{D}$

$$\mathscr{U}(a,\Lambda) \phi^{\text{in}}(F) \mathscr{U}^*(a,\Lambda) = \phi^{\text{in}}(F^{a,\Lambda}) \tag{3.51}$$

where

$$F^{a,\Lambda}(x) \equiv F(\Lambda^{-1}(x-a)). \tag{3.52}$$

Thus ϕ^{in} satisfies the Wightman axioms. Note that in view of (3.40) π^{in} is Lorentz non-covariant.

B. The Evolution Operator and S-Operator in $\mathscr{F}_s(\mathscr{H})$

We now assume that the magnetic field vanishes. It then follows from I Theorem 4.4 that the hypothesis of I Theorem 2.8 is satisfied, and from I Corollary 4.5 that $U_\lambda(T_2, T_1)$ is implementable in $\mathscr{F}_s(\mathscr{H})$ for any $(\lambda, T_2, T_1) \in R \times \tilde{R}^2$. Denoting the resulting 3-parameter family of unitary operators by $\mathscr{U}_\lambda(T_2, T_1)$ it follows in particular that on $\tilde{D}$, for any $f \in W_{1/2}(R^3)$ and $t \in \tilde{R}$,

$$\psi_{\text{int},t}(f) = \mathscr{U}_1^*(t, -\infty)\psi_{\text{in}}(f)\mathscr{U}_1(t, -\infty) \qquad \psi = \pi, \phi. \tag{3.53}$$

If (λ, T_2, T_1) is such that (2.61) holds then we require that (2.62) hold. The operator Λ in the next theorem is defined by I (2.49).

Theorem 3.3. (i) *For any* $(\lambda, T_2, T_1) \in R \times \tilde{R}^2$ (2.61) *holds true. For these values of the arguments one has for any* $\phi \in D$:

$$\mathscr{U}\phi = \det(1_{--} - \Lambda_{+-}{}^*\Lambda_{+-})^{\frac{1}{2}} : \exp(\Lambda_{+-}a^*b^* + \Lambda_{++}a^*a + \Lambda_{--}bb^* + \Lambda_{-+}ba) : \phi. \tag{3.54}$$

(ii) $\mathscr{U}_\lambda(T_2, T_1)$ *is strongly continuous on* $\tilde{R}^2$ *for any* $\lambda \in R$ *and on* R *for any* $(T_2, T_1) \in \tilde{R}^2$.

(iii) *On* $R \times \tilde{R}^2$ (2.64) *holds true.*

(iv) *For any* $(T_2, T_1) \in \tilde{R}^2$ *and* $\psi, \phi \in D$ *the function* $(\psi, \mathscr{U}_\lambda(T_2, T_1)\phi)$ *on* $(-\frac{1}{2}l, \frac{1}{2}l)$ *has an analytic continuation to* $D_{\frac{1}{2}l}$.

Proof. The statements (i)–(iii) follow from I and B by the arguments used in the proof of Theorem 2.3. To prove (iv) we first note that on $R \times \tilde{R}^2$

$$1_{--} - \Lambda_{+-}{}^*\Lambda_{+-} = U_{--}^{*\,-1} U_{--}{}^{-1} \tag{3.55}$$

so

$$(1_{--} - \Lambda_{+-}{}^*\Lambda_{+-})(1_{--} + U_{-+}U_{-+}{}^*) = 1_{--}. \tag{3.56}$$

Continuing (3.55–56) to D_l we conclude that

$$g(\lambda) \equiv \det(1_{--} - \Lambda_{\bar{\lambda}+-}{}^*\Lambda_{\lambda+-}) = \det(1_{--} + U_{\lambda-+}U_{\bar{\lambda}-+}{}^*)^{-1} \tag{3.57}$$

is a non-vanishing analytic function in D_l. Thus, its positive square root on $(-l, l)$ has a (unique) analytic continuation to D_l, which we denote by $v(\lambda)$.

We now observe, using I (2.8), that for any $\lambda \in D_{\frac{1}{2}l}$,

$$\|\Lambda_{\lambda+-}\| < 1. \tag{3.58}$$

Thus, defining for any $(\lambda, T_2, T_1) \in D_{\frac{1}{2}l} \times \tilde{R}^2$ and ϕ of the form (2.73)

$$\mathscr{U}_\lambda \phi \equiv v(\lambda) \prod_{i=1}^{n} (a^*(U_{\lambda_{++}} f_i) - b(\overline{U_{\lambda_{-+}} f_i})) \prod_{j=1}^{r} (b^*(\overline{U_{\lambda_{--}} g_j}) - a(U_{\lambda_{+-}} g_j))$$
$$\cdot \exp(A_{\lambda_{+-}} a^* b^*) \Omega \,, \tag{3.59}$$

it follows from B that the r.h.s. of (3.59) belongs to $\mathscr{F}_s$ and equals $\mathscr{U}_\lambda \phi$ if $\lambda \in (-\frac{1}{2}l, \frac{1}{2}l)$. The statement now easily follows. ■

Of course, $(\psi, \mathscr{U}_\lambda(T_2, T_1)\phi)$ can be analytically continued to a larger set which is determined both by $E(T_2, T_1)$ and by the requirement (3.58), but we shall not pursue this.

Using the properties of D_∞ mentioned in B and relations like B (2.6), (2.8), it can be seen that $\mathscr{U}_\lambda \phi$ is analytic in $D_{\frac{1}{2}l}$ if for any $k \in N^+$ and $\alpha < \frac{1}{2}l$

$$\sup_{|\lambda| = \alpha} \| N^k \exp(A_{\lambda_{+-}} a^* b^*) \Omega \| < \infty \,. \tag{3.60}$$

However, we do not know whether (3.60) holds true.

We further observe that $\mathscr{U}_\lambda \Omega$ is analytic in $D_{\frac{1}{2}l}$, but that it has no analytic continuation to C unless

$$U_{\lambda_{\pm\mp}} = 0 \qquad \forall \lambda \in C \,. \tag{3.61}$$

[Indeed, if it has, $A_{\lambda_{+-}}$ is $\| \cdot \|_2$-entire and satisfies (3.58) on C, so by Liouville's theorem,

$$A_{\lambda_{+-}} = 0 \qquad \forall \lambda \in C \,, \tag{3.62}$$

from which (3.61) easily follows.]

We shall now consider the Fock space S-operator, defined by (2.75) (which should hold on $\tilde{D}$), which corresponds to (real-valued) scalar and electromagnetic fields in $S(R^4)$. It follows from [20, 21, l.c.] that $l_s > 0$, where l_s is defined as in Subsection 2B. Thus, $\mathscr{S}_\lambda$ exists for $\lambda \in (-l_s, l_s)$. If $\lambda \in (-l_s, l_s)$ is such that (2.76) holds then we require (2.77). Denoting the supremum of the numbers $\alpha > 0$ such that

$$\int dt \| O(t, \lambda) \| < \tfrac{1}{2} \qquad \forall \lambda \in D_\alpha \tag{3.63}$$

(cf. I (4.36)) by l' and setting

$$l_c = \min(l', l_s) \tag{3.64}$$

we have

Theorem 3.4. (i) *For any* $\lambda \in (-l_s, l_s)$ (2.76) *holds true. For these* λ *one has for any* $\phi \in D$:

$$\mathscr{S}\phi = \det(1_{--} - A_{+-}{}^* A_{+-})^{\frac{1}{2}} : \exp(A_{+-} a^* b^* + A_{++} a^* a + A_{--} b b^*$$
$$+ A_{-+} b a) : \phi \,. \tag{3.65}$$

(ii) $\mathscr{S}_\lambda$ *is strongly continuous on* $(-l_s, l_s)$.

(iii) *For any* $\psi, \phi \in D$ *the function* $(\psi, \mathscr{S}_\lambda \phi)$ *on* $(-l_c, l_c)$ *has an analytic continuation to* D_{l_c}.

(iv) *For any* $\lambda \in (-l_s, l_s)$ $\mathscr{S}_\lambda$ *is causal, up to a phase factor, and Lorentz covariant.*

Proof. It suffices to prove (iv). However, this statement easily follows from I Theorem 4.1. ■

We remark that, by I Theorems 4.4, 4.1, $l_s=\infty$ if A_μ is such that $A=0$ in some inertial frame. Hence, $l_c=l'$ for these fields.

We further mention that Bellissard [20] arrived at the expression (3.65) for $\mathscr{S}_\lambda$ by using renormalization theory. He then showed that it can be defined on coherent states and proved several properties, like unitarity, causality up to a phase factor, Lorentz covariance and analyticity.

C. The Connection with the Feynman-Dyson Series

According to Theorem 3.4 $(\psi, \mathscr{S}_\lambda\phi)$ can, for any $\psi, \phi\in D$, be expanded in a power series in λ, the convergence radius of which is greater than or equal to l_c. We will now derive explicit expressions for the expansions of $v(\lambda)$ [defined by (2.80)] and of $(\psi, \mathscr{R}_\lambda\phi)$ [where $\mathscr{R}_\lambda$ is defined by (2.81)], and compare the result with the expressions obtained from the F.D. series [25].

We introduce a formal operator M_λ $(\lambda\in D_{l_c})$ by (2.82), in which

$$\Lambda_{\lambda\varepsilon\varepsilon'}(\boldsymbol{p},\boldsymbol{q})=\sum_{n=1}^{\infty}\lambda^n\Lambda^{(n)}{}_{\varepsilon\varepsilon'}(\boldsymbol{p},\boldsymbol{q}) \tag{3.66}$$

where

$$\begin{aligned}\Lambda^{(n)}{}_{\varepsilon\varepsilon'}(\boldsymbol{p},\boldsymbol{q})=2\pi i\int dk_1\ldots dk_{n-1}(2E_p)^{-\frac{1}{2}}[\tilde{V}(\varepsilon p,k_1)\tilde{\Delta}_F(k_1)\tilde{V}(k_1,k_2)\\ \ldots\tilde{\Delta}_F(k_{n-1})\tilde{V}(k_{n-1},\varepsilon' q)+\text{all }\tilde{A}_\mu\tilde{A}^\mu\text{-contractions}](2E_q)^{-\frac{1}{2}}\end{aligned} \tag{3.67}$$

$$\tilde{V}(k,k')\equiv\tilde{A}_\mu(k-k')(k^\mu+k'^\mu)+\tilde{A}_4(k-k') \tag{3.68}$$

[cf. I § 4B, I (2.49)]. Arguing as in Subsection 2C one infers that (2.85–91) hold true, with $-\Lambda^T_{\lambda--}\to\Lambda^T_{\lambda--}$ in (2.88). Hence,

Theorem 3.5. *For any $\psi, \phi\in D$ and $\lambda\in D_{l_c}$:*

$$(\psi,\mathscr{R}_\lambda\phi)=\lim_{N\to\infty}\sum_{n=0}^{N}\lambda^n(\psi,\mathscr{R}^{(n)}\phi) \tag{3.69}$$

where $\mathscr{R}^{(n)}$ is defined by (2.93). ■

The F.D. S-operator is given by (2.96), where (cf. [25])

$$\mathscr{L}_I(x)\equiv\lambda:\phi^{0*}(x)V(x)\phi^0(x):+\lambda^2:\phi^{0*}(x)(A_\mu A^\mu)(x)\phi^0(x):, \tag{3.70}$$

$$V\equiv-i\overleftarrow{\partial}_\mu A^\mu+iA_\mu\overrightarrow{\partial}^\mu+A_4 \tag{3.71}$$

and

$$(\Omega, T(\partial^k_{x_\mu}\phi^0(x)\partial^l_{y_\nu}\phi^{0*}(y))\Omega)\equiv\partial^k_{x_\mu}\partial^l_{y_\nu}(\Omega, T(\phi^0(x)\phi^{0*}(y))\Omega)\quad k,l=0,1. \tag{3.72}$$

Proceeding as in Subsection 2C, one obtains as the analogue of (2.99), using the relation

$$(\Omega, T(\phi^0(x)\phi^{0*}(y))\Omega) = -i\Delta_F(x-y) \tag{3.73}$$

and combinatorial arguments,

$$M^{(n,L)}_{\text{F.D.}} = i^L \sum_{\substack{j_1,\dots,j_L=1 \\ j_1+\dots+j_L=n}}^{n-L+1} \int dx_1 \dots dx_n [:(\phi^{0*}(x_1) V(x_1)\Delta_F(x_1-x_2)V(x_2) \\ \dots \Delta_F(x_{j_1-1}-x_{j_1})V(x_{j_1})\phi^0(x_{j_1})) \\ \dots (\phi^{0*}(x_{j_1+\dots+j_{L-1}+1})V(x_{j_1+\dots+j_{L-1}+1}) \\ \dots V(x_n)\phi^0(x_n)): + \text{all } A_\mu A^\mu\text{-contractions}]. \tag{3.74}$$

An $A_\mu A^\mu$-contraction of the term in brackets is by definition the same term where one or several different triplets $V(x_i)\Delta_F(x_i - x_{i+1})V(x_{i+1})$ are replaced by $A_\mu(x_i)\delta(x_i - x_{i+1})A^\mu(x_{i+1})$ $(i=1, \dots, n-1)$. Since

$$i\int dx_1 \dots dx_l [\phi^{0*}(x_1)V(x_1)\Delta_F(x_1-x_2)\dots V(x_l)\phi^0(x_l) \\ + \text{all } A_\mu A^\mu\text{-contractions}] = M^{(l)} \tag{3.75}$$

(2.101) follows. Thus, (2.102) holds. Regarding the meaning of this equality and regarding the relative S-matrix elements the same remarks can be made as in Subsection 2C.

The analogue of Theorem 2.6 is:

Theorem 3.6. *For any* $\lambda \in D_{l_c}$ $v(\lambda)$ *is given by* (2.105), *where* d_n *is defined by* (2.106), *and*

$$a_k \equiv \sum_{n=1}^{[\frac{1}{2}k]} -(2n)^{-1} \sum_{\substack{i_1,\dots,j_n=1 \\ i_1+\dots+j_n=k}}^{k-2n+1} \text{Tr}\Lambda^{(i_1)}{}_{+-} {}^*\Lambda^{(j_1)}{}_{+-} \dots \Lambda^{(i_n)}{}_{+-} {}^*\Lambda^{(j_n)}{}_{+-} \quad k \geqq 2\,. \tag{3.76}$$

Proof. This follows as in Subsection 2C from the relation (for $|\lambda|$ small enough)

$$v(\lambda) = det(1_{--} - \Lambda_{\bar{\lambda}+-} {}^*\Lambda_{\lambda+-})^{\frac{1}{2}} = \exp\left(-\tfrac{1}{2}\sum_{n=1}^{\infty} n^{-1}\sigma_n(\lambda)\right) \tag{3.77}$$

where σ_n is defined by (2.109). ■

As the analogue of (2.112) one obtains

$$a_2^{\text{F.D.}} = 2^{-1}\int dx_1 dx_2 V(x_1)\Delta_F(x_1-x_2)V(x_2)\Delta_F(x_2-x_1)\,, \tag{3.78}$$

$$a_k^{\text{F.D.}} = k^{-1}\int dx_1 \dots dx_k [V(x_1)\Delta_F(x_1-x_2)V(x_2)\dots\Delta_F(x_k-x_1) \\ + \text{all } A_\mu A^\mu\text{-contractions}] \quad k \geqq 3\,. \tag{3.79}$$

In time-momentum variables the integral is absolutely convergent if $k \geqq 4$ and $A_\mu = 0$. However, we do not know whether in this case its real part equals a_k. But for this circumstance, similar remarks on $(\text{Re})a_k^{\text{F.D.}}$ and $|(\Omega, \mathscr{S}_\lambda^{\text{F.D.}}\Omega)|$ can be made as in Subsection 2C. In order to see that (2.114) holds, transform (3.78–79) to energy-momentum variables and substitute

$$\tilde{\Delta}_F = \tilde{\Delta}_R + \tilde{\Delta}_-\,. \tag{3.80}$$

Using the relation

$$\tilde{\Delta}_-(p) = 2\pi i\theta(-p^0)\delta(p^2 - m^2) \tag{3.81}$$

[cf. I (3.17)] and I (4.47) it then follows as in Subsection 2C that

$$(\Omega, \mathscr{S}_\lambda^{\text{F.D.}}\Omega) = \det(1_{--} + R_{\lambda --})^{-1}. \tag{3.82}$$

(Again, $R_{\lambda --}$ is actually not $\|\cdot\|_1$-analytic in a neighbourhood of the origin.) Thus, in view of I (2.47) and (3.55), (2.114) holds true.

We close this section with the following Furry type theorem.

Theorem 3.7. *Let* $A_4 = 0$. *Then* (2.124) *holds true.*

Proof. If $A_4 = 0$ then the charge conjugation operator satisfies (2.126), with $-\Lambda \to \Lambda$ at the r.h.s. (cf. I § 4B). Thus, using (2.127), (2.124) follows. ■

4. Concluding Remarks

(1) It follows from I that time-independent electric and "pseudo-electric" fields (in the spin-$\frac{1}{2}$ case) resp. electric and scalar fields (in the spin-0 case), which are real-valued functions in $S(R^3)$, give rise to an evolution operator $U_\lambda(T_2, T_1)$ which is implementable in $\mathscr{F}_a$ resp. $\mathscr{F}_s$ for any $(\lambda, T_2, T_1) \in R^3$. It is easily seen that the resulting Fock space evolution operator (after normalization) has properties analogous to those mentioned in Theorems 2.3, 3.3 (mutatis mutandis: l now depends on $|T_2 - T_1|$). Similarly, the (pseudo-)unitary 1-parameter group $\exp(-iHt)$ [$H \equiv H(1)$, cf. I (2.104)] leads to a family of unitary operators $\mathscr{U}(t)$, forming a projective representation of R; after normalization $\mathscr{U}(t)$ is strongly continuous for $t \in R(|t| < \|V\|^{-1})$ in the spin-0 (spin-$\frac{1}{2}$) case. Since such a representation is equivalent to a vector representation [30] there exists a phase function $c(t)$ such that

$$c(t)\mathscr{U}(t) = \exp(-iBt) \qquad \forall t \in R \tag{4.1}$$

with B self-adjoint. B can be regarded as the perturbed Hamiltonian in Fock space.

Provided that the classical S-operator

$$S = s\cdot\lim_{t\to\infty} U(t,0)\; s\cdot\lim_{t'\to-\infty} U(0,t') \tag{4.2}$$

exists and is unitary (and, in the spin-0 case, pseudo-unitary as well), one has

$$S_{\pm\mp} = 0 \tag{4.3}$$

so the Fock space S-operator $\mathscr{S}$ then exists and

$$\mathscr{S} = \Gamma(\tilde{S}) \tag{4.4}$$

[cf. B (4.23–26)]. Thus, for time-independent external fields, perturbation theory for $\mathscr{S}$ amounts to investigating the Born series connected with S. It can be seen that the Feynman-Dyson series formally leads to the same result if vacuum diagrams are omitted.

(2) There exists a remarkable symmetry between the operators R and F (cf. I § 2): If for some $(\lambda, T_2, T_1) \in (-l, l) \times \tilde{R}^2$ $U_\lambda(T_2, T_1)$ is implementable in Fock space the operator $U'_\lambda(T_2, T_1)$ defined by I (2.50) resp. I (2.58) is implementable in the "wrong statistics Fock space" in virtue of I Theorems 2.10–11 and B (and vice versa). The resulting unitary operator $\mathcal{U}'$ is given by the r.h.s. of (2.63) (spin-0) resp. (3.54) (spin-$\frac{1}{2}$) with $\Lambda \to \Lambda'$, where Λ' is defined by I (2.51) resp. I (2.59). Clearly, one could prove analogues of Theorems 2.3–4 resp. Theorems 3.3–4 for $\mathcal{U}'$. Observe that $\mathcal{U}'$ does not satisfy (2.64) and that the wrong statistics "S-operator" $\mathcal{S}'$ is Lorentz covariant, but non-causal; the perturbation expansion of its matrix elements is determined by the functions at the r.h.s. of (3.67) resp. (2.84) with $\tilde{\Delta}_F$ resp. $\tilde{S}_F$ replaced by $\tilde{\Delta}_R$ resp. $\tilde{S}_R$.

(3) It would be worthwhile to use second-quantized operators like the momentum cutoff interaction Hamiltonian as starting point for an investigation of the problems considered in this paper. (In the time-independent case an interesting result in this context has been obtained by Palmer [16].) The methods and ideas from constructive quantum field theory which could then be used might in particular lead to a deeper understanding of the Feynman-Dyson series (especially of the divergent vacuum diagrams).

Acknowledgement. This investigation is part of the research program of the "Stichting voor Fundamenteel Onderzoek der Materie (F.O.M.)", which is financially supported by the "Organisatie voor Zuiver Wetenschappelijk Onderzoek (Z.W.O.)".

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Communicated by A. S. Wightman

Received May 3, 1976