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Dimensionally Renormalized Green's Functions for Theories with Massless Particles. II.

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Abstract. In the framework of dimensional renormalization the existence of Green's functions to all orders of perturbation theory is proved for theories with massless particles, provided all terms in the interaction Lagrangian have infrared degree $\Omega \geqq 4$. If the vanishing of masses is enforced by some symmetry and this symmetry is respected by dimensional regularization, Schwinger's action principle holds for these Green's functions as in the massive case.

I. Introduction

In this work we continue the discussion of the dimensional renormalization. In two previous publications [1, 2] we have outlined the method, or rather a possible version of it, for theories with exclusively massive or massless particles. The purpose of the present paper is to relax these restrictions on the mass. The reason why we have anticipated the case of purely massless theories is that they can be treated completely within the framework of dimensional renormalization, that is to say, the subtraction of pole terms in $n-4$ (n = dimensional regularization parameter) is sufficient to obtain Feynman amplitudes which are well-defined distributions in their external momenta (which clearly is not to say that we can prove anything about the existence of the S-matrix). The reason is that the massless particles do not develop a mass or super-renormalizable couplings through their interaction or the UV-counterterms. This is in contrast to the generalized $BPHZ$-method [3] where mass-counterterms are employed even in theories with massless particles only.

Since it is not entirely self-evident how to define dimensionally regularized Feynman amplitudes in the presence of massless fields, let us give the definition we are going to use. This definition differs from other ones given in the literature [4].

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As extensively discussed in [1], we can construct a dimensionally regularized Feynman integral for each Feynman graph G

$$\mathcal{T}_{G,\varepsilon}(\underline{p}, n) = c_G \int_0^\infty d\underline{\alpha}\, I_{G,\varepsilon}(\underline{p}, \underline{u}, \underline{\alpha}, n)|_{\underline{u}=0} \tag{1}$$

where the regularized integrand $I_{G,\varepsilon}$ is element of an abstract algebra of Lorentz covariants. Interpreting the momenta $\underline{p}$ as 4-dimensional ones, this integral converges absolutely for $\varepsilon > 0$ and $\mathrm{Re}(n) \ll 4$, and defines a C^∞ function of $\underline{p}$ which is analytic in n and can be analytically continued to a meromorphic function. All poles correspond to UV-divergences; in particular those at $n=4$ reflect the lack of convergence of the original integral in 4 space-time dimensions. This function is polynomially bounded in $\underline{p}$ and can, therefore, be interpreted as distribution over $\mathcal{S}$. The limit $\varepsilon \to 0$ may or may not exist (in $\mathcal{S}'$), depending on whether there are IR-singularities or not. As will be shown subsequently, there are no such IR-singularities near $n=4$, provided the vertex functions satisfy certain normalization conditions in the tree approximation and these normalization conditions are enforced in every order of perturbation theory by suitable finite renormalizations.

Using dimensional renormalization it turns out that these normalization conditions are automatically satisfied as long as all particles are massles. Unfortunately the situation changes, once Lagrangians with both massive and massless particles are considered. The massless particles will then in general develop a self-mass, resp. super-renormalizable couplings via their interaction with the massive ones. If nothing is done to enforce the correct normalization of vertex functions containing massless fields, perturbation theory breaks down as can be easily seen from the example of the propagator $(p^2+i0)^{-1}\Sigma(p)(p^2+i0)^{-1}+\ldots$ which is ill-defined as a distribution if the self-mass $\Sigma(p=0)$ does not vanish. Consequently any Feynman integral containing this propagator may fail to converge for small momenta (of this propagator). Consider as an example the integral

$$\int \frac{d^4k}{[k^2+i\varepsilon]^2\,[(p-k)^2-m^2+i\varepsilon]} \tag{2}$$

corresponding to the graph in Figure 1 with $\Sigma(p)=$ const. For small ε it has an IR-singular behavior $\sim (p^2-m^2)^{-1} \ln\varepsilon$ and the limit $\varepsilon \to 0$ certainly does *not* exist. Similarly (the UV-finite part of) the self-energy integral

$$\int \frac{d^4k}{[k^2+i0]\,[(p-k)^2+i0]} \tag{3}$$

corresponding to the graph in Figure 2 reveals the consequence of a trilinear coupling for massless scalar fields. Its small momentum behavior is $\sim \ln p^2$ and there is no finite renormalization such that $\Sigma_R(p=0)$ vanishes. Insertion of this self-energy into graphs such as the one in Figure 1 leads again to an IR-singularity.

Even in cases where some symmetry (gauge invariance or spontaneously broken symmetry) prevents the formation of a self-mass for the massless field, one will encounter IR-singularities in the individual Feynman graphs, although

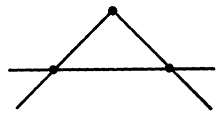

Fig. 1. A graph with $\Delta_i=2$ for one internal vertex, leading to an IR-singularity

Fig. 2. A self-energy graph with $\Delta_i=3$ for two internal vertices, leading to IR-singularities when inserted into graphs such as the one in Figure 1

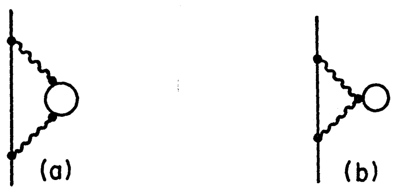

Fig. 3. Two self-energy graphs in scalar QED, which individually lead to an IR-singularity

Fig. 4. Two vacuum polarization graphs in scalar QED, which individually need an IR-subtraction

these cancel when all contributions to some Green's function are summed up. As an example Figure 3 shows two graphs contributing to the self energy in scalar QED. Each one contains a vacuum polarization subgraph (Fig. 4), whose amplitude does not have the correct small momentum behavior. Due to gauge invariance their sum does, however, vanish for small photon momenta. It is easy to see that the contribution of the graph shown in Figure 3b has indeed an IR-singularity $\sim \ln \varepsilon$.

If one wants to avoid these IR-divergencies one has to subtract Feynman amplitudes for graphs with IR-degree $\Omega > 0$ (for the definition of Ω see Sect. 2) at the origin in momentum space. Again these subtractions cancel order by order if the contributions of different graphs to the same Green's function are added, *if* there is some symmetry principle at work which forbids the corresponding counterterms in the Lagrangian *and* dimensional regularization respects this symmetry (examples are: QED, Yang-Mills field + fermions without axial coupling). This has the gratifying effect that there will be no radiative corrections to the equations of the action principle in such cases.

Since the existence proof for the renormalized Green's functions resp. Feynman integrals is admittedly somewhat complicated due to the interplay of both UV- and IR-subtractions we have divided it up in a series of smaller steps. These are

i) the definition of infrared degrees,

ii) the introduction of combined UV- and IR-subtractions and how they can be considered separately,

iii) the decomposition of the domain of integration into sectors corresponding to π-complete forests and

iv) the resolution of the singularity structure of the IR-subtracted integrals.

We make wide use of the notations and results of Refs. [1, 2].

2. Infrared Degrees

In addition to the renormalization that removes UV-divergencies of Feynman amplitudes we have to perform finite renormalizations that maintain the small momentum behavior (up to logarithms) of the tree approximation.

Similar to the superficial degree of divergence (UV degree) ω characterizing the large momentum behavior we shall need an infrared (IR) degree Ω for each 1PI graph describing its low momentum behavior. This IR-degree will decide about the counterterms we add to the Lagrangian in order to guarantee convergence of Feynman integrals at low momenta.

Let us consider some field ϕ with propagator $G_\phi(p) = \langle T\tilde{\phi}(p)\phi^*(0)\rangle_c^0$ behaving like $|p|^D$ for $|p| \to 0$. We assign an IR-degree $\Delta_\phi = (4+D)/2$ to ϕ in complete analogy to the UV-degree $\delta_\phi = (4+d)/2$ corresponding to the behavior $G_\phi(p) \sim |p|^d$ for $|p| \to \infty$.

As an example, a massive scalar field gets $\Delta_\phi = 2$, whereas a massless scalar field has $\Delta_\phi = 1$ (both have $\delta_\phi = 1$). This allows us to assign a "canonical" IR-degree $\Delta^c_{\mathcal{M}}$ to each monomial $\mathcal{M}$ composed of (quantized) fields and their derivatives by adding their respective degrees plus the number of derivatives.

We require $\Delta_\phi \geqq \frac{1}{2}$ for all fields, thus excluding IR-singularities in the free theory. For the existence of Green's functions it will turn out to be sufficient that all terms in the interaction Lagrangian of the form $\int \mathcal{M}(x)dx$ (i.e. without external fields) have a degree $\Delta^c_{\mathcal{M}} \geqq 4$. As an example this rules out a ϕ^3-coupling for a massless field, whereas it allows this coupling for two massless and one massive field (which appears in the σ-model).

One possibility to maintain an admissible low momentum behavior beyond the tree approximation would be to subtract from all vertex functions their Taylor series at $p=0$ up to degree $\Omega - 1$, where Ω is determined in the same way by the Δ's as the UV-degree ω is by the δ's. This procedure would, however, in general lead to counterterms of an unnecessary high degree $\Omega > 4$ and may convert a renormalizable theory into a non-renormalizable one via these counterterms.

In order to avoid this situation we shall allow an arbitrary assignment of IR-degrees $\Delta_{\mathcal{M}} \leqq \Delta^c_{\mathcal{M}}$ for each monomial, as long as $4 \leqq \Delta_{\mathcal{M}}$ for each integrated (non-source) term. We shall simply denote a pair $(\mathcal{M}, \Delta_{\mathcal{M}})$ by $\mathcal{M}_\Delta$.

If we consider a connected graph G contributing to $\left\langle T \prod_{i=1}^{n} (\mathcal{M}_i)_{\Delta_i}(x_i) \right\rangle_c^0$, we can assign an IR-degree Ω_G to it by $\Omega_G = 4 + \sum_{i=1}^{n} (\Delta_i - 4)$. We shall also need

IR-degrees Ω_H for all 1PI subgraphs $H \subset G$. Let H correspond to $\left\langle T \prod_{i=1}^{k} \mathcal{M}_i^{(\alpha^i)}(x_i) \right\rangle_c^0$ where $\mathcal{M}_i^{(\alpha^i)} = \delta^{|\alpha^i|} \mathcal{M} / \delta\phi^{\alpha^i}$ then we put

$$\Omega_H = 4 + \sum_{i=1}^{k} (\Delta_i^{(\alpha^i)} - 4)$$

with $\Delta_i^{(\alpha^i)} = \Delta_i - \sum_j \alpha_j^i \Delta_{\phi_j}$ the sum running over the set of fields removed from $\mathcal{M}_i$. This is not the only possible choice, but it is the most convenient one for our purposes and leads to the simple rules

$$\frac{\delta}{\delta\phi}(\mathcal{M}_\Delta) = \left(\frac{\delta\mathcal{M}}{\delta\phi}\right)_{\Delta - \Delta_\phi}$$

and

$$\Delta_i^{(\alpha^i)c} - \Delta_i^{(\alpha^i)} = \Delta_i^c - \Delta_i \, .$$

3. UV- and IR-Subtractions

Renormalization can be expressed equivalently by adding appropriate counterterms to the Lagrangian or by performing suitable subtractions (corresponding to 1PI subgraphs) on the amplitudes [5]. In particular dimensional renormalization consists in the subtraction of pole terms in $n-4$ arising through the singular behavior of $I_{G,\varepsilon}$ [Eq. (1.1)] when some subsets of α's tend to zero. In the present case this is not sufficient; we have to perform additional subtractions to guarantee the correct behavior of $I_{G,\varepsilon}$ when some α's tend to ∞. This is done in the form of Taylor series subtractions in the external momenta for 1PI subgraphs H with $\Omega_H > 0$.

To simplify the discussion we will disentangle these two kinds of subtractions. To be able to do so we have to adapt the machinery of additive renormalization to the present situation. The difference to the formulation given by Hepp [5] is that we use counterterms corresponding to subgraphs rather than to generalized vertices. The necessary modifications are so straightforward that we feel justified to proceed rather quickly without giving proofs.

Given some Feynman graph G, which we keep fixed for the following discussion, we define sets $\mathcal{F}_G$ resp. $\mathcal{F}'_G$ of all forests of non-overlapping non-trivial 1PI subgraphs contained resp. properly contained in G. Furthermore we denote by $\hat{\mathcal{F}}_G$ resp. $\hat{\mathcal{F}}'_G$ the subsets of forests in $\mathcal{F}_G$ resp. $\mathcal{F}'_G$ containing only pairwise disjoint subgraphs. Obviously $\mathcal{F}_G = \bigcup_{F \in \mathcal{F}'_G} \{F, F \cup \{G\}\}$ and $\hat{\mathcal{F}}_G = \hat{\mathcal{F}}'_G \cup \{\{G\}\}$ if G is non-trivial and 1PI; otherwise $\mathcal{F}_G = \mathcal{F}'_G$ and $\hat{\mathcal{F}}_G = \hat{\mathcal{F}}'_G$.

A pair (G, F) with $F \in \hat{\mathcal{F}}_G$ is called a *generalized Feynman graph*. A *vertex part* for a non-trivial 1PI subgraph $H \subset G$ is a distribution of the form

$$X_H = D\delta(x_1 - x_2) \dots \delta(x_{m-1} - x_m) \quad (m = |\mathcal{V}_H|) \tag{1}$$

where D is a covariant differential operator with constant coefficients. A *generalized Feynman amplitude* for (G, F) is an expression of the form

$$\mathscr{T}_{(G,F,X)} = \prod_{H\in F} X_H \cdot \mathscr{T}_{G/F} = \prod_{H\in F} X_H \cdot \prod_{V_i\in\mathscr{V}_G\setminus\bigcup_{H\in F}\mathscr{V}_H} X_i \cdot \prod_{\ell\in\mathscr{L}_{G/F}} \Delta_\ell \tag{2}$$

where G/F is the reduced graph with lines $\mathscr{L}_G \setminus \bigcup_{H\in F} \mathscr{L}_H$.

$$\mathscr{R}_G^X = \sum_{F\in\hat{\mathscr{F}}_G} \mathscr{T}_{(G,F,X)} \tag{3}$$

is called an *additively renormalized Feynman amplitude.* The correspondence $\mathscr{T}_G \to \mathscr{R}_G^X$ defines a *renormalization* if the Hepp axioms [5] are satisfied. As is well know $\mathscr{R}_G^X$ can also be obtained as a sum of (unrenormalized) Feynman amplitudes with respect to a modified Lagrangian.

A *subtraction operator* S is a linear operator determining vertex parts X by the recursive construction

$$X_H^S = \begin{cases} -S\bar{\mathscr{T}}_H^S & \text{if} \quad H \quad \text{is 1PI and nontrivial} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

with

$$\bar{\mathscr{T}}_H^S = \sum_{F\in\hat{\mathscr{F}}'_H} \left(\prod_{H'\in F} X_{H'}^S\right) \mathscr{T}_{G/F}$$

S determines an additively renormalized Feynman amplitude by

$$\mathscr{R}_G^S = \mathscr{R}_G^{X^S} = \sum_{F\in\hat{\mathscr{F}}_G} \left(\prod_{H\in F} X_H^S\right) \mathscr{T}_{G/F}\,. \tag{5}$$

As known from Zimmermann's work [6], where S is a Taylor series subtraction, the recursive construction of $\mathscr{R}_G^S$ may be resolved into the forest formula

$$\mathscr{R}_G^S = \sum_{F\in\mathscr{F}_G} \prod_{H\in F} (-S_H)\, \mathscr{T}_G \ \text{resp.}\ \bar{\mathscr{T}}_G^S = \sum_{F\in\mathscr{F}'_G} \prod_{H\in F} (-S_H)\, \mathscr{T}_G \tag{6}$$

assuming that one succeeds in defining the action S_H of the subtraction operator S on the Feynman integrand I_G.

In order to be able to describe a change of the renormalization prescription we have to introduce the concept of a *"finite" renormalization,* which is the analogue of a finite renormalization as defined by Hepp [5], although we do not require that the Y_H (see below) remain finite when the regularization is removed.

Starting from a set of vertex parts Y_H and a subtraction operator S we construct new vertex parts Y_H^S by

$$Y_H^S = \begin{cases} -S\,\bar{\mathscr{T}}_H^{S,Y} + Y_H & \text{for} \quad H \quad \text{nontrivial, 1PI} \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

with

$$\bar{\mathscr{T}}_H^{S,Y} = \sum_{F\in\hat{\mathscr{F}}'_H} \mathscr{T}_{(H,F,Y^S)}$$

resulting in a new renormalization

$$\mathscr{T}_G = \prod_{\mathscr{V}_G} X_i \prod_{\mathscr{L}_G} \Delta_\ell \to \mathscr{R}_G^{S,Y} = \sum_{F \in \hat{\mathscr{F}}_G} \mathscr{T}_{(G,F,Y^S)} \,. \tag{8}$$

There is a second way [7] to describe such a "finite" renormalization which makes direct contact with the counterterms put into the Lagrangian. For any generalized subgraph (H, F) of G we define vertex parts

$$X_{(H,F)} = \begin{cases} Y_H & \text{if } \; F = \{H\} \\ -S\bar{\mathscr{T}}^{S,Y}_{(H,F)} & \text{if } \; H/F \; \text{ is nontrivial}, \; \text{1PI} \\ 0 & \text{if } \; H/F \; \text{ is } \; \text{1PR} \end{cases} \tag{9}$$

with

$$\bar{\mathscr{T}}^{S,Y}_{(H,F)} = \sum_{F' \in \hat{\mathscr{F}}'_H}' \left(\prod_{H' \in F'} X_{(H', F_{H'})} \right) \mathscr{T}_{H/F'}$$

where the sum runs over all $F' \in \hat{\mathscr{F}}'_H$ which are coarser than F, i.e. $F = \bigoplus_{H' \in F'} F_{H'}$ with $F_{H'} \in \hat{\mathscr{F}}_{H'}$. The corresponding

$$\mathscr{R}^{S,Y}_{(G,F)} = \sum_{F' \in \hat{\mathscr{F}}_G}' \left(\prod_{H \in F'} X_{(H, F_H)} \right) \mathscr{T}_{G/F'} \tag{10}$$

are the renormalizations of the amplitudes $\left(\prod_{H \in F} Y_H \right) \mathscr{T}_{G/F}$ for the reduced graphs G/F with vertex parts Y_H, which are naturally obtained from a Lagrangian modified by counterterms computed from the Y's. The equivalence of both constructions is expressed by

Proposition 1.

$$\mathscr{R}_G^{S,Y} = \sum_{F \in \hat{\mathscr{F}}_G} \mathscr{R}^{S,Y}_{(G,F)} \tag{11}$$

the proof of which is completely analogous to the one in [7].

As already indicated we shall use the combined subtraction operator $S = C + T(1 - C)$ where C extracts the singular part of the Laurant expansion at $n = 4$ from $\bar{\mathscr{T}}^S_G$ (compare [1]) whereas T picks up the Taylor series in p at $p = 0$ up to order $\Omega_G - 1$. This subtraction procedure is based on the assumption—to be proven below—that $\bar{\mathscr{T}}^S_G$ is meromorphic at $n = 4$, the poles being polynomials in the external momenta and that the Taylor series exists up to the required order.

Obviously T and C commute and we may as well write $S = T + C(1 - T)$. It is very convenient to make use of both decompositions for S, interpreting $T(1 - C)$ resp. $C(1 - T)$ as "finite" renormalizations (which are actually UV- resp. IR-finite). Let us write $S = S_1 + S_2$ and put

$$Y_H = \begin{cases} -S_2 \bar{\mathscr{T}}^S_H & \text{if } \; H \; \text{ is nontrivial}, \; \text{1PI} \\ 0 & \text{otherwise}. \end{cases} \tag{12}$$

According to Proposition 1 we have

$$\mathscr{R}^S_G = \mathscr{R}_G^{S_1,Y} = \sum_{F \in \hat{\mathscr{F}}_G} \mathscr{R}^{S_1,Y}_{(G,F)} \tag{13}$$

which we can use in two ways:

i) We put $S_1 = C$ and $S_2 = T(1 - C)$.

As long as $\varepsilon \neq 0$ we know according to Ref. [1] that all $\mathscr{R}^{S_1,Y}_{(G,F)}$ are analytic at $n=4$, since S_2 defines obviously vertex parts Y_H which are analytic at $n=4$. Therefore $\mathscr{R}^S_G$ is analytic at $n=4$.

ii) This time we put $S_1 = T$ and $S_2 = C(1-T)$.

Assigning IR-degrees to the vertex parts Y_H originating from subgraphs H by

$$\Delta^G_H = 4 + \sum_{V_i \in \mathscr{V}_H} (\Delta^G_i - 4) \tag{14}$$

it is easily verified that the IR-degree of any reduced graph G/H equals the IR-degree of G and that this assignment is admissible. In particular the operator $1-T$ contained in S_2 ensures that $(\Delta^G_H)^c \geqq \Delta^G_H$. If all vertices of H are internal ones of G then $\Delta^G_H \geqq 4$. Considering the Y_H's as arising from a modification of the Lagrangian by counterterms the latter ones are of an admissible IR-type (i.e. $\Delta_{\mathscr{M}} \geqq 4$). Provided the Taylor series subtraction $S_1 = T$ does what it is supposed to do—what we actually claim to prove in the following—each $\mathscr{R}^{S_1,Y}_{(G,F)}$ has a limit for $\varepsilon \to 0$ as a distribution in p which is meromorphic at $n=4$, entailing the same properties for $\mathscr{R}^S_G$. For this purpose we shall also prove that the $Y_H(p,n)$ exist for $\varepsilon > 0$ and have a limit for $\varepsilon \to 0$ as meromorphic functions in n. All poles at $n=4$ arise from UV-singularities already present for $\varepsilon > 0$.

Combining these results we conclude that $\mathscr{R}^S_{G,\varepsilon}$ has a limit $\varepsilon \to 0$ as a distribution in p which is analytic at $n=4$. It should, however, be noted that the existence of this limit is restricted to a small neighborhood of $n=4$.

Thus we are led to the conclusion that it is sufficient to show that $\lim_{\varepsilon \to 0} \mathscr{R}^T_{G}, (p,n)$ exists as a distribution, meromorphic at $n=4$, for any Lagrangian satisfying $\Delta \leqq \Delta^c$ for all monomials and $4 \leqq \Delta$ for all integrated monomials.

Due to the linearity of our subtraction operators T and C, $\mathscr{R}$ is multilinear in all vertex factors X_i and spin polynomials Z_ℓ. Decomposing them into monomials, we may define for each combination of such monomials effective UV- and IR-degrees. The most general propagator, we are going to consider, is of the form

$$\Delta_\ell(p) = \frac{\sum_j M^j_\ell(p)}{(-ip^2+0)^{s^0_\ell}(-ip^2+im^2_\ell+0)^{s^m_\ell}} \tag{15}$$

where $m^2_\ell > 0$, s^0_ℓ and s^m_ℓ are non-negative integers with $s_\ell = s^0_\ell + s^m_\ell > 0$ and $M^j_\ell(p)$ are monomials of degree r^j_ℓ.

For each monomial M^j_ℓ we define

$$\delta^{\mathrm{eff}}_\ell = \tfrac{1}{2}(4+r^j_\ell) - s_\ell \leqq \delta_\ell \quad \text{and} \quad \Delta^{\mathrm{eff}}_\ell = \tfrac{1}{2}(4+r^j_\ell) - s^0_\ell \geqq \Delta^c_\ell \geqq \Delta_\ell\,. \tag{16}$$

$\Delta^{\mathrm{eff}}_\ell = \delta^{\mathrm{eff}}_\ell + s^m_\ell$ and for each term contributing to the amplitude for a graph G, $\Omega^{\mathrm{eff}}_G = \omega^{\mathrm{eff}}_G + 2 \sum_{\mathscr{L}_G} s^m_\ell$. The actual UV-divergencies arise from ω^{eff}_G and not from ω_G, which is the maximal value ω^{eff}_G can possibly assume.

For the denominator of Equation (15) we use the integral representation

$$\frac{1}{(-ip^2+0)^{s^0}(-ip^2+im^2+0)^{s^m}} = \lim_{\varepsilon \to 0} \int d\mu(z) \int_0^\infty d\alpha\, \alpha^{s-1} e^{i\alpha(p^2-z+i\varepsilon)} \tag{17}$$

where $d\mu(z)/dz$ is either $\delta(z)/\Gamma(s^0)$, $\delta(z-m^2)/\Gamma(s^m)$ or $\theta(z)\theta(m^2-z)(z/m^2)^{s^m-1}(1-z/m^2)^{s^0-1}/\Gamma(s^0)\Gamma(s^m)$ depending on whether s^0, s^m or both are non-zero.

Remark. For propagators with several massive poles we can use the same integral representation with a different $d\mu(z)$.

In either case $\int d\mu(z)z^{-\varrho s^m}$ converges for $0<\varrho<1$. Keeping this in mind we can forget the integral over z and write again m^2 instead of z.

We claim that it is sufficient to prove the existence of $\lim\limits_{\varepsilon\to 0}\mathscr{R}^T_{G,\varepsilon}$, employing the maximal possible Taylor subtraction for each combination of monomials in $\mathscr{T}_{G,\varepsilon}$. Denoting by T resp. T^{eff} the Taylor operators corresponding to degrees Δ resp. Δ^{eff}, we may rewrite our subtraction operator S as $S=T^{\text{eff}}+[(T-T^{\text{eff}})+ + C(1-T)]$ and interpret the vertex parts Y_H arising from $[(T-T^{\text{eff}})+C(1-T)]$ as IR-finite renormalizations. Again these vertex parts allow us to assign IR-degrees Δ^G_H to Y_H (when inserted into G/H). The proof of the existence of Y_H originating from T^{eff} implies automatically the existence of Y_H originating from T.

These oversubtractions T^{eff} will introduce artificial UV-divergencies besides those already introduced by the Taylor subtractions T; yet they drop out again if we add all contributions to $\mathscr{R}^S_{G,\varepsilon}$. Nevertheless it is in general true that the necessary IR-subtractions for subgraphs $H\subset G$ will increase the UV-degree ω_G.

Anticipating that we know how T^{eff}_H acts on the Feynman integrand $I_{G,\varepsilon}$ we may resolve the recursion and obtain the IR-subtracted amplitudes originating from subtractions recursively defined by

$$Y_{G,\varepsilon}=\begin{cases}-T^{\text{eff}}\bar{\mathscr{T}}^{\text{IR}}_{G,\varepsilon} & \text{if} \quad G \quad \text{is nontrivial}, \quad 1\text{PI}\\ 0 \quad \text{otherwise}\end{cases}$$

with

$$\bar{\mathscr{T}}^{\text{IR}}_{G,\varepsilon}=\int\limits_0^\infty d\underline{\alpha}\sum_{F\in\mathscr{F}'_G}\prod_{H\in F}(-T^{\text{eff}}_H)I_{G,\varepsilon}(\underline{p},\underline{\alpha},n) \tag{18}$$

resp.

$$\mathscr{R}^{\text{IR}}_{G,\varepsilon}=\int\limits_0^\infty d\underline{\alpha}\sum_{F\in\mathscr{F}_G}\prod_{H\in F}(-T^{\text{eff}}_H)I_{G,\varepsilon}(\underline{p},\underline{\alpha},n)\,.$$

In the following sections we will prove the existence of $\lim\limits_{\varepsilon\to 0}\mathscr{R}^{\text{IR}}_{G,\varepsilon}$ and $\lim\limits_{\varepsilon\to 0}Y_{H,\varepsilon}$. For simplicity we will drop from now on the superscript 'eff' from Taylor operators and degrees.

4. Complete Forests

In order to take advantage of the Taylor subtractions we have to rearrange the $\sum\limits_{F\in\mathscr{F}}\prod\limits_{H\in F}(-T_H)$ such that $(1-T_H)$ appears in the right places. Following the original treatment of Hepp [8] we consider sectors $\mathscr{D}_\pi$ in Feynman parameter space.

Let G be some connected graph with 1PI components G_i.

Definition 1. $(\mathscr{C}, \mathscr{B}, \sigma)$ is called a *labelled forest with basis* $\mathscr{B}$ for G iff $(\mathscr{C}, \sigma)$ is a labelled forest for G (compare [1]) and $\mathscr{B} \subset \mathscr{C}' = \mathscr{C} \setminus \{G_i\}$. For any $H \in \mathscr{C}$ we denote by $\mathscr{M}(H)$ the set of all maximal elements of $\mathscr{C}$ properly contained in H, $\bar{H} = H/\mathscr{M}(H)$ and for any $H \in \mathscr{C}'$ by H^+ the uniquely determined element of $\mathscr{C}$ with $H \in \mathscr{M}(H^+)$. For any line ℓ let H_ℓ be the element of $\mathscr{C}$ with $\ell \in \mathscr{L}_{\bar{H}_\ell}$.

Definition 2.

$$\begin{aligned}\mathscr{D}(\mathscr{C}, \mathscr{B}, \sigma) = \{\underline{\alpha} : \alpha_\ell \geqq 0 \ &\text{for all } \ell \in \mathscr{L}_G;\\ \alpha_\ell \leqq \alpha_{\sigma(H)} \ &\text{for } \ell \in \mathscr{L}_{\bar{H}}, H \in \mathscr{C};\\ \alpha_{\sigma(H)} \leqq \alpha_{\sigma(H^+)} \ &\text{for } H \in \mathscr{C}' \setminus \mathscr{B};\\ \alpha_{\sigma(H)} \geqq \alpha_{\sigma(H^+)} \ &\text{for } H \in \mathscr{B}\}.\end{aligned}$$

Remark. $\mathscr{D}(\mathscr{C}, \emptyset, \sigma) = \mathscr{D}(\mathscr{C}, \sigma)$ as defined in [1].

Definition 3. For any permutation π of the lines of G we put

$$\mathscr{D}_\pi = \{\underline{\alpha} : \alpha_{\pi(1)} \geqq \ldots \geqq \alpha_{\pi(L)} \geqq 0\}.$$

$(\mathscr{C}, \mathscr{B}, \sigma)$ is called *π-complete* iff $\mathscr{D}_\pi \subset \mathscr{D}(\mathscr{C}, \mathscr{B}, \sigma)$; let us denote by $\mathscr{F}_\pi$ the set of all π-complete forests (for G).

This concept corresponds to that of a complete forest with respect to a hyperplane in Zimmermann's work [6].

Lemma 1. *Let π be any permutation of lines and F any forest in $\mathscr{F}_G$, then there exists a unique π-complete forest $(\mathscr{C}, \mathscr{B}, \sigma)$ with $\mathscr{B} \subset F \subset \mathscr{C}$.*

Proof. Assume $\{G_i\} \subset F$ (otherwise we replace F by $F \cup \{G_i\}$) and put $F' = F \setminus \{G_i\}$. As shown in the proof of Lemma 3.f [1], there is a unique labelled forest $(\mathscr{C}, \sigma)_H$ for each $\bar{H}$ with $H \in F$ ($\bar{H}$ is to be taken with respect to F!) such that in $\mathscr{D}_\pi$ the α_ℓ, $\ell \in \mathscr{L}_{\bar{H}}$ are ordered as required by $\mathscr{D}(\mathscr{C}, \sigma)_H$. Lemma 3.c [1] tells us that there exists a unique labelled forest $(\mathscr{C}, \sigma)$ for G associated with these $(\mathscr{C}, \sigma)_H$ such that $F \subset \mathscr{C}$. We put $\mathscr{B} = \{H \in \mathscr{C}' : \pi^{-1}(\sigma(H)) < \pi^{-1}(\sigma(H^+))\}$. By construction $\mathscr{B} \subset F' \subset F \subset \mathscr{C}$ and $\mathscr{D}_\pi \subset \mathscr{D}(\mathscr{C}, \mathscr{B}, \sigma)$. Assume that there are two such forests $(\mathscr{C}, \mathscr{B}, \sigma) \neq (\mathscr{C}', \mathscr{B}', \sigma')$. Either $(\mathscr{C}, \sigma) = (\mathscr{C}', \sigma')$ but then $\mathscr{D}(\mathscr{C}, \mathscr{B}, \sigma) \cap \mathscr{D}(\mathscr{C}, \mathscr{B}', \sigma)$ is a set of Lebesgue measure zero for $\mathscr{B} \neq \mathscr{B}'$ and therefore cannot contain $\mathscr{D}_\pi$. Hence $(\mathscr{C}, \sigma) \neq (\mathscr{C}', \sigma')$ and therefore $(\mathscr{C}, \sigma)_H \neq (\mathscr{C}', \sigma')_H$ for at least one $H \in F$ and $\mathscr{D}(\mathscr{C}, \sigma)_H \cap \mathscr{D}(\mathscr{C}', \sigma')_H$ again has Lebesgue measure zero. This is a contradiction because

$$\mathscr{D}_\pi \subset \mathscr{D}(\mathscr{C}, \mathscr{B}, \sigma) \cap \mathscr{D}(\mathscr{C}', \mathscr{B}', \sigma') \subset \mathop{\times}_{H \in F} (\mathscr{D}(\mathscr{C}, \sigma)_H \cap \mathscr{D}(\mathscr{C}', \sigma')_H)$$

Lemma 2. *Let π be a fixed permutation of the lines of G, then*

$$\sum_{F \in \mathscr{F}_G^\#} \prod_{H \in F} (-T_H) = \sum_{(\mathscr{C}, \mathscr{B}, \sigma) \in \mathscr{F}_\pi} \prod_{H \in \mathscr{C}^\# \setminus \mathscr{B}} (1 - T_H) \prod_{H \in \mathscr{B}} (-T_H) \tag{19}$$

where $\mathscr{F}^\#$ is either $\mathscr{F}$ or $\mathscr{F}'$ and similar for $\mathscr{C}^\#$.

Proof. Follows immediately from Lemma 1 and the fact that any subset $F \subset \mathscr{C}^\#$ is an element of $\mathscr{F}_G^\#$.

Corollary 1.

$$\int_0^\infty d\underline{\alpha} \sum_{F\in\mathscr{F}_G^\sharp} \prod_{H\in F} (-T_H) I_G = \sum_{(\mathscr{C},\mathscr{B},\sigma)} \int_{\mathscr{D}(\mathscr{C},\mathscr{B},\sigma)} d\underline{\alpha} \prod_{H\in\mathscr{C}^\sharp\setminus\mathscr{B}} (1-T_H) \prod_{H\in\mathscr{B}} (-T_H) I_G . \tag{20}$$

Proof. We first decompose

$$\int_0^\infty d\underline{\alpha} = \sum_{\pi\in S_L} \int_{\mathscr{D}_\pi} d\underline{\alpha}$$

use Lemma 2 and finally collect all π's for which a given $(\mathscr{C},\mathscr{B},\sigma)$ is π-complete.

Definition 4. Let G be 1PI and $(\mathscr{C},\sigma)$ a labelled forest for G. Instead of $\underline{\alpha}$ we introduce scaling variables $(\underline{t},\underline{\beta})=(t_H, H\in\mathscr{C}, \beta_\ell, \ell\in\mathscr{L}'_G=\mathscr{L}_G\setminus\{\sigma(\mathscr{C})\})$ as well as auxiliary variables ξ_H, η_H, $H\in\mathscr{C}$

$$\alpha_\ell = \begin{cases} \prod\limits_{H\subset H'\in\mathscr{C}} t_{H'}^2 = \xi_H^2 = t_G^2\eta_H^2 & \text{if } \ell=\sigma(H), \quad H\in\mathscr{C}; \\ \beta_\ell \xi_H^2 & \text{if } \ell\in\mathscr{L}'_{\bar{H}} = \mathscr{L}_H\setminus\{\sigma(H)\}, \quad H\in\mathscr{C}, \end{cases} \tag{21}$$

and define $\beta_\ell=1$, $\ell\in\sigma(\mathscr{C})$ for convenience. The image of $\mathscr{D}(\mathscr{C},\mathscr{B},\sigma)$ under this substitution is the set $\{(\underline{t},\underline{\beta}): 0\leqq t_G<\infty; 0\leqq t_H\leqq 1 \text{ for } H\in\mathscr{C}'\setminus\mathscr{B}; 1\leqq t_H \text{ for } H\in\mathscr{B}; 0\leqq\beta_\ell\leqq 1 \text{ for } \ell\in\mathscr{L}'_G\}$.

5. Singularity Structure of the IR-Subtracted Feynman Amplitudes

From now on we follow essentially the analysis given in Ref. [2], the only difference is that we take the Taylor subtraction operators into account.

We restrict ourselves to connected graphs G which have the property that i) $\Omega_i\geqq 4$ for all internal vertices and ii) every external vertex $V_i\in\mathscr{U}_G$ is connected to the rest of G by exactly one line which we call external; graphs with arbitrary external vertices can be obtained by 'amputation' of external lines. In addition we assume that the propagators for these external lines are of the form

$$k_G!/(-ip^2+im^2+0)^{k_G+1}$$

with some nonnegative integer k_G, to be determined later.

From G we obtain a new graph G_∞ by collapsing all its external vertices to a new one called V_∞. Neglecting graphs with trivial c-components, there is clearly a one-to-one correspondence between subgraphs H of G and subgraphs H_∞ of G_∞. Notice that H need not be connected for H_∞ to be 1PI.

Definition 5. A labelled c_∞-family with basis $\mathscr{B}$ for G is a triple $(\mathscr{C}_\infty,\mathscr{B},\sigma)$ such that its image under the above mentioned correspondence is a labelled forest with basis $\mathscr{B}$ for G_∞.

These c_∞-families with basis allow us to analyze simultaneously the UV- and IR-singularities of the Feynman integral corresponding to G in the presence of Taylor subtraction operators.

Remark. G_∞ is 1PI iff G contains no tadpoles. We assume for the moment that this is the case. Later on we will prove that the amplitude for any graph with tadpoles vanishes (even before the limit $\varepsilon\to 0$ is taken).

Definition 6. For future reference we define some subsets of $\mathscr{C}_\infty$ and $\mathscr{L}_G$

$$\mathscr{C}'_\infty = \mathscr{C}_\infty \setminus \{G\};$$
$$\mathscr{C}''_\infty = \{H \in \mathscr{C}_\infty : V_\infty \notin \mathscr{V}_{H\infty} \text{ or equivalently } H \text{ is 1PI}\};$$
$$\mathscr{C}'''_\infty = \{H \in \mathscr{C}''_\infty : \Omega_H > 0\}; \qquad \mathscr{C}'''_\infty \subset \mathscr{C}''_\infty \subset \mathscr{C}'_\infty \subset \mathscr{C}_\infty .$$
$$\mathscr{H} = \{H \in \mathscr{C}_\infty : \bar{H} \text{ is a tree (in } G/\mathscr{M}(H))\}; \qquad \mathscr{H} \subset \mathscr{C}_\infty \setminus \mathscr{C}''_\infty .$$
$$\mathscr{L}'_G = \mathscr{L}_G \setminus \sigma(\mathscr{C}_\infty);$$
$$\mathscr{L}_G^{\text{ext}} = \{\ell \in \mathscr{L}_G : V_\infty \text{ is one of the endpoints of } \ell \text{ in } G_\infty\}.$$

We want to use the decomposition of Corollary 1 adapted to labelled c_∞-families with basis for the graph G. This can be done because the subgraph H_∞ of G_∞ corresponding to any 1PI $H \subset G$ is 1PI as well. We just have to put the Taylor operator $T_H = 0$ for all $H \in \mathscr{C}_\infty \setminus \mathscr{C}'''_\infty$, which means that we may ignore all $(\mathscr{C}_\infty, \mathscr{B}, \sigma)$ except those satisfying $\mathscr{B} \subset \mathscr{C}'''_\infty$.

Before we can apply formula (4.20) we have to give a representation of the Taylor operators that exhibits their action on the Feynman integrand $I_{G,\varepsilon}(\underline{p}, \underline{\alpha})$. Since this has already been discussed in the literature [9, 10], let us just indicate the essential idea. Using the formulae

$$T^{\Omega-1} f(p) = \sum_{\ell=1}^{\Omega} \frac{1}{(\Omega-\ell)!} \left(\frac{\partial}{\partial\lambda}\right)^{\Omega-\ell} f(\lambda p)\bigg|_{\lambda=0} \tag{1}$$

and

$$(1 - T^{\Omega-1}) f(p) = \int_0^1 d\lambda \frac{(1-\lambda)^{\Omega-1}}{(\Omega-1)!} \left(\frac{\partial}{\partial\lambda}\right)^{\Omega} f(\lambda p) \tag{2}$$

one has to keep track of all λ's while one performs the Gaussian integrals leading to the α-space integral (1.1). The result is

$$\begin{aligned} &\prod_{H \in \mathscr{C}'''_\infty \setminus \mathscr{B}} (1 - T_H^{\Omega_H - 1}) \prod_{H \in \mathscr{B}} (-T_H^{\Omega_H - 1}) I_{G,\varepsilon}(\underline{p}, \underline{u}, \underline{\alpha}, n)\bigg|_{\underline{u}=0} \\ &= \prod_{H \in \mathscr{C}'''_\infty \setminus \mathscr{B}} \int_0^1 d\lambda_H \frac{(1-\lambda_H)^{\Omega_H - 1}}{(\Omega_H - 1)!} \left(\frac{\partial}{\partial\lambda_H}\right)^{\Omega_H} \prod_{H \in \mathscr{B}} \sum_{\ell=1}^{\Omega_H} \frac{1}{(\Omega_H - \ell_H)!} \left(\frac{\partial}{\partial\lambda_H}\right)^{\Omega_H - \ell_H} \hat{\hat{I}}\bigg|_{\underline{u}=\underline{\lambda}_{\mathscr{B}}=0} \end{aligned} \tag{3}$$

with

$$\hat{\hat{I}} = \prod_{H \in \mathscr{C}'''_\infty} \lambda_H^{\omega_H - \nu h_H + |\mathscr{L}_H|} \hat{I}_{G,\varepsilon}(\underline{p}, \underline{u}, \underline{\alpha}, \underline{\lambda}, n)$$

where $\underline{\lambda}_{\mathscr{B}} = (\lambda_H, H \in \mathscr{B})$, $\nu = 4 - n$ and $\hat{I}_{G,\varepsilon}$ is obtained from $I_{G,\varepsilon}$ by multiplying each α_ℓ except those in $\sum \alpha_\ell (m_\ell^2 + i\varepsilon)$ by a factor $\prod_{H_\ell \subset H \in \mathscr{C}_\infty} \lambda_H^2$. In terms of scaling variables (4.21) that means $t_H \to \lambda_H t_H$ for all $H \in \mathscr{C}'''_\infty$. For convenience we define $\lambda_H = 1$ for $H \notin \mathscr{C}'''_\infty$.

Remark. We are allowed to interchange the evaluation of the Taylor operators with the $\underline{\alpha}$-integral, because the latter is defined by analytic continuation from a region where it converges absolutely.

Next we put all auxiliary momenta p_i associated with internal vertices $V_i \in \mathscr{V}_G \setminus \mathscr{U}_G$ to zero and introduce linear combinations $(q_H, H \in \mathscr{H})$ of the external ones (defined in Ref. [2]).

Integrating over $\mathscr{D}(\mathscr{C}_\infty, \mathscr{B}, \sigma)$ and using the results of Appendix A of Ref. [2], we obtain a sum of terms

$$\int_0^\infty \frac{dt_G}{t_G} t_G^{\nu h_G - \omega_G + r} \int_{\hat{\mathscr{D}}} d\mu(\underline{t}, \underline{\beta}, \underline{\lambda}) \prod_{H \in \mathscr{C}'_\infty} t_H^{\nu h_H - \omega_H} \prod_{\ell \in \mathscr{L}_G} \beta_\ell^{s_\ell} \prod_{H \in \mathscr{C}'''_\infty} \left(\frac{\partial}{\partial \lambda_H}\right)^{\Omega_H - \ell_H}$$
$$\cdot g_{\varepsilon, r}(\eta_H q_H, \lambda_H t_H, \underline{\beta}, \nu) \exp[-t_G^2(-id(\eta_H q_H, \lambda_H t_H, \underline{\beta}) + i\tilde{M} + \tilde{\varepsilon})] \tag{4}$$

where

$$\int_{\hat{\mathscr{D}}} d\mu(\underline{t}, \underline{\beta}, \underline{\lambda}) = \prod_{H \in \mathscr{C}'_\infty \setminus \mathscr{B}} \int_0^1 \frac{dt_H}{t_H} \prod_{H \in \mathscr{B}} \int_1^\infty \frac{dt_H}{t_H} \prod_{\ell \in \mathscr{L}_G} \int_0^1 \frac{d\beta_\ell}{\beta_\ell} \prod_{H \in \mathscr{C}'''_\infty \setminus \mathscr{B}} \int_0^1 d\lambda_H \frac{(1-\lambda_H)^{\Omega_H - 1}}{(\Omega_H - 1)!}. \tag{5}$$

$g_{\varepsilon, r}$ is a homogeneous polynomial of degree r in $(\underline{\eta q}, \varepsilon^{1/2})$, $d(\underline{\eta q}, \underline{\lambda t}, \underline{\beta}) = \sum_{H, H' \in \mathscr{H}} d_{H,H'} \eta_H \eta_{H'} q_H q_{H'}$ a quadratic form in $\underline{\eta q}$ with a positive definite matrix $d_{H,H'}$; $d_{H,H'}$ and the coefficients of $g_{\varepsilon, r}$ are C^∞ in $(\underline{\lambda t}, \underline{\beta})$, independent of t_G and independent of resp. analytic in ν. Here we have used that $\lambda_H = 1$ for $H \in \mathscr{H}$, $\ell_H = 0$ for $H \in \mathscr{C}'''_\infty \setminus \mathscr{B}$ and $0 < \ell_H \leqq \Omega_H$ for $H \in \mathscr{B}$. Furthermore $i\tilde{M} + \tilde{\varepsilon} = \sum_{\mathscr{L}_G} \beta_\ell \eta_{H_\ell}^2 (im_\ell^2 + \varepsilon)$.

Remark. Since $\hat{\hat{I}}$ [Eq. (3)] is an even resp. odd function of λ_H if Ω_H is even resp. odd we may restrict the sum over ℓ_H to even values. For the same reason $r - \omega_G$ is always even.

Evaluation of the derivatives with respect to $\underline{\lambda}$ yields a similar expression with $\partial/\partial\lambda_H$ replaced by t_H; thereby r may increase without changing the properties of $g_{\varepsilon, r}$. Putting $\underline{\lambda}_{\mathscr{B}} = 0$ leaves C^∞ functions $g_{\varepsilon, r}(\underline{\eta q}, \underline{\lambda t})$ and $d(\underline{\eta q}, \underline{\lambda t})$ with $\lambda_H t_H \leqq 1$.

Choosing k_G sufficiently large, we achieve $\omega_G < 0$ and the t_G-integral converges absolutely (for $\mathrm{Re}(\nu) \geqq 0$, $\varepsilon > 0$), leaving expressions of the form

$$\Gamma\left(\frac{\nu h_G - \omega_G + r}{2}\right) \int_{\hat{\mathscr{D}}} d\mu(\underline{t}, \underline{\beta}, \underline{\lambda}) \prod_{\mathscr{C}'_\infty} t_H^{\nu h_H - \omega_H + \Omega_H - \ell_H} \prod_{\mathscr{L}_G} \beta_\ell^{s_\ell}$$
$$\cdot \frac{g_{\varepsilon, r}(\underline{\eta q}, \underline{\lambda t}, \underline{\beta}, \nu)}{[-id(\underline{\eta q}, \underline{\lambda t}, \underline{\beta}) + i\tilde{M} + \tilde{\varepsilon}]^{(\nu h_G - \omega_G + r)/2}} \tag{6}$$

with

$$\ell_H \begin{cases} = 0 & \text{for} \quad H \in \mathscr{C}'''_\infty \setminus \mathscr{B} \\ \geqq 2 & \text{even for} \quad H \in \mathscr{B} \\ = \Omega_H & \text{for} \quad H \in \mathscr{C}'_\infty \setminus \mathscr{C}'''_\infty \end{cases} \tag{7}$$

Since $\Omega_H \leqq 0$ for all $H \in \mathscr{C}''_\infty \setminus \mathscr{C}'''_\infty$, we may write $\ell_H \leqq 0$ for all $H \in \mathscr{C}''_\infty \setminus \mathscr{B}$.

In Ref. [2] we have proven the following

Lemma 3 (ε-Lemma). *Let $\{\eta_j, \varrho_j\}_{j=1}^J$ and $\{\gamma_\ell, k_\ell, \kappa_\ell\}_{\ell=1}^L$ be real numbers in the domain $0 \leqq \eta_J \leqq \ldots \leqq \eta_1$; $0 < \varrho_j < 4$ for $j = 1, \ldots, J$; $\gamma_\ell \geqq 0$ and $k_\ell, \kappa_\ell > 0$ for $\ell = 1, \ldots, L$. In addition be $\varrho \in \mathbb{C}$, $\{d_{ij}\}_{i,j=1}^J$ a positive definite matrix defining the quadratic* *form* $d(\underset{\sim}{q})=\sum d_{ij}q_iq_j$ *over* J *copies of 4-dimensional Minkowski space,* $Q(\underset{\sim}{q})$ *a homogeneous polynomial of degree* r *in* $\underset{\sim}{q}$,

$$\Delta=\sum_{j=1}^{J}\varrho_j+r+2\sum_{\ell=1}^{L}k_\ell-\mathrm{Re}(\varrho)>0$$

and $f(\underset{\sim}{q},\varrho,\varepsilon)$ *the expression*

$$f(\underset{\sim}{q},\varrho,\varepsilon)=\frac{Q(\underset{\sim}{\eta}\underset{\sim}{q})\prod\eta_j^{\varrho_j}\prod\gamma_\ell^{k_\ell}}{[-id(\underset{\sim}{\eta}\underset{\sim}{q})+i\sum\gamma_\ell\kappa_\ell+\varepsilon]^{\varrho/2}}. \tag{8}$$

Then $f(\underset{\sim}{q},\varrho,0)=\lim_{\varepsilon\to0}f(\underset{\sim}{q},\varrho,\varepsilon)$ *exists in* $\mathscr{S}'(\mathbb{R}^{4J})$, *is continuous in* $(\underset{\sim}{\eta},\underset{\sim}{\gamma})$ *and* C^∞ *in the coefficients of* Q *and* d. *Furthermore* $\lim_{\varepsilon\to0}\varepsilon^k f(\underset{\sim}{q},\varrho+2k,\varepsilon)=0$ *for* $k>0$.

Corollary 2. *Using the same notation,* $f(\underset{\sim}{q},\varrho,\varepsilon)/(1+\sum\eta_j^2+\sum\gamma_\ell+\varepsilon)^{\Delta/2}$ *is uniformly bounded (as element of* $\mathscr{S}'$*) in* $(\underset{\sim}{\eta},\underset{\sim}{\gamma},\varepsilon)$.

Proof. This is an immediate consequence of Lemma 3 and of the homogeneity of $f(\underset{\sim}{q},\varrho,\varepsilon)$ in $(\underset{\sim}{\eta}^2,\underset{\sim}{\gamma},\varepsilon)$.

First we decompose the polynomial $g_{\varepsilon,r}$

$$g_{\varepsilon,r}(\underset{\sim}{\eta}\underset{\sim}{q},\underset{\sim}{\lambda}\underset{\sim}{t},\underset{\sim}{p},\nu)=\sum_{r'=0}^{r}Q_{r-r'}(\underset{\sim}{\eta}\underset{\sim}{q},\underset{\sim}{\lambda}\underset{\sim}{t},\underset{\sim}{\beta},\nu)\varepsilon^{r'/2}. \tag{9}$$

The last statement of the ε-Lemma guarantees that if the limit ε→0 of Equation (6) exists, then only the term with $r'=0$ contributes. In addition we can neglect all ε's in the numerator when we "amputate" external lines or derive the equations of the action particle.

All we have to do now, is to rearrange the integrand to display the numerator required for the ε-Lemma. We rewrite Equation (6) as

$$\Gamma\left(\frac{\nu h_G-\omega_G+r}{2}\right)\int_{\mathfrak{D}}d\mu\prod_{\mathscr{C}_\infty'}t_H^{n_H}\prod_{\mathscr{L}_G}\beta_\ell^{s_\ell-k_\ell}\frac{Q_r(\underset{\sim}{\eta}\underset{\sim}{q})\prod_{\mathscr{H}}\eta_H^{\varrho_H}\prod_{\mathscr{L}_G}(\beta_\ell\eta_{H_\ell}^2)^{k_\ell}}{[-id(\underset{\sim}{\eta}\underset{\sim}{q})+i\tilde{M}+\tilde{\varepsilon}]^{\frac{\nu h_G-\omega_G+r}{2}}} \tag{10}$$

where

$$n_H=\nu h_H+\Omega_H-\omega_H-\sigma_H-\ell_H-2\tau_H\,,\qquad \sigma_H=\sum_{H\subset H'\in\mathscr{H}}\varrho_{H'}\quad\text{and}\quad\tau_H=\sum_{\mathscr{L}_H}k_\ell\,.$$

We have to demonstrate the existence of ϱ_H's and k_ℓ's such that the hypothesis of the ε-Lemma is satisfied and the t_H-, β_ℓ- and λ_H-integrals converge. This is greatly facilitated by our choice of maximal (=effective) IR-degrees. Let us first collect some knowledge about the subgraphs $H\in\mathscr{C}_\infty$ and their IR-degrees Ω_H.

We know that all elements $H\in\mathscr{C}_\infty\setminus\mathscr{C}_\infty''$ are linearly ordered by inclusion and that the smallest one, $H_{\min}$, is an element of $\mathscr{H}$. For any $H\supset H_{\min}$ resp. $H\supsetneq H_{\min}$ let H^- resp. H_- be the largest element of $\mathscr{H}$ satisfying $H^-\subset H$ resp. $H_-\subsetneq H$. The elements of $\mathscr{C}_\infty\setminus\mathscr{C}''$ are not necessarily connected, but i) every c-component has at least 2 external vertices and ii) every c-component of any $H\supset H_{\min}$ contains exactly one c-component of H^- with the same number of external vertices.

In particular $H_{\min}$ is connected and has two external vertices. For all external resp. internal vertices of G, $\Delta_i^G=2$ resp. $\Delta_i^G \geqq 4$ and for any vertex V_i contained in an arbitrary subgraph H of G, $\Delta_i^H \leqq \Delta_i^G$. If $H \subsetneqq G$ then $\Delta_i^H < \Delta_i^G$ for at least one vertex in each c-component of H.

For any graph H with c-components H_i, the IR-degree Ω_H is the sum of the degrees Ω_{H_i}.

Definition 7. We split the degree $\Omega_G = 4 - \sum_{\mathscr{V}_G}(4-\Delta_i^G)$ into the *internal degree* $\Omega_G^{\text{int}} = -\sum_{\mathscr{V}_G \setminus \mathscr{U}_G}(4-\Delta_i^G)$ and the *external degree* $\Omega_G^{\text{ext}} = \Omega_G - \Omega_G^{\text{int}}$. For any $H \subset G$ with c-components H_i we define the *relative external degree of H in G* by

$$\Omega_{H,G}^{\text{ext}} = \sum_i \Omega_{H_i,G}^{\text{ext}} = \sum_i \left[4 - \sum_{\mathscr{U}_{H_i}} (4-\Delta_i^G)\right]. \tag{11}$$

Remark. These degrees are completely analogous to those defined in Ref. [2] using UV-degrees δ and ω.

Ω_G^{int} is a sum of non-negative terms, $\Omega_G^{\text{ext}} = \Omega_{G,G}^{\text{ext}}$ and for each connected graph H_i, $\Omega_{H_i,G}^{\text{ext}} = 4 - 2|\mathscr{U}_{H_i}|$. For any $H \subsetneqq G$ with c-components H_i

$$\Omega_H \leqq \sum_i (\Omega_{H_i,G}^{\text{ext}} - \tfrac{1}{2}) + \Omega_G^{\text{int}} \tag{12}$$

and most importantly for any $H \subset G$, $\Omega_H = \omega_H + 2s_H$ with $s_H = \sum_{\mathscr{L}_H} s_\ell^m$.

We can now proceed to the definition of the k_ℓ's and ϱ_H's. Let a and δ be real numbers with $0 < a < \frac{1}{2}$ and $0 < \delta < a/2(s_G + h_G)$. For $\ell \in \mathscr{L}_G$ we choose

$$k_\ell = \begin{cases} (1-\delta)s_\ell^m + x_\ell - \Omega_G^{\text{int}}/4 & \text{if} \quad \ell \in \mathscr{L}_G^{\text{ext}} \cap \mathscr{L}_{H_{\min}} \ (s_\ell^m = k_G + 1) \\ (1-\delta)s_\ell^m + x_\ell & \text{otherwise} \end{cases} \tag{13}$$

where

$$x_\ell = \begin{cases} x(v)/2 = \max\left(0, \operatorname{Re}\dfrac{v}{2} - \delta\right) & \text{if} \quad \ell \in \sigma(\mathscr{C}_\infty \setminus \mathscr{H}) \\ 0 \quad \textit{otherwise}. \end{cases} \tag{14}$$

The v-dependence of k_ℓ via $x(v)$ allows us to show the convergence of the decomposed integral Equation (6) for $\operatorname{Re}(v) \gg 0$. In a neighborhood of $v=0$, where the limit $\varepsilon \to 0$ is studied, $x(v)$ vanishes. These k_ℓ all satisfy $k_\ell > 0$ and (unless $\beta_\ell \equiv 1$ *and* $\operatorname{Re}(v) \geqq 2\delta$) $k_\ell < s_\ell^m$ which guarantees convergence of the β_ℓ-integrals. The corresponding τ_H's are

$$\tau_H = \begin{cases} (1-\delta)s_H + h_H x(v)/2 - \Omega_G^{\text{int}}/2 & \text{if} \quad H \in \mathscr{C}_\infty \setminus \mathscr{C}_\infty'' \\ (1-\delta)s_H + h_H x(v)/2 & \text{otherwise}. \end{cases} \tag{15}$$

For $H \in \mathscr{C}_\infty$ we choose $\sigma_H = \max\left[0, \sum_i (a - \Omega_{H_i,G}^{\text{ext}})\right]$ which first of all implies that σ_H vanishes for all $H \in \mathscr{C}_\infty''$ and otherwise depends only on H^- as it should. The ϱ_H's corresponding to this choice of σ_H's are

$$\varrho_H = \begin{cases} \sigma_H & \text{if} \quad H = H_{\min} \\ \sigma_H - \sigma_{H_-} & \text{if} \quad H \in \mathscr{H} \setminus \{H_{\min}\}. \end{cases} \tag{16}$$

There are altogether four possibilities:

i) $|\mathscr{U}_{H_-}|=|\mathscr{U}_H|$ (i.e. H_- has one c-component more than H) and $\varrho_H=4-a$;

ii) $|\mathscr{U}_{H_-}|=|\mathscr{U}_H|-1$ (i.e. H_- and H have the same number of c-components) and $\varrho_H=2$;

iii) $|\mathscr{U}_{H_-}|=|\mathscr{U}_H|-2$ (i.e. H_- has one c-component less than H) or

iv) $H=H_{\min}$ and $\varrho_H=a$.

In either case $0<\varrho_H<4$ is satisfied.

With these k_ℓ's and ϱ_H's we obtain

$$\begin{aligned}\Delta&=\sum\varrho_H+2\sum k_\ell+\omega_G-vh_G=\sigma_G+2\tau_G+\omega_G-vh_G\\&=a-2\delta s_G+h_G(x(v)-v)\,.\end{aligned}\tag{17}$$

Due to our choice of $x(v)$ in Equation (14), $\mathrm{Re}(x(v)-v)=\max(-\mathrm{Re}(v),-2\delta)$ and therefore $\mathrm{Re}(\Delta)>0$.

Combining everything we get the following exponents n_H for t_H:

i) $\mathrm{Re}\,n_H\geqq h_H(\mathrm{Re}\,v-x(v))+2\delta s_H+\sum\limits_i(\tfrac{1}{2}-a)$ for $H\in\mathscr{C}'_\infty\setminus\mathscr{C}''_\infty$;

ii) $\mathrm{Re}\,n_H\geqq h_H(\mathrm{Re}\,v-x(v))+2\delta s_H$ for $H\in\mathscr{C}''\setminus\mathscr{B}$;

iii) $\mathrm{Re}\,n_H\leqq h_H(\mathrm{Re}\,v-x(v))+2\delta s_H-2$ for $H\in\mathscr{B}$.

Proposition 2. *The* IR-*renormalized Feynman amplitude* $\mathscr{R}^{\mathrm{IR}}_{G,}(p,n)$ *for a connected graph* G *with* $\Delta_i\geqq4$ *for all internal vertices, as defined by Equation* (3.18), *can be decomposed into a sum of integrals*

$$\delta(\sum p_i)\Gamma\left(\frac{vh_G-\omega_G+r}{2}\right)\int\limits_{\hat{\mathscr{D}}}d\mu(\underline{t},\underline{\beta},\underline{\lambda})f_\varepsilon(\underline{p},\underline{t},\underline{\beta},\underline{\lambda},n)\tag{18}$$

which—considered as a distribution over $\mathscr{S}(\mathrm{IR}^{4K})$ *in the p's—converge absolutely for* $\mathrm{Re}(4-n)$ *positive and can be analytically continued into a finite neighborhood of* $n=4$ *where* $\mathscr{R}^{\mathrm{IR}}_G(p,n)=\lim\limits_{\varepsilon\to0}\mathscr{R}^{\mathrm{IR}}_{G,\varepsilon}(p,n)$ *exists in* $\mathscr{S}'(\mathrm{IR}^{4K})$ *and is meromorphic in n.*

Proof. With our choice of k_ℓ's and ϱ_H's the integrand of Equation (10) satisfies the hypothesis of the ε-Lemma. The λ_H- and β_ℓ-integrals converge due to our choice of k_ℓ's, whereas Corollary 2 supplies an estimate which is needed to prove the convergence of the t_H-integrals. The variables t_H for $H\in\mathscr{B}$ are the only unbounded ones, however $(1+\sum\eta_H^2+\sum\gamma_\ell+\tilde{\varepsilon})^{1/2}$ is bounded by some multiple of $\prod\limits_{H\in\mathscr{B}}t_H$. This effectively increases the exponent n_H of these t_H by Δ. If $\mathrm{Re}(v)>0$ then $\mathrm{Re}(n_H)>0$ for $H\in\mathscr{C}'_\infty\setminus\mathscr{B}$ and $\mathrm{Re}(n_H+\Delta)<0$ for $H\in\mathscr{B}$ and therefore all t_H-integrals converge. For $\mathrm{Re}(v)\to0$ only the t_H-integrals for $H\in\mathscr{C}'''_\infty\setminus\mathscr{B}$ with $s_H=0$ may fail to converge. Since the remaining integrand is C^∞ in these variables, the integral can be analytically continued in v and has poles at $v=0$. All these poles arise from UV-divergent 1PI subgraphs with massless lines only and are present whether $\varepsilon>0$ or $\varepsilon=0$.

Remark. The whole amplitude certainly can have poles for $\mathrm{Re}(v)>0$. Due to our choice of maximal (effective) IR-degree they are, however, all absorbed into IR-finite but possibly UV-divergent counterterms.

We still have to show the existence of the IR-counterterms $\lim_{\varepsilon\to 0} T^{\Omega_G-1}\bar{\mathscr{T}}^{\mathrm{IR}}_{G,\varepsilon}$ for 1PI graphs G with $\Omega_G>0$.

Proposition 3. *Let G be a* 1PI *graph with $\Omega_G>0$ and $\Delta_i\geqq 4$ for all internal vertices. Then $Y_{G,\varepsilon}=T^{\Omega_G-1}\bar{\mathscr{T}}^{\mathrm{IR}}_{G,\varepsilon}(\underline{p},n)$ is a polynomial in $\underline{p}$ with coefficients which are meromorphic in ν. In a finite neighborhood of $\nu=0$ these coefficients have a limit $\varepsilon\to 0$.*

Proof. Let us first concentrate on the highest derivatives

$$\left(\frac{\partial}{\partial\lambda}\right)^{\Omega_G-2}\bar{\mathscr{T}}^{\mathrm{IR}}_{G,\varepsilon}(\lambda\underline{p},n)|_{\lambda=0}\,. \tag{19}$$

Decomposing it according to c-families with basis for G (here we use c- and not c_∞-families since there are no external momenta) yields integrals of the form

$$\Gamma\left(\frac{\nu h_G}{2}+s_G-1\right)\int_{\mathscr{D}} d\mu\prod_{\mathscr{C}'} t^{\nu h_H+2s_H-\ell_H}\prod_{\mathscr{L}_G}\beta_\ell^{s_\ell}\,\frac{g(\lambda\underline{t},\underline{\beta},\nu)}{(i\tilde{M}+\tilde{\varepsilon})^{\frac{\nu h_G}{2}+s_G-1}} \tag{20}$$

with

$$\ell_H\begin{cases}\leqq 0 & \text{for}\quad H\in\mathscr{C}'\setminus\mathscr{B}\\ \geqq 2 & \text{even for}\quad H\in\mathscr{B}\end{cases} \tag{21}$$

multiplying monomials of degree Ω_G-2 in $\underline{p}$. Reorganizing the powers in the integrand in a familiary fashion we obtain

$$\begin{aligned}&\Gamma\left(\frac{\nu h_G}{2}+s_G-1\right)\int_{\mathscr{D}} d\mu\prod_{\mathscr{C}'} t_H^{\nu h_H+2s_H-\ell_H-2\tau_H}\prod_{\mathscr{L}_G}\beta_\ell^{s_\ell-k_\ell}\\ &\cdot g(\lambda\underline{t},\underline{\beta},\nu)\frac{\prod(\beta_\ell\eta^2_{H_\ell})^{k_\ell}}{(i\tilde{M}+\tilde{\varepsilon})^{\frac{\nu h_G}{2}+s_G-1}}\end{aligned} \tag{22}$$

with

$$k_\ell=\begin{cases}(1-\delta)s_\ell^m+x(\nu)/2 & \text{if}\quad \ell\in\sigma(\mathscr{C})\\ (1-\delta)s_\ell^m & \text{if}\quad \ell\in\mathscr{L}'_G\end{cases} \tag{23}$$

where $x(\nu)$ is given by Equation (14) and

$$\tau_H=\sum_{\mathscr{L}_H}k_\ell=(1-\delta)s_H+h_Hx(\nu)/2 \tag{24}$$

for all $H\in\mathscr{C}$. Application of the ε-Lemma with $\varrho=\nu h_G+2s_G-2$ shows the existence of this integral for $\mathrm{Re}(\nu)>0$. Again the only possible singularities at $\nu=0$ (apart from the singularity of the Γ-function) develop from the t_H-integrals with $H\in\mathscr{C}'\setminus\mathscr{B}$ and $s_H=0$, leading to a finite order pole at $\nu=0$.

Next we consider all the remaining terms in $T^{\Omega_G-1}\bar{\mathscr{T}}^{\mathrm{IR}}_{G,\varepsilon}(\underline{p},n)$. The coefficients of the polynomial in $\underline{p}$ arising from

$$\left(\frac{\partial}{\partial\lambda}\right)^{\Omega_G-2k}\bar{\mathscr{T}}^{\mathrm{IR}}_{G,\varepsilon}(\lambda\underline{p},n)|_{\lambda=0} \tag{25}$$

where $k > 1$ are integrals of the form

$$\Gamma\left(\frac{\nu h_G}{2}+s_G-k\right)\int_{\hat{\mathscr{D}}} d\mu \prod_{\mathscr{C}'} t_H^{\nu h_H+2s_H-\ell_H} \prod_{\mathscr{L}_G} \beta_\ell^{s_\ell} \frac{g(\underline{\lambda t}, \underline{\beta}, \nu)}{(i\tilde{M}+\tilde{\varepsilon})^{\frac{\nu h_G}{2}+s_G-k}} \tag{26}$$

with an exponent of the denominator which is in general too small for a direct application of the ε-Lemma. We can, however, multiply numerator and denominator by

$$M_G = i\tilde{M}+\tilde{\varepsilon} \tag{27}$$

and decompose M_G in the numerator into

$$M_G = M_{G/\{H_i\}} + \sum_{H\in\{H_i\}} t_H^2 M_H \tag{28}$$

where $\{H_i\}$ is the set of all maximal elements of $\mathscr{B}$. $M_{G/\{H_i\}}$ is bounded and can be absorbed into $g(\underline{\lambda t}, \underline{\beta}, \nu)$. For each $H\in\{H_i\}$ we obtain a term with ℓ_H replaced by ℓ_H-2. If $\ell_H=2$ we replace $\int_1^\infty$ by $\int_0^\infty - \int_0^1$. The first term can be integrated to

$$\int_0^\infty \frac{dt_H}{t_H} t_H^{\nu h_H+2s_H} \frac{M_H}{(M_{G/H}+t_H^2 M_H)^{\frac{\nu h_G}{2}+s_G-k+1}}$$

$$= \frac{1}{2}\Gamma\left(\frac{\nu h_G}{2}+s_G-k+1\right)^{-1} \frac{\Gamma\left(\frac{\nu h_{G/H}}{2}+s_{G/H}-k+1\right)\Gamma\left(\frac{\nu h_H}{2}+s_H\right)}{(M_{G/H})^{\frac{\nu h_{G/H}}{2}+s_{G/H}-k+1}(M_H)^{\frac{\nu h_H}{2}+s_H-1}} \tag{29}$$

In the second one, where we have reduced the number of integrals from 0 to ∞ (i.e. members of $\mathscr{B}$), or if $\ell_H > 2$ we may further decompose M_H as above. In any case we obtain one or several integrals of the type we started from, but with k reduced. Continuing this process we obtain eventually integrals of the form (20), converging for $\varepsilon\to 0$ and $\mathrm{Re}(\nu)>0$. They all have a limit $\varepsilon\to 0$ in some neighborhood of $\nu=0$ and all poles for $\mathrm{Re}(\nu)>0$ are made explicit by the various Γ-functions occurring in the reduction.

Combining Propositions one, two and three we have proved:

Theorem 1. *For any connected Feynman graph G, satisfying $\Delta_i \geqq 4$ for all internal vertices, the renormalized amplitude $\mathscr{R}^S_{G,\varepsilon}(\underline{p}, n)$ as defined by Equation* (3.5) *is analytic at $n=4$ and has a limit $\varepsilon\to 0$ (as distribution over $\mathscr{S}$) in some finite neighborhood of $n=4$.*

Up to this point we have assumed, that G contains no tadpoles, i.e. that G_∞ is 1PI. Any 1PI tadpole has $\Omega>0$ and its amplitude does not depend on any momenta. Proposition 3 guarantees the existence of $T^{\Omega-1}\mathscr{T}^{\mathrm{IR}}$ and consequently $(1-T)\mathscr{T}^{\mathrm{IR}}$ vanishes identically in ε and ν. Let G be an arbitrary (connected) graph containing tadpoles and G_0 the graph obtained from G by removing all tadpoles as well as the lines connecting them to the rest of G. The condition $\Delta_i \geqq 4$ may be violated for those internal vertices of G_0, to which the tadpoles are attached. Nevertheless the amplitude for G_0 is given by a convergent integral if $\varepsilon > 0$ and $\mathrm{Re}(v)$ sufficiently large. Thus the IR-renormalized amplitude for any graph containing tadpoles vanishes.

Remark. As in Ref. [2] it can be shown that $\mathscr{R}_G^S(\underline{p}, n)$ exists even if *one* internal vertex has $\Delta_i = 3$.

Theorem 2. *For any connected graph G with* IR-*degree* Ω_G *assigned to it via the degrees* Δ_i^G *of its vertices* V_i *(i.e.* Ω_G *is not the effective degree) and for any* $\varrho > 0$.

$$\lim_{\lambda \to 0} \lambda^{\varrho - \Omega_G} \mathscr{R}_G^S(\lambda \underline{p}, n) = 0 \tag{30}$$

(as a distribution) in a finite neighborhood of $n = 4$.

Remark. Using the method of Pohlmeyer [11] it should be possible to show that

$$\mathscr{R}_G^S(\lambda \underline{p}, 4) = \lambda^{\Omega_G} \sum_{k=0}^{h_G} (\ln \lambda)^k (r_k(\underline{p}) + 0(\lambda)) \tag{31}$$

with some $r_k(\underline{p}) \in \mathscr{S}'$.

Proof. First we observe that $\mathscr{R}_G^S$ is a sum of $\mathscr{R}^{\mathrm{IR}}$'s for various reduced graphs of G and that the effective IR-degree of each of them is not less than the IR-degree assigned to G. Next we "augment" the graph G by "external" lines to a graph G' and note that $\Omega_{G'} = \Omega_G$. Choosing λm as mass of the external lines we obtain

$$\begin{aligned} \lambda^{\varrho - \Omega_G} \mathscr{R}_G^{\mathrm{IR}}(\lambda \underline{p}, n) &= \lambda^{\varrho - \Omega_G} \prod_{\mathscr{L}_{G'}^{\mathrm{ext}}} \frac{[\lambda^2(-ip^2 + im^2)]^{k_G + 1}}{k_G!} \mathscr{R}_{G'}^{\mathrm{IR}}(\lambda \underline{p}, n) \\ &= \lambda^{\varrho - \Omega_G + 2s_{G'} - 2s_G} \prod_{\mathscr{L}_{G'}^{\mathrm{ext}}} \frac{[(-ip^2 + im^2)]^{k_G + 1}}{k_G!} \mathscr{R}_{G'}^{\mathrm{IR}}(\lambda \underline{p}, n) \,. \end{aligned} \tag{32}$$

Starting from $\lambda^{\varrho - \Omega_G + 2s_{G'} - 2s_G}$ times Equation (6), multiplying all momenta and all masses of external lines with λ and proceeding in a by now familiar way we obtain $\lambda^{\varrho - a + 2\delta(s_{G'} - s_G)}$ times Equation (10) with the replacements $\underline{\eta q} \to \lambda \underline{\eta q}$ and, for $\ell \in \mathscr{L}_{G'}^{\mathrm{ext}}$, $\beta_\ell \eta_{H_\ell}^2 \to \lambda^2 \beta_\ell \eta_{H_\ell}^2$. Choosing a small enough we see that the limit $\varepsilon \to 0$ exists in some neighborhood of $v = 0$, is continuous in λ and vanishes for $\lambda = 0$.

6. IR-Subtractions and the Action Principle

Like the BPHZ-renormalization the Taylor series subtractions performed to improve the IR-convergence of Feynman integrals are in general not compatible with the equations of the action principle, i.e. lead to radiative corrections in these equations. In principle it is possible to determine these corrections explicitly although in many applications their precise form turns out to be irrelevant [12].

There are however important cases in which the Taylor series subtractions completely cancel if all contributions from different graphs to one given Green's function are combined. This is so if (but not iff) some symmetry principle (gauge invariance, spontaneous symmetry breaking) *respected* by dimensional regularization enforces the vanishing of the Taylor expansion of the corresponding vertex function up to the required order. Relevant examples are QED, the Goldstone model and certain non-abelian gauge theories.

Under these circumstances a proof of the Ward identities expressing the symmetry at the level of Green's (or vertex) functions can be given to all orders of perturbation theory inductively hand in hand with a proof of the absence of IR-counterterms in the Lagrangian. Assuming that the Ward identities hold at the tree level and are respected by dimensional regularization, they also hold for the regularized one-loop Green's functions (see Ref. [1] for a discussion of this point). Assuming further that the validity of the Ward identities implies the vanishing of the "massless" vertex functions at zero external momenta up to the required order there will be no IR-counterterms in the Lagrangian to this order. Notice however that individual two-loop graphs may still require one-loop IR-counterterms in order to be IR-finite, it is just that they cancel out if one puts them together to an operator counterterm in the Lagrangian. The argument given above can be clearly extended to any order in perturbation theory, thus showing at the same time the validity of the Ward identities and the absence of IR-counterterms in the Lagrangian.

To illustrate this we consider the lowest order contributions to the vacuum polarization in scalar QED, originating from the two graphs shown in Figure 4. Both of them are needed in order to obtain a gauge invariant amplitude. The regularized amplitudes are

$$\Pi^a_{\mu\nu}(p) = e^2\pi^{\frac{n}{2}}\left[2g_{\mu\nu}\Gamma\left(1-\frac{n}{2}\right)(m^2-i\varepsilon)^{\frac{n}{2}-1} + (g_{\mu\nu}p^2 - p_\mu p_\nu)\,\Gamma\left(2-\frac{n}{2}\right)\int\limits_0^1 d\beta(1-2\beta)^2(m^2-\beta(1-\beta)p^2-i\varepsilon)^{\frac{n}{2}-2}\right] \tag{1}$$

resp.

$$\Pi^b_{\mu\nu}(p) = -2e^2\pi^{\frac{n}{2}}g_{\mu\nu}\Gamma\left(1-\frac{n}{2}\right)(m^2-i\varepsilon)^{\frac{n}{2}-1} \tag{2}$$

and neither one vanishes for $p=0$, their sum however does. The Ward identity requires $p^\mu\Pi_{\mu\nu}(p)=0$ and therefore $\Pi_{\mu\nu}(p)=(g_{\mu\nu}p^2-p_\mu p_\nu)\Pi(p^2)$ with

$$\Pi(p^2) = e^2\pi^{\frac{n}{2}}\,\Gamma\left(2-\frac{n}{2}\right)\int\limits_0^1 d\beta(1-2\beta)^2(m^2-\beta(1-\beta)p^2-i\varepsilon)^{\frac{n}{2}-2} \tag{3}$$

In higher orders $\Pi(p^2)$ will develop a (weak) singularity at $p^2=0$. Theorem 2, however, guarantees that $\Pi_{\mu\nu}(p)$ and $\partial\Pi_{\mu\nu}(p)/\partial p_\varrho$ vanish at $p=0$.

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