Commun. math. Phys. 51, 163—182 (1976)

Communications in
Mathematical
Physics

A Symplectic Structure on the Set of Einstein Metrics

A Canonical Formalism for General Relativity

Wiktor Szczyrba

Institute of Mathematics, Polish Academy of Sciences, 00-950 Warsaw, Poland

Abstract. A symplectic structure i.e. a symplectic form Γ on the set of all solutions of the Einstein equations on a given 4-dimensional manifold is defined. A degeneracy distribution of Γ is investigated and its connection with an action of the diffeomorphism group is established. A multiphase formulation of General Relativity is presented. A superphase space for General Relativity is proposed.

1. Introduction

It is known that the Hamilton formulation of mechanics is an appropriate tool for the quantization of classical systems. In the sixties this formulation was elegantly presented in a general theory of symplectic manifolds cf. [1, 22]. A basic concept in that approach is a $2n$-dimensional manifold $\mathscr{P}_{2n}$ — a phase space of a dynamical system and a non-degenerate closed 2 form $\overset{2}{\gamma}$ on $\mathscr{P}_{2n}$. The differential form $\overset{2}{\gamma}$ defines a bilinear skewsymmetric form $\{\cdot, \cdot\}$ on the vector space $\mathscr{F}$ of all smooth functions on $\mathscr{P}_{2n}$. The form $\{\cdot, \cdot\}$ is called a Poisson bracket. It defines a Lie algebra structure on the set $\mathscr{F}$. Very often $\mathscr{P}_{2n}$ is the cotangent bundle to an n-dimensional manifold V (a configuration space of a system). Then $\overset{2}{\gamma}$ is the canonical differential 2-form on T^*V and if (q^i) are local coordinates in V, (p_j, q^i) are local coordinates in $\mathscr{P}_{2n} = T^*V$ then

$$\overset{2}{\gamma} = \sum_{i=1}^{n} dp_i \wedge dq^i \tag{1.1}$$

and for $f_1, f_2 \in \mathscr{F} = C^\infty(\mathscr{P}_{2n})$

$$\{f_1, f_2\} = \sum_{i=1}^{n} ((\partial f_1/\partial p_i)(\partial f_2/\partial q^i) - (\partial f_2/\partial p_i)(\partial f_1/\partial q^i)). \tag{1.2}$$

In recent years was found a generalization of the notion of the symplectic manifold which is useful in classical field theories [15–17, 23]. This construction is based on a geometric theory of the calculus of variations formulated by Dedecker

[7] cf. also [16]. It turns out that for any variational problem with a fixed boundary in a space-time M there exists a multisymplectic manifold $(\mathscr{P}, \overset{5}{\gamma})$ i.e. a manifold $\mathscr{P}$ with a closed 5-form $\overset{5}{\gamma}$ $(5=\dim M+1)$. Field equations of the theory have the form:

$$(X \lrcorner \overset{5}{\gamma})|\Omega=0 \tag{1.3}$$

where Ω is a 4-dimensional submanifold of $\mathscr{P}$, X is an arbitrary vector field on Ω tangent to $\mathscr{P}$ and $|$ denotes the pull-back of the 4-form $X \lrcorner \overset{5}{\gamma}$ to the submanifold Ω.

To see that (1.3) really generalizes the Hamilton equations of mechanics we have to consider a homogeneous description of mechanics. Let $\mathscr{P}_{\text{hom}}$ be a $2n+1$ dimensional submanifold of the cotangent bundle of $V \times \mathbb{R}$ given by a constraint equation $H=H(p_j, q^i, t)$, where t is a coordinate in $\mathbb{R}$ and $-H$ is a coordinate conjugate to t. We have

$$\overset{2}{\gamma}_{\text{hom}}=\sum_{i=1}^{n} dp_i \wedge dq^i - dH \wedge dt\,. \tag{1.1'}$$

If Ω is a 1-dimensional submanifold of $\mathscr{P}_{\text{hom}}$ given by a parametrization $\Omega=\{(t, p_j(t), q^i(t)): t\in\mathbb{R}\}$ then the equation $(X \lrcorner \overset{2}{\gamma}_{\text{hom}})|\Omega=0$ is equivalent to the system of Hamilton equations:

$$dq^i/dt=\partial H/\partial p_i, \qquad dp_i/dt=-\,\partial H/\partial q^i\,. \tag{1.4}$$

Notion of a multiphase space was intoduced by Kijowski [17] who gave its axiomatic definition. For Lagrangian theories a couple $(\mathscr{P}, \overset{5}{\gamma})$ can be constructed by means of the Legendre transformation [16, 23]. In the paper [17] a Lie algebra $\mathscr{F}_{\text{loc}}$ of "local functionals" has been defined. These functionals are represented by integrals of differential 3-forms on 3-dimensional Cauchy surfaces in $\mathscr{P}$. Unfortunately for non-linear theories the algebra $\mathscr{F}_{\text{loc}}$ is too poor. It was proved in [17] that for a non-linear scalar field $\lambda\varphi^n$ $n>2$ the algebra $\mathscr{F}_{\text{loc}}$ consists only of the generators of the Poincaré group. Similar results concerning the algebra of "local functionals" were almost simultaneously presented by Goldschmidt and Sternberg [16]. The theory of multiphase spaces has been investigated later by Gawedzki [15] who has found a partial solution of that problem considering only physical quantities (functionals) localized on a given space-like surface in the space-time M. However this approach does not enable to compute a Poisson bracket of two physical quantities at "different instants of time".

The essential progress in the canonical formalism was achieved recently by Kijowski and the author in the paper [18]. In this paper has been found a natural symplectic structure on the set of all solutions of the field equations for a given field theory. A starting point in [18] is a given multiphase space (a multisymplectic manifold) $(\mathscr{P}, \overset{5}{\gamma})$. In the set $\mathscr{H}$ of all solutions of the field equations $(X \lrcorner \overset{5}{\gamma})|\Omega=0$ we define a pseudodifferential structure, i.e. a pseudodifferential structure in a subset of all 4-dimensional submanifolds of $\mathscr{P}$. This structure, called the structure of an "inductive differential manifold" is a generalization of the notion of an infinite dimensional manifold. It enables to define in $\mathscr{H}$ standart notions of differential geometry as: vector fields on $\mathscr{H}$, differential forms, commutators of vector fields, the exterior derivative ....

In turns out that there exists on $\mathscr{H}$ a closed differential 2-form Γ (naturally defined by $\overset{5}{\gamma}$). Using the 2-form Γ we can define a Lie algebra of physical quantities. In general the form Γ is degenerate i.e. there exists such a vector $\hat{Y} \in T_\Omega(\mathscr{H})$ that for every $\hat{X} \in T_\Omega(\mathscr{H})$

$$\Gamma(\hat{Y}, \hat{X}) = 0 . \tag{1.5}$$

A degeneration of Γ imposes restrictions on the definition of physical quantities: a physical quantity F is such a smooth functional on $\mathscr{H}$ that there exists a vector field $\hat{Y}$ on $\mathscr{H}$ such that for every vector field $\hat{X}$ on $\mathscr{H}$

$$dF(\hat{X}) = -\Gamma(\hat{Y}, \hat{X}) = -\hat{Y}^\flat(\hat{X}) . \tag{1.6}$$

For instance in electodynamics a degeneration of Γ is connected with an invariance of the Makswell equations with respect to the gradient gauge: $A_\mu \to A_\mu + \partial_\mu \varphi$ and A_μ are not physical quantities but $\bar{B}$ and $\bar{E}$ are (cf. [18]).

A subspace $W_\Omega \subset T_\Omega(\mathscr{H})$ which contains all vectors $\hat{Y}$ satisfying (1.5) is called a gauge subspace and the corresponding distribution W is called the gauge distribution of Γ. It is involutive [18].

The gauge distribution enables us to eliminate physically irrelevant variables of the theory. We can try to pass to the quotient space $\tilde{\mathscr{H}}$ such that $T(\tilde{\mathscr{H}}) = T(\mathscr{H})/W$ and then we have on $\tilde{\mathscr{H}}$ a closed nondegenerate form $\tilde{\Gamma}$.

In the present paper we apply the general theory developed in [18] to the General Relativity. In the Section 2 we construct a multiphase space $(\mathscr{P}, \overset{5}{\gamma})$ such that the Equation (1.3) is equivalent ti the system of Einstein equations:

$$\begin{aligned} \Gamma^\lambda_{\mu\nu} &= \tfrac{1}{2} g^{\lambda\sigma}(\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu}) \\ R_{\mu\nu} &- \tfrac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 0 . \end{aligned} \tag{1.7}$$

The Section 3 is devoted to a brief discussion of the Cauchy problem for General Relativity. In the Section 4 we derive an effective formula for the form Γ. If we choose the ADMW coordinate system [3, 24] connected with a given space-like surface σ in M we obtain a "diagonal" form of Γ in terms of the infinitesimal translations δg_{ij}, $\delta \pi^{ij}$, where g_{ij} is a metric tensor of σ and π^{ij} is expressed by its second fundamental form K_{ij} by the formula

$$\pi^{ij} = -\sqrt{\bar{g}}(K_{pq} - g_{pq} K_{rs} \bar{g}^{rs}) \bar{g}^{ip} \bar{g}^{jq} . \tag{1.8}$$

The diagonal form of Γ enables to give the full discussion of a gauge distribution of Γ. In the Section 5 we prove that the gauge distribution W is closely related with an invariance of the Einstein equations with respect to an action of the diffeomorphism group of the space-time M (coordinate transformations). If we divide the space of states $\mathscr{H}$ by the gauge equivalency relation, we obtain a superphase space $\tilde{\mathscr{H}}$ for General Relativity. This construction of the superphase space $\tilde{\mathscr{H}}$ justifies the proposition made by Fischer and Marsden in the paper [13], where a similar object has been proposed as a superphase space for the Einstein dynamics.

In this paper we present in a more general framework some results obtained earlier by Dirac [9], Arnowitt-Deser-Misner [3], De Witt [8], and Fadeev [11]. It seems interesting to give a detailed comparison of their results with ours. We hope to do it in another paper.

During the work on this subject the author was supported financially by the Italian Scientific Agency – Consiglio Nazionale delle Ricerche and was a guest scientifist at the University of Trieste and at International Centre for Theoretical Physics.

I would like to express my thanks to these institutions for hospitality and excellent conditions for the scientific work.

2. A Multisympletic Structure of General Relativity

The purpose of this section is to construct a manifold $\mathscr{P}$ and a closed differential 5-form γ on $\mathscr{P}$ such that γ-singular 4 dimensional submanifolds of $\mathscr{P}$ are in a one to one correspondence with the set of Einstein metrics on a given 4-dimensional, smooth manifold M. Let $\pi_1 : S^2_* T^*M \to M$ be the bundle of symmetric, 2-covariant nondegenerate tensors (metrics) on M with a negative determinant $g = \det g_{\mu\nu}$. Let $\pi_1 - \operatorname{tr} G^4(S^2_* T^*M)$ be the Grassmannian bundle of π_1-transversal planes tangent to $S^2_* T^*M$ [π_1-transversality means that for $v \in \pi_1 - \operatorname{tr} G^4(S^2_* T^*M)$, $\pi_{1*} v \neq 0$]. If (x^μ) are local coordinates in M, $(x^\mu, g_{\mu\nu})$ are local coordinates in $S^2_* T^*M$ then local coordinates in $\pi_1 - \operatorname{tr} G^4(S^2_* T^*M)$ are $(x^\mu, g_{\mu\nu}, \gamma_{\mu\nu\lambda})$, where $\gamma_{\mu\nu\lambda} = \gamma_{\nu\mu\lambda}$. They have the following transformation properties:

$$
\begin{aligned}
g_{\mu'\nu'} &= (\partial x^\mu / \partial x^{\mu'})(\partial x^\nu / \partial x^{\nu'}) g_{\mu\nu} \\
\gamma_{\mu'\nu'\lambda'} &= (\partial x^\lambda / \partial x^{\lambda'})(\partial x^\mu / \partial x^{\mu'})(\partial x^\nu / \partial x^{\nu'}) \gamma_{\mu\nu\lambda} + (\partial((\partial x^\mu / \partial x^{\mu'})(\partial x^\nu / \partial x^{\nu'})) / \partial x^{\lambda'}) g_{\mu\nu}
\end{aligned}
\tag{2.1}
$$

For purposes of General Relativity it is more convenient to introduce the bundle of Christoffel symbols Ch (bundle of the Riemannian connection) which is isomorphic (as a bundle over $S^2_* T^*M$) to the bundle $\pi_1 - \operatorname{tr} G^4(S^2_* T^*M)$. If local coordinates in Ch are $(x^\mu, g_{\mu\nu}, \Gamma^\lambda_{\mu\nu})$ with a coordinate transformation law:

$$
\Gamma^{\lambda'}_{\mu'\nu'} = (\partial x^\mu / \partial x^{\mu'})(\partial x^\nu / \partial x^{\nu'})(\partial x^{\lambda'} / \partial x^\lambda) \Gamma^\lambda_{\mu\nu} + (\partial^2 x^\sigma / \partial x^{\mu'} \partial x^{\nu'})(\partial x^{\lambda'} / \partial x^\sigma)
\tag{2.2}
$$

then an isomorphism between Ch and $\pi_1 - \operatorname{tr} G^4(S^2_* T^*M)$ is given by the formulas:

$$
\begin{aligned}
\Gamma^\lambda_{\mu\nu} &= \tfrac{1}{2} g^{\lambda\sigma}(\gamma_{\mu\sigma\nu} + \gamma_{\nu\sigma\mu} - \gamma_{\mu\nu\sigma}) \\
\gamma_{\mu\nu\lambda} &= g_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} + g_{\nu\sigma} \Gamma^\sigma_{\mu\lambda} \, .
\end{aligned}
\tag{2.3}
$$

We define $\mathscr{P} = Ch$ and

$$
\begin{aligned}
\omega = {} & g^{\alpha\beta} \sqrt{-g}\, dx^0 \wedge \ldots \wedge \underbrace{d\Gamma^\tau_{\alpha\beta}}_{\tau} \wedge \ldots \wedge dx^3 \\
& + -g^{\alpha\tau} \sqrt{-g}\, dx^0 \wedge \ldots \wedge \underbrace{d\Gamma^\beta_{\alpha\beta}}_{\tau} \wedge \ldots \wedge dx^3 \\
& + -(g^{\nu\varrho}(\Gamma^\mu_{\tau\nu} \Gamma^\tau_{\mu\varrho} - \Gamma^\mu_{\tau\mu} \Gamma^\tau_{\nu\varrho}) + 2\lambda) \sqrt{-g}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3
\end{aligned}
\tag{2.4}
$$

(λ is a real constant).

Proposition 1. *The Formula* (2.4) *defines a 4-form on* $\mathscr{P}$.

Proof. We have to check that (2.4) is covariant with respect to the coordinate transformations (2.1) and (2.2).

Definition. $\gamma = d\omega$.

$$\begin{aligned}\gamma = &\left(-(\Gamma^{\mu}_{\beta\lambda} g^{\beta\nu} + \Gamma^{\nu}_{\beta\lambda} g^{\beta\mu}) + \tfrac{1}{2}(\Gamma^{\mu}_{\beta\varrho} g^{\beta\varrho}\delta^{\nu}_{\lambda} + \Gamma^{\nu}_{\beta\varrho} g^{\beta\varrho}\delta^{\mu}_{\lambda}) + g^{\mu\nu}\Gamma^{\beta}_{\lambda\beta}\right)\\ &\cdot \sqrt{-g}\, d\Gamma^{\lambda}_{\mu\nu} \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3\\ &+ \left(\tfrac{1}{2}(g^{\alpha\mu} g^{\beta\nu} + g^{\alpha\nu} g^{\beta\mu})\delta^{\varrho}_{\lambda} - \tfrac{1}{4}(g^{\alpha\mu} g^{\beta\varrho}\delta^{\nu}_{\lambda} + g^{\alpha\nu} g^{\beta\varrho}\delta^{\mu}_{\lambda} + g^{\alpha\varrho} g^{\beta\mu}\delta^{\nu}_{\lambda} + g^{\alpha\varrho} g^{\beta\nu}\delta^{\mu}_{\lambda})\right)\\ &\cdot \sqrt{-g}\, d\Gamma^{\lambda}_{\mu\nu} \wedge dx^0 \wedge \ldots \wedge \underbrace{dg_{\alpha\beta}}_{\varrho} \wedge \ldots \wedge dx^3\\ &+ \left(-\tfrac{1}{2} g^{\alpha\beta} g^{\mu\nu}\delta^{\varrho}_{\lambda} + \tfrac{1}{4} g^{\alpha\beta}(g^{\mu\varrho}\delta^{\nu}_{\lambda} + g^{\nu\varrho}\delta^{\mu}_{\lambda})\right)\sqrt{-g}\, d\Gamma^{\lambda}_{\mu\nu} \wedge dx^0 \wedge \ldots \wedge \underbrace{dg_{\alpha\beta}}_{\varrho} \wedge \ldots \wedge dx^3\\ &+ \left((\Gamma^{\mu}_{\tau\nu}\Gamma^{\tau}_{\mu\varrho} - \Gamma^{\mu}_{\tau\mu}\Gamma^{\tau}_{\nu\varrho}) g^{\nu\alpha} g^{\varrho\beta} - \tfrac{1}{2}(\Gamma^{\mu}_{\tau\nu}\Gamma^{\tau}_{\mu\varrho} - \Gamma^{\mu}_{\tau\mu}\Gamma^{\tau}_{\nu\varrho}) g^{\nu\varrho} g^{\alpha\beta} - \lambda g^{\alpha\beta}\right)\\ &\cdot \sqrt{-g}\, dg_{\alpha\beta} \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 .\end{aligned} \tag{2.5}$$

Proposition 2. *γ is locally an exterior derivarive of a form ψ i.e. $\gamma = d\psi$ locally where:*

$$\begin{aligned}\psi = &\left(g^{\alpha\tau} g^{\beta\xi}\Gamma^{\varrho}_{\tau\xi} - \tfrac{1}{2}(g^{\alpha\tau} g^{\beta\varrho}\Gamma^{\varepsilon}_{\tau\varepsilon} + g^{\alpha\varrho} g^{\beta\tau}\Gamma^{\varepsilon}_{\tau\varepsilon}) - \tfrac{1}{2} g^{\alpha\beta} g^{\tau\xi}\Gamma^{\varrho}_{\tau\xi} + \tfrac{1}{2} g^{\alpha\beta} g^{\tau\varrho}\Gamma^{\xi}_{\tau\xi}\right)\\ &\cdot \sqrt{-g}\, dx^0 \wedge \ldots \wedge \underbrace{dg_{\alpha\beta}}_{\varrho} \wedge \ldots \wedge dx^3\\ &+ -\left(g^{\nu\varrho}(\Gamma^{\mu}_{\tau\nu}\Gamma^{\tau}_{\mu\varrho} - \Gamma^{\mu}_{\tau\mu}\Gamma^{\tau}_{\nu\varrho}) + 2\lambda\right)\sqrt{-g}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 .\end{aligned} \tag{2.6}$$

The expression (2.6) is not covariant and therefore ψ is determined only locally (in one coordinate chart). Formally (2.6) can be obtained by a general procedure from a Lagrangian function $\mathscr{L}$ (cf. [23]). We know [2] that the formula

$$\mathscr{L}(g_{\mu\nu}, \Gamma^{\lambda}_{\mu\nu}) = \left((\Gamma^{\mu}_{\tau\nu}\Gamma^{\tau}_{\mu\varrho} - \Gamma^{\mu}_{\tau\mu}\Gamma^{\tau}_{\nu\varrho}) g^{\nu\varrho} + 2\lambda\right)\sqrt{-g} \tag{2.7}$$

locally gives a non-covariant lagrangian density for Einstein equations. Hence (locally)

$$\begin{aligned}\psi = &\sum_{\mu \leqq \nu} (\partial\mathscr{L}/\partial g_{\mu\nu,\lambda}) dx^0 \wedge \ldots \wedge \underbrace{dg_{\mu\nu}}_{\lambda} \wedge \ldots \wedge dx^3\\ &- \left(\sum_{\mu \leqq \nu} (\partial\mathscr{L}/\partial g_{\mu\nu,\lambda}) g_{\mu\nu,\lambda} - \mathscr{L}\right) dx^0 \wedge \ldots \wedge dx^3\end{aligned} \tag{2.8}$$

c.f. [23] the formula 4.27.

Let $\varphi: M \to \mathscr{P}$ be a global section of the bundle $\pi: \mathscr{P} \to M$ such that $\Omega = \varphi(M)$ is a 4-dimensional embedded submanifold of $\mathscr{P}$. Let X be a π-vertical vector field tangent to $\mathscr{P}$ and defined on Ω (a vector field on Ω). We say (cf. [17]) that Ω is γ-singular if for every such X:

$$(X \lrcorner \gamma) | \Omega = 0 \tag{2.9}$$

In local coordinates we have

$$\varphi(x^{\lambda}) = \{(x^{\lambda}, g_{\mu\nu}(x^{\lambda}), \Gamma^{\tau}_{\mu\nu}(x^{\lambda}))\} \tag{2.10}$$

$$X = \sum_{\alpha \leqq \beta} Q_{\alpha\beta}\, \partial/\partial g_{\alpha\beta} + \sum_{\mu \leqq \nu} P^{\lambda}_{\mu\nu}\, \partial/\partial\Gamma^{\lambda}_{\mu\nu} \tag{2.11}$$

and the Equation (2.9) is equivalent to:

$$\Gamma^{\lambda}_{\mu\nu} = \tfrac{1}{2} g^{\lambda\sigma}(\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu}), \tag{2.12}$$

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 0 \tag{2.13}$$

where

$$\begin{aligned} &R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}, \qquad R = g^{\mu\nu} R_{\mu\nu} \\ &R^{\beta}_{\mu\alpha\nu} = \partial_\alpha \Gamma^{\beta}_{\mu\nu} - \partial_\nu \Gamma^{\beta}_{\mu\alpha} + \Gamma^{\beta}_{\tau\alpha}\Gamma^{\tau}_{\mu\nu} - \Gamma^{\beta}_{\tau\nu}\Gamma^{\tau}_{\mu\alpha}. \end{aligned} \tag{2.14}$$

In this way we have a one to one correspondence between γ-singular 4-dimensional submanifolds of $\mathscr{P}$ and solutions of the Einstein equations (2.12), (2.13). This correspondence gives us a multisymplectic structure of General Relativity cf. [17]. The Proposition 2 and the Formula (2.8) give a connection between the multisymplectic description and the classical lagrangian formulation of General Relativity. However this connection is not exactly the same as in lagrangian theories (cf. [23]), the form ψ is not determined globally.

The couple $(\mathscr{P}, \gamma)$ will be a starting point of our further considerations. γ-singular submanifolds of $\mathscr{P}$ form a prephase space (space of states in the terminology of [18]) $\mathscr{H}$ for General Relativity. It turns out that the space $\mathscr{H}$ is too large. In the sequel sections we show that $\mathscr{H}$ should be divided by an equivalency relation.

3. The Cauchy Problem and the ADMW Coordinates in General Relativity

In this section we discuss briefly the initial problem for the Einstein equations. Main results in that direction have been obtained by Lichnerowicz [20], Choquet-Bruhat [5], Choquet-Bruhat and Geroch [6], Fischer and Marsden [13, 14]. It turns out that an appropriate choice of coordinates in the space $\mathscr{P}$ is very important for a discussion of the problem. It has been shown by Arnowitt -Deser -Misner and Wheeler [3, 24] that a 3+1 decomposition of geometrical objects connected with a given space-like surface σ in M provides an elegant description of the Cauchy problem. A profound discussion of the ADMW coordinates has been recently done by Fischer and Marsden [13]. We shall describe briefly those coordinates.

Let $(g_{\mu\nu})$ be a metric tensor on M having a signature $(-,+,+,+)$. Let σ be a 3-dimensional surface in M which is space-like. We assume that there exists a neighbourhood $\mathcal{O}$ of σ in M and local coordinates (x^0, x^1, x^2, x^3) in $\mathcal{O}$ having the transformation properties:

$$\begin{aligned} &x^{0'} = x^{0'}(x^0, x^k); \qquad x^{s'} = x^{s'}(x^k); \qquad x^{0'}(0, x^k) = 0 \\ &(\partial x^{0'}/\partial x^0)(0, x^k) = 1 \quad \text{and} \quad \sigma = \{x : x^0 = 0\}. \end{aligned} \tag{3.1}$$

Because σ is space-like the 3×3 tensor g_{ij} is positively defined and has a positively defined inverse tensor $\bar{g}^{ij}$. It is easy to prove that $g^{00} < 0$ and we can define:

$$N = (\sqrt{-g^{00}})^{-1}; \qquad N_k = g_{0k}; \qquad N^k = \bar{g}^{ks} N_s. \tag{3.2}$$

It follows by (3.1) that N is a scalar function on σ and that N^k, N_k are components of a vector (covector) field on σ. We call N a lapse function and N^k a shift vector (cf. [24]). We have formulas:

$$g^{0k}=N^k/N^2\,;\qquad g_{00}=-N^2+N^kN_k\,;\qquad g^{sp}=\bar{g}^{sp}-N^sN^p/N^2\,. \tag{3.3}$$

$$\sqrt{-g}=N\sqrt{\bar{g}},\quad \text{where}\quad g=\det g_{\mu\nu}\,,\qquad \bar{g}=\det g_{ij}\,. \tag{3.4}$$

The second fundamental form of σ is defined by (cf. [19]):

$$K_{ij}=-g_{j\mu}\nabla_i n^\mu \tag{3.5}$$

where $n^\mu=(1,-N^k)N^{-1}$ is a unit normal vector to σ. Let

$$\bar{\Gamma}^k_{ij}=\tfrac{1}{2}\bar{g}^{ka}(\partial_j g_{ia}+\partial_i g_{ja}-\partial_a g_{ij})$$

then

$$\begin{aligned}
&\Gamma^k_{rs}=\bar{\Gamma}^k_{rs}-N^k\Gamma^0_{rs}\\
&\Gamma^k_{r0}=N^2\bar{g}^{kp}\Gamma^0_{rp}+\bar{\nabla}_r N^k-N^kN^p\Gamma^0_{rp}-N^k\cdot N^{-1}\partial_r N\\
&\Gamma^\lambda_{p\lambda}=\partial_p N/N+\bar{\Gamma}^s_{ps}\\
&\Gamma^0_{p0}=N^s\Gamma^0_{ps}+\partial_p N/N
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
&\partial_0 g_{ij}=\bar{\nabla}_i N_j+\bar{\nabla}_j N_i+2N^2\Gamma^0_{ij}\\
&\partial_k g_{00}=2(-N\partial_k N+N^p\bar{\nabla}_k N_p)
\end{aligned} \tag{3.7}$$

where $\bar{\nabla}_r$ is the covariant derivative with respect to the affinity $\bar{\Gamma}^k_{ij}$. By (3.5) and (3.6) we have

$$K_{ij}=-N\Gamma^0_{ij} \tag{3.8}$$

Let

$$\pi^{ij}=-\sqrt{\bar{g}}(K_{pq}-g_{pq}K_{rs}\bar{g}^{rs})\bar{g}^{pi}\bar{g}^{qj} \tag{3.9}$$

then

$$\pi^{ij}=\sqrt{-g}(\Gamma^0_{pq}-g_{pq}\Gamma^0_{rs}\bar{g}^{rs})\bar{g}^{pi}\bar{g}^{qj} \tag{3.10}$$

$$\Gamma^0_{pq}=(-g)^{-1/2}(g_{ip}g_{jq}\pi^{ij}-\tfrac{1}{2}g_{pq}\,\mathrm{tr}\,\pi),\quad \text{where}\quad \mathrm{tr}\,\pi=g_{ij}\pi^{ij} \tag{3.11}$$

Now we shall express the Einstein equations in terms of g_{ij}, π^{ij}, N_k, N. It is known [2] that the system of the Einstein equations

$$G_{\mu\nu}=R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R+\lambda g_{\mu\nu}=0 \tag{3.12}$$

is equivalent to the system:

$$R_{ks}=\lambda g_{ks}\,, \tag{3.13a}$$

$$G^0_\mu=0\,. \tag{3.13b}$$

The Einstein tensor $G_{\mu\nu}$ satisfies always the contracted Bianchi identities

$$\nabla_\nu G^\nu_\mu=0 \tag{3.14}$$

It follows by (3.14) that the system (3.13) is equivalent to the system:

$$R_{ks}=\lambda g_{ks}\,, \tag{3.15a}$$

$$G^0_\mu|\sigma=0\,. \tag{3.15b}$$

In local coordinates of the type (3.1) G^0_μ does not depend on $g_{0\nu,0}$, $g_{ij,00}$. Therefore G^0_μ on σ depends only on g_{ij}, π^{ij}, N_k, N and their spatial derivatives. In this way the system (3.15) consists of 6 dynamical Equations (3.15a) and four conditions (3.15b) on initial data.

In the ADMW coordinates the system (3.15) reads (cf. [3]):

$$\begin{aligned}\partial_0\pi^{ij}= & -N\sqrt{\bar g}(\bar R^{ij}-\bar g^{ij}\bar R)-2N\bar g^{-1/2}(\pi^i_q\pi^{qj}-\tfrac{1}{2}\operatorname{tr}\pi\cdot\pi^{ij})\\ & +\sqrt{\bar g}(\bar\nabla^i\bar\nabla^j N-\bar g^{ij}\bar\nabla^s\bar\nabla_s N)+\bar\nabla_u(N^u\pi^{ij})\\ & +-\bar\nabla_s N^i\pi^{sj}-\bar\nabla_s N^j\pi^{si}-2\lambda N\sqrt{\bar g}\bar g^{ij}\,,\end{aligned} \tag{3.16a}$$

$$\bar\nabla_i\pi^{ij}=0 \qquad \text{on } \sigma\,, \tag{3.16b$'$}$$

$$\bar R-2\lambda-\bar g^{-1}(\pi_{pq}\pi^{pq}-\tfrac{1}{2}(\operatorname{tr}\pi)^2)=0 \qquad \text{on } \sigma\,. \tag{3.16b$''$}$$

If we choose g_{ij} and π^{ij} on σ such that the Equations (3.16b) are satisfied, we can look for a solution of (3.16a) with those initial values. However the Equations (3.16a) do not contain time derivatives of N_k, N and therefore N_k, N have to be chosen arbitrary not only on σ but also beyond it. If N_k, N are chosen in a neighbourhood of σ in M then there exists a unique solution of (3.16a) satisfying the Cauchy data g_{ij}, π^{ij} on σ. For details see [5, 6, 13, 14, 20].

4. A Symplectic Structure of the Set of Einstein Metrics

Let λ be a real number (a cosmological constant). We consider the space of states $\mathscr{H}(M,\lambda)$ (or briefly $\mathscr{H}$) i.e. an infinite dimensional space of all γ-singular sections of the bundle $\pi:\mathscr{P}\to M$. According to the results of the section 2 γ-singular submanifolds of $\mathscr{P}$ are in a one to one correspondence with the set of Einstein metrics i.e. metrics fulfilling the Einstein equations

$$R_{\mu\nu}=\lambda g_{\mu\nu}\,.$$

In this section we prove that the set $\mathscr{H}(M,\lambda)$ has a natural (pre)-symplectic structure. At any point $\Omega\in\mathscr{H}$ we define a vector space $T_\Omega(\mathscr{H})$ tangent to $\mathscr{H}$ at Ω and a bilinear skewsymmetric map $\Gamma: T_\Omega(\mathscr{H})\times T_\Omega(\mathscr{H})\to\mathbb{R}$. Moreover we define a notion of the exterior derivative of the differential 2-form Γ. It turns out that $d\Gamma=0$. The definition of $T_\Omega(\mathscr{H})$ and Γ follows the paper [18], in which Kijowski and the author have elaborated a general approach to any field theory with a multisymplectic structure $(\mathscr{P},\gamma)$. We do not need a differentiable structure in the set $\mathscr{H}$. This set has a natural pseudodifferential structure of so called "inductive differential manifold". Axioms of that theory have been given in [18]. A detailed discussion of them for General Relativity will be done in another paper. We recall that according to [18] a vector $\hat X$ tangent to $\mathscr{H}$ at Ω can be represented by a π-vertical vector field X tangent to $\mathscr{P}$ and defined on Ω (a π-vertical vector field on Ω), which additionally satisfies some system of linear differential equations.

If

$$X=\sum_{\mu\leqq\nu}\delta g_{\mu\nu}\partial/\partial g_{\mu\nu}+\sum_{\mu\leqq\nu}\delta\Gamma^{\lambda}_{\mu\nu}\partial/\partial\Gamma^{\lambda}_{\mu\nu} \tag{4.1}$$

then $\delta g_{\mu\nu}$, $\delta\Gamma^{\lambda}_{\mu\nu}$ satisfy equations:

$$\begin{aligned}\delta\Gamma^{\lambda}_{\mu\nu}&=\delta(\tfrac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma}+\partial_\nu g_{\mu\sigma}-\partial_\sigma g_{\mu\nu}))\\&=\tfrac{1}{2}g^{\lambda\sigma}(\nabla_\mu\delta g_{\nu\sigma}+\nabla_\nu\delta g_{\mu\sigma}-\nabla_\sigma\delta g_{\mu\nu})=0\,,\end{aligned} \tag{4.2a}$$

$$\delta(R_{\mu\nu}-\lambda g_{\mu\nu})=\sum_{\alpha\leqq\beta}(\partial R_{\mu\nu}/\partial g_{\alpha\beta})\delta g_{\alpha\beta}+\sum_{\alpha\leqq\beta}(\partial R_{\mu\nu}/\partial\Gamma^{\tau}_{\alpha\beta})\delta\Gamma^{\tau}_{\alpha\beta}-\lambda\delta g_{\mu\nu}=0\,. \tag{4.2b}$$

A vector field X on Ω satisfying (4.2) transforms infinitesimaly the solution Ω of the Einstein equations into a solution of the Einstein equations.

Let φ be a γ-singular section of $\pi:\mathscr{P}\to M$ which corresponds to an Einstein metric $g_{\mu\nu}$. Let σ be a 3-dimensional space-like surface in M and $c=\varphi(\sigma)$. For any two tangent vectors $\hat{X}_1, \hat{X}_2$ at $\Omega=\varphi(M)$ we define:

$$\Gamma(\hat{X}_1,\hat{X}_2)=\int_c(X_1\wedge X_2)\lrcorner\gamma=\int_c X_2\lrcorner X_1\lrcorner\gamma \tag{4.3}$$

where vector fields X_1, X_2 on Ω represent the vectors $\hat{X}_1, \hat{X}_2$.

Remarks. 1. If σ is properly chosen a submanifold $c=\varphi(\sigma)\subset\Omega$ is called an admissible initial surface in $\mathscr{P}$ (cf. [17]). We know (Sect. 3) that through such a submanifold pass many solutions of the Einstein equations.

2. To provide a convergence of the integral in (4.3) we have to assume that fields X_1, X_2 have compact supports on c. Of course, one can consider X_1, X_2 having no compact supports on c but impose some vanishing conditions at the "spatial infinity".

3. It has been proved in [18] that if $c=\varphi(\sigma)$ is an admissible initial surface then the integral in (4.3) does not depend on a particular choice of a space-like surface σ in M.

Proposition 3. *The form Γ is closed i.e. $d\Gamma=0$.*

For a definition of the operator d and a proof of the proposition see [18].

We shall now express the formula (4.3) in local coordinates of the type (3.1) which are determined by a lapse function N on σ and a shift covector N_k on σ (cf. [3, 13, 24]). If $\sigma=\{x:x^0=0\}$ then submanifolds $\sigma_t=\{x:x^0=t\}-\varepsilon<t<\varepsilon$ are also spacelike (at least in a neighbourhood of a compact subset of σ). Therefore we can use the coordinates $(N, N_k, g_{ij}, \pi^{ij}, M_\mu, M_{\mu k}, \bar{\Gamma}^j_{ks})$ (where $M_\mu=\partial_\mu N$, $M_{\mu k}=\partial_\mu N_k$) in a neighbourhood of a compact subset of $c=\varphi(\sigma)\subset\mathscr{P}$. A connection between the coordinates $(g_{\mu\nu}, \Gamma^{\lambda}_{\mu\nu})$ and $(N, N_k, g_{ij}, \pi^{ij}, M_\mu, M_{\mu k}, \bar{\Gamma}^j_{ks})$ is given by the formulas (3.4), (3.6), (3.7), (3.10), (3.11).

A π-vertical vector field X on Ω representing a vector $\hat{X}$ tangent to $\mathscr{H}$ at Ω has in these coordinates a form:

$$\begin{aligned}X=\delta N\partial/\partial N+\delta N_k\partial/\partial N_k+\sum_{i\leqq j}\delta g_{ij}\partial/\partial g_{ij}+\sum_{i\leqq j}\delta\pi^{ij}\partial/\partial\pi^{ij}+\delta M_\mu\partial/\partial M_\mu\\+\delta M_{\mu k}\partial/\partial M_{\mu k}+\sum_{k\leqq s}\delta\bar{\Gamma}^j_{ks}\partial/\partial\bar{\Gamma}^j_{ks}\,,\end{aligned} \tag{4.4}$$

with conditions [cf. (3.7)]:

$$\delta M_\mu = \partial_\mu \delta N\,, \qquad \delta M_{\mu k} = \partial_\mu \delta N_k\,, \tag{4.5}$$

$$\delta \bar{\Gamma}^j_{ks} = \tfrac{1}{2}\bar{g}^{ja}(\bar{V}_k \delta g_{sa} + \bar{V}_s \delta g_{ka} - \bar{V}_a \delta g_{ks}) \tag{4.6}$$

$$\partial_0 \delta g_{ij} = \delta(\bar{V}_i N_j + \bar{V}_j N_i + (2N/\sqrt{\bar{g}}(g_{ip} g_{jq} \pi^{pq} - \tfrac{1}{2} g_{ij} \operatorname{tr} \pi))\,. \tag{4.7}$$

The equations (4.5), (4.6), (4.7) form a set of kinematical conditions. We have also a set of dynamical conditions obtained by a linearization of the Equations (3.16):

$$\begin{aligned} \partial_0 \delta \pi^{ij} = {} & \delta(-N\sqrt{\bar{g}}(\bar{R}^{ij} - \bar{g}^{ij}\bar{R}) - (2N/\sqrt{\bar{g}})(\pi^i_q \pi^{qj} - \tfrac{1}{2} \operatorname{tr} \pi \pi^{ij})) \\ & + \delta(\sqrt{\bar{g}}(\bar{V}^i \bar{V}^j N - \bar{g}^{ij} \bar{V}^s \bar{V}_s N) + \bar{V}_u(N^u \pi^{ij})) \\ & + \delta(-\bar{V}_s N^i \pi^{sj} - \bar{V}_s N^j \pi^{si} - 2\lambda N \sqrt{\bar{g}}\bar{g}^{ij})\,, \end{aligned} \tag{4.8a}$$

$$\delta(\bar{V}_j \pi^{ij}) = \bar{V}_j \delta \pi^{ij} + \delta \bar{\Gamma}^i_{ks} \pi^{ks} = 0 \qquad \text{on } \sigma\,, \tag{4.8b'}$$

$$\delta(\bar{R} - 2\lambda - \bar{g}^{-1}(\pi_{pq}\pi^{pq} - \tfrac{1}{2}(\operatorname{tr}\pi)^2) = 0 \qquad \text{on } \sigma\,. \tag{4.8b''}$$

Combining results of the Section 3 with the above formulas we conclude that a vector $\hat{X}$ tangent to $\mathscr{H}$ at Ω determines 12 quantities $(\delta\pi^{ij}, \delta g_{ij})$ on $c = \varphi(\sigma)$ (or equivalently on $\sigma \subset M$), which satisfy the constraint Equations (4.8b) and 4 arbitrary quantities δN, δN_k given in a neighbourhood of c in Ω (or in a neighbourhood of σ in M). Conversely, if we have on $c = \varphi(\sigma)$ 12 quantities $(\delta\pi^{ij}, \delta g_{ij})$ fulfilling the Equations (4.8b) and 4 arbitrary quantities δN, δN_k given in a neighbourhood of c in Ω, we have a uniquely determined vector field X on an open subset of Ω. We obtain its components solving Equations (4.7), (4.8a) with the Conditions (4.5) and (4.6). Of course, a problem arises, whether X can be extended on the whole Ω such that (4.2) hold. This is the problem of finding of a global solution of the linearized Einstein equations.

The following subspace of $T_\Omega(\mathscr{H})$ plays an important role in our considerations:

Definition. $\mathring{T}_{(\Omega,c)}(\mathscr{H})$ is a linear subspace of $T_\Omega(\mathscr{H})$ consisting of these $\hat{Y} \in T_\Omega(\mathscr{H})$ that there exists a vector field Y on Ω of the form (4.4) such that:

1. $\delta N, \delta N_k, \delta M_0, \delta M_{0k}$ are arbitrary on c,
2. $\delta M_k = \partial_k \delta N, \delta M_{sk} = \partial_s \delta N_k$ on c,
3. $\delta\pi^{ij} = 0\,, \delta g_{ij} = 0$ on c. (4.9)

Remark. Let us notice that the Conditions (4.9) are consistent with (4.5) and (4.8b).

Proposition 4. *For any $\hat{X} \in T_\Omega(\mathscr{H})$ and $\hat{Y} \in \mathring{T}_{(\Omega,c)}(\mathscr{H})$ we have $\Gamma(\hat{X}, \hat{Y}) = 0$.*

Proof in the Section 7.

We see that the subspace $\mathring{T}_{(\Omega,c)}(\mathscr{H})$ belongs to a gauge distribution of the form Γ. It is connected with the fact that the Einstein equations do not determine N, N_k by their initial values (cf. the Sec. 3). That in turn is related to an invariance of the Einstein equations with respect to an action of the diffeomorphism group of the space-time M.

The main result of this section is a "diagonal" expression for the 2-form Γ in the ADMW coordinate system:

Theorem 1. *Let $\hat{X}_1, \hat{X}_2 \in T_\Omega(\mathscr{H})$ be represented by vector fields X_1, X_2 of the form* (4.4) *then:*

$$\Gamma(\hat{X}_1, \hat{X}_2) = \int_c (\underset{1}{\delta}\pi^{ks}\underset{2}{\delta}g_{ks} - \underset{2}{\delta}\pi^{ks}\underset{1}{\delta}g_{ks})dx^1 \wedge dx^2 \wedge dx^3 \tag{4.10}$$

Proof in the Section 7.

The Theorem 1 shows that entities π^{ks} are in some sense conjugate to the spatial components of a metric tensor $g_{\mu\nu}$. However we must remember that π^{ks} and g_{ks} are not independent, they fulfil constraint equations (3.16b). These four equations are an essential feature of the theory. In the next section we show that they determine a gauge distribution of the form Γ i.e. such a maximal linear subspace $W_\Omega \subset T_\Omega(\mathscr{H})$ that for every $\hat{Y} \in W_\Omega$ and $\hat{X} \in T_\Omega(\mathscr{H})$, $\Gamma(\hat{Y}, \hat{X}) = 0$.

The Proposition 4 is a first step in that direction.

5. The Gauge Distribution and an Action of the Diffeomorphism Group

It is known [1, 22] that the symplectic 2-form $\overset{2}{\gamma} = \sum_{i=1}^{n} dp_i \wedge dq^i$ in mechanics is non-degenerate and provides an isomorphism between the tangent and the cotangent space of the phase space $\mathscr{P}_{2n}$. This isomorphism plays an essential role in the definition of physical quantities as functions on $\mathscr{P}_{2n}$. It has been shown in [18] that in general the 2-form Γ is degenerate. In the present section we investigate the gauge distribution of Γ. As we can expect the gauge distribution of Γ is closely related to an invariance of the Einstein equations with respect to an action of the diffeomorphism group of the space time M. In the language of the classical physics that diffeomorphism group action is called "a change of coordinates".

Definition. The gauge distribution

$$W_\Omega = \{\hat{Y} \in T_\Omega(\mathscr{H}) : \Gamma(\hat{Y}, \hat{X}) = 0 \quad \text{for every} \quad \hat{X} \in T_\Omega(\mathscr{H})\} \tag{5.1}$$

Definition. $\overset{1}{T}_\Omega(\mathscr{H})$ is a subspace of $T_\Omega(\mathscr{H})$ consisting of all vectors $\hat{Y}$ which are represented by a vector field Y on Ω such that $\delta N = 0$, $\delta N_k = 0$ on Ω and $\delta\pi^{ij}$, δg_{ij} are arbitrary on c fulfilling only (4.8b).

Remark. The definition of $\overset{1}{T}_\Omega(\mathscr{H})$ does not depend on a choice of an admissible initial surface $c \subset \Omega$ (for such c that $\pi(c) = \{x \in M : x^0 = \text{const}\}$).

The constraint Equations (4.8b) do not contain entities δN, δN_k and therefore each $\hat{X} \in T_\Omega(\mathscr{H})$ can be uniquely decomposed into $X_1 \in \overset{1}{T}_\Omega(\mathscr{H})$ and $X_2 \in \overset{\circ}{T}_{(\Omega, c)}(\mathscr{H})$, i.e.

$$T_\Omega(\mathscr{H}) = \overset{1}{T}_\Omega(\mathscr{H}) \oplus \overset{\circ}{T}_{(\Omega, c)}(\mathscr{H}) \quad \text{(a direct sum)}. \tag{5.2'}$$

The decomposition (5.2') together with the proposition 4 allow to consider only vectors belonging to the subspace $\overset{1}{T}_\Omega(\mathscr{H})$.

For a given space-like surface $\sigma \subset M$ (or equivalently for an admissible initial surface $c \subset \Omega$) we define $C_\sigma = C^\infty(\operatorname{den} S^2 T(\sigma) \oplus S^2 T^*(\sigma))$, i.e. a vector space consisting of pairs $(\delta\pi^{ij}, \delta g_{ij})$ where $\delta\pi^{ij}$ is a symmetric 2-contravariant tensor density on σ and δg_{ij} is a symmetric 2-covariant tensor on σ. We know that a vector $\hat{X}$ tangent to $\mathscr{H}$ at Ω determines an element $\underset{1}{\mathfrak{X}}\in C_\sigma$ and every element $(\delta\pi^{ij}, \delta g_{ij})\in C_\sigma$ which fulfils (4.8b) represents a vector $\hat{X}\in T_\Omega(\mathscr{H})$.

For any $\Omega\in\mathscr{H}$ we define a scalar product on C_σ:

$$g_{(\sigma,\Omega)}((\underset{1}{\delta\pi^{ij}}, \underset{1}{\delta g_{ij}}), (\underset{2}{\delta\pi^{ij}}, \underset{2}{\delta g_{ij}})) = \int_\sigma (\bar{g}^{-1/2}\underset{1}{\delta\pi^{ij}} g_{ip}g_{jq}\underset{2}{\delta\pi^{pq}} + \sqrt{\bar{g}}\underset{1}{\delta g_{ij}}\bar{g}^{ip}\bar{g}^{jq}\underset{2}{\delta g_{pq}})dx^1\wedge dx^2\wedge dx^3 \tag{5.3}$$

and an operator $J: C_\sigma\to C_\sigma$

$$J(\delta\pi^{ij}, \delta g_{ij}) = (\sqrt{\bar{g}}\delta g_{pq}\bar{g}^{pi}\bar{g}^{qj}, -(\bar{g})^{-1/2}\delta\pi^{pq}g_{pi}g_{qj}). \tag{5.4}$$

It is easy to see that $J^2=-\mathrm{id}$.

The scalar product $g_{(\sigma,\Omega)}$ defines a scalar product $\tilde{g}_{(\sigma,\Omega)}$ in $T_\Omega(\mathscr{H})$

$$\tilde{g}_{(\sigma,\Omega)}(\hat{X}_1, \hat{X}_2) = g_{(\sigma,\Omega)}(\mathfrak{X}_1, \mathfrak{X}_2) \tag{5.5}$$

For $\hat{X}_1, \hat{X}_2\in T_\Omega(\mathscr{H})$ we have by (4.10), (5.3), and (5.4)

$$-g_{(\sigma,\Omega)}(J\mathfrak{X}_1, \mathfrak{X}_2) = \Gamma(\hat{X}_1, \hat{X}_2) = +g_{(\sigma,\Omega)}(\mathfrak{X}_1, J\mathfrak{X}_2). \tag{5.6}$$

Remark. The definition of $g_{(\sigma,\Omega)}$ and J depend on a choice of $\sigma\subset M$. For a given $\Omega\in\mathscr{H}$ [i.e. if (π^{ij}, g_{ij}) satisfy (3.16b)] we have a differential operator generated by the constraint Equations (4.8b):

$$C_\sigma\ni\mathfrak{X}\to A\mathfrak{X}\in C^\infty(T(\sigma)\oplus\mathbb{R}).$$

If $\mathfrak{X}=(\delta\pi^{ij}, \delta g_{ij})$ then:

$$A\mathfrak{X} = (\bar{g}^{-1/2}(\bar{\nabla}_j\delta\pi^{ij} + \delta\bar{\Gamma}^i_{ks}\pi^{ks}), \delta\bar{R} + \bar{g}^{-1}(\bar{R}-2\lambda)\delta\bar{g} - \bar{g}^{-1}\delta(\pi^{ks}\pi_{ks} - \tfrac{1}{2}(\mathrm{tr}\,\pi)^2)) \tag{5.7}$$

where

$$\begin{aligned} \bar{\nabla}_j\delta\pi^{ij} &= \partial_j\delta\pi^{ij} + \bar{\Gamma}^i_{js}\delta\pi^{js} \\ \delta\bar{\Gamma}^k_{ij} &= \tfrac{1}{2}\bar{g}^{ka}(\bar{\nabla}_j\delta g_{ia} + \bar{\nabla}_i\delta g_{ja} - \bar{\nabla}_a\delta g_{ij}) \\ \delta\bar{R} &= -\bar{R}^{pq}\delta g_{pq} + \bar{\nabla}^j\bar{\nabla}^k\delta g_{jk} - \bar{\nabla}^k\bar{\nabla}_k\delta g_{ij}\bar{g}^{ij} \\ \delta\bar{g} &= \bar{g}\bar{g}^{jk}\delta g_{jk}. \end{aligned} \tag{5.8}$$

The vector space $C^\infty(T(\sigma)\oplus\mathbb{R})$ consists of pairs $U=(u^j,\chi)$, where u^j is a vector field on σ and χ is a scalar function on σ. It has a natural scalar product:

$$g_{1(\sigma,\Omega)}((\underset{1}{u^j}, \underset{1}{\chi}), (\underset{2}{u^j}, \underset{2}{\chi})) = \int_\sigma (\underset{1}{u_j}\underset{2}{u^j}\sqrt{\bar{g}} + \underset{1}{\chi}\underset{2}{\chi}\sqrt{\bar{g}})dx^1\wedge dx^2\wedge dx^3 \tag{5.9}$$

where $u_j = g_{js}u^s$.

By means of the scalar products (5.3) and (5.9) we define the adjoint operator $A^*: C^\infty(T(\sigma)\oplus\mathbb{R})\to C_\sigma$

$$g_{(\sigma,\Omega)}(A^*U, \mathfrak{X}) = g_{1(\sigma,\Omega)}(U, A\mathfrak{X})\,; \qquad U\in C^\infty(T(\sigma)\oplus\mathbb{R}),\ \mathfrak{X}\in C_\sigma. \tag{5.10}$$

Integrating (5.10) by parts we have:

$$A^*(u^j,\chi) = (\delta\pi^{ij}, \delta g_{ij}),$$

where:

$$\begin{aligned}\delta\pi^{ij} &= -\tfrac{1}{2}(\bar{\nabla}^i u^j + \bar{\nabla}^j u^i)\sqrt{\bar{g}} - 2(\pi^{ij} - \tfrac{1}{2}\operatorname{tr}\pi\bar{g}^{ij})\chi \\ \delta g_{ij} &= -(2\sqrt{\bar{g}})^{-1}(\pi_{ai}\bar{\nabla}^a u_j + \pi_{aj}\bar{\nabla}^a u_i - \bar{\nabla}_a(\pi_{ij}u^a)) \\ &\quad + -2\bar{g}^{-1}(\pi_{is}\bar{g}^{sk}\pi_{kj} - \tfrac{1}{2}\operatorname{tr}\pi\pi_{ij})\chi + \bar{\nabla}_i\bar{\nabla}_j\chi - g_{ij}\bar{\nabla}^k\bar{\nabla}_k\chi - \bar{R}_{ij}\chi \\ &\quad + g_{ij}(\bar{R} - 2\lambda)\chi\,. \end{aligned} \tag{5.11}$$

Remark. The definition of the operators A, A^* depends on a choice of a state $\Omega\in\mathscr{H}$ and on a choice of a space-like surface $\sigma\subset M$.

Proposition 5. $\operatorname{im} JA^*\subset\ker A$.

The proof in the Section 7.

The Proposition 5 allows us to construct vector fields on Ω, which represent elements of $\overset{1}{T}_\Omega(\mathscr{H})$. Indeed, every $\mathfrak{X}\in\operatorname{im} JA^*$ generates such a field.

Definition. $\overset{\circ}{W}_{(\Omega,c)}\subset\overset{1}{T}_\Omega(\mathscr{H})$ consists of such vectors $\hat{Y}\in\overset{1}{T}_\Omega(\mathscr{H})$ which are represented by vector fields on Ω generated by $\operatorname{im} JA^*$ (5.12)

Proposition 6. $\overset{\circ}{W}_{(\Omega,c)}\subset W_\Omega$.

Proof. This proposition follows immediately from (5.6) and the orthogonality of $\ker A$ and $\operatorname{im} A^*$.

Definition. $\overset{\circ}{W}_\Omega=\overset{\circ}{W}_{(\Omega,c)}\oplus\overset{\circ}{T}_{(\Omega,c)}$ (a direct sum) (5.13)

Remark. $\overset{\circ}{W}_\Omega$ does not depend on a choice of $c\subset\Omega(\sigma\subset M)$. Combining the Propositions 4 and 6 we have:

Proposition 7. $\overset{\circ}{W}_\Omega\subset W_\Omega$.

We cannot assert that $\overset{\circ}{W}_\Omega=W_\Omega$ because we do not know whether $\ker A\oplus\operatorname{im} A^*$ is equal to the whole space C_σ. It is certainly true when σ is compact.

Proposition 8. *If $\sigma\subset M$ is compact then for every $\Omega\in\mathscr{H}$ (for which σ is space-like)*

$$C^\infty(\operatorname{den} S^2T(\sigma)\oplus S^2T^*(\sigma))=\ker A\oplus\operatorname{im} A^* \quad \text{(an orthogonal sum)}\,. \tag{5.14}$$

The proof of the proposition is based on the theory of differential operators with injective symbols ([21, 4]) and is given in the Section 7. It seems that (5.14) can be also proved in a non-compact space, but we have to impose appropriate boundary conditions.

Corollary 1. *If* (5.14) *is fulfilled then:*

$$\ker A=(\ker A\cap\ker AJ)\oplus\operatorname{im} JA^* \quad \text{(an orthogonal sum)}\,. \tag{5.15}$$

For the proof see the Section 7.

Corollary 2. $\overset{\circ}{W}_\Omega=W_\Omega$.

$$T_\Omega(\mathscr{H})=F_\Omega\oplus W_\Omega \quad \text{(a direct sum)} \tag{5.16}$$

where F_Ω is a subspace of $\overset{1}{T}_\Omega(\mathscr{H})$ generated by elements belonging to $\ker A\cap\ker AJ$.

The subspace $F_\Omega \subset T_\Omega(\mathscr{H})$ determines degrees of freedom for the gravitational field (cf. the Sec. 6).

We shall explain now a connection between the gauge distribution W and an action of the diffeomorphism group of the manifold M in the space $\mathscr{H}$. Let Diff(M) be the diffeomorphism group of M. This group acts on the right on the set of all Riemannian metrics on M with a signature $(-,+,+,+)$

$$\mathrm{Diff}(M)\times S^2_L TM \ni (\varphi, g)\to R_\varphi g = \varphi^* g \in S^2_L T^*M\,. \tag{5.17}$$

The action (5.17) can be naturally extended on the bundle $\mathscr{P}$ and the multisymplectic form γ, defined by (2.5) is invariant with respect to this action. Therefore we can define an action of Diff(M) in the space $\mathscr{H}$:

$$\mathrm{Diff}(M)\times \mathscr{H} \ni (\varphi, \Omega)\to \hat{R}_\varphi(\Omega) = \varphi^*\Omega \in \mathscr{H}\,. \tag{5.18}$$

[In local coordinates if $R_{\mu\nu}(g)=\lambda g_{\mu\nu}$ then $R_{\mu\nu}(\varphi^*g)=\lambda(\varphi^*g)_{\mu\nu}$.] It is known [10] that the Lie algebra of the group Diff(M) can be identified with the Lie algebra of smooth vector fields on M (with the commutator as a Lie bracket). Therefore the action (5.18) generates an action in Lie algebras:

$$C^\infty(TM)\ni v \to d\hat{R}_{\mathrm{id}}(\Omega)v \in T_\Omega(\mathscr{H})\,, \tag{5.19}$$

where $d\hat{R}_{\mathrm{id}}(\Omega)$ is the derivative of (5.18) with respect to φ at the point $(\mathrm{id}, \Omega)\in \mathrm{Diff}(M)\times\mathscr{H}$.

It turns out, that the image of the map (5.19) is equal to the subspace $\mathring{W}_\Omega$ defined by (5.13).

Proposition 9. $\mathrm{im}\, d\hat{R}_{\mathrm{id}}(\Omega) = \mathring{W}_\Omega$.

For a proof see the Section 7.

If the manifold M has compact spatial sections i.e. if admissible initial surfaces in $\mathscr{P}$ are compact then combining results of the Propositions 8 and 9 we have:

Theorem 2. *For a manifold M with compact spatial sections*

$$\mathrm{im}\, d\hat{R}_{\mathrm{id}}(\Omega) = W_\Omega\,.$$

In this case the gauge distribution of Γ is fully determined by the action (5.18).

6. Degrees of Freedom and a Superphase Space for General Relativity

The space $\mathscr{H}$ introduced in previous sections is too large for a description of the dynamics in General Relativity. The action (5.18) devides $\mathscr{H}$ into equivalency classes. At first we shall discuss this problem locally in terms of the tangent bundle $T(\mathscr{H})$. A complete discussion is possible if admissible initial surfaces in $\mathscr{P}$ are compact. Then using Corollary 2 of Proposition 8 we conclude that only the subspace F_Ω is of a real interest. Therefore for a given admissible initial surface $c\subset\Omega$ we can assign 12 quantities $(\delta\pi^{ij}, \delta g_{ij})$, where $\delta\pi^{ij}$ is a 2-contravariant tensor density on $\sigma=\pi(c)$ and δg_{ij} is a 2-covariant tensor field on σ. These quantities fulfil 8 linear differential equations: 4 constraint Equations (4.8b) and 4 equations obtained from (4.8b) by the transformation:

$$\begin{aligned}&\delta\pi^{ij}\to \sqrt{\bar{g}}\,\bar{g}^{pi}\bar{g}^{qj}\delta g_{pq}\\ &\delta g_{ij}\to -(\bar{g})^{-1/2} g_{pi} g_{qj}\delta\pi^{pq}\,.\end{aligned} \tag{6.1}$$

In this sense we say that there are 4 independent degrees of freedom for the gravitational field.

Let us discuss briefly a global problem. We consider a 3-dimensional compact submanifold σ of M and a subset $S^2_{(E,\sigma)}T^*M$ of Einstein metrics on M for which σ is space-like. We assume also that σ determines correctly the Cauchy problem i.e. for any $g \in S^2_{(E,\sigma)}T^*M$ the natural lift of σ to $\mathscr{P}$ is an admissible initial surface. If $\mathcal{O}$ is a sufficiently small neighbourhood of identity in Diff(M) (in a suitable topology) we have an action:

$$\mathcal{O} \times S^2_{(E,\sigma)}T^*M \ni (\varphi, g) \to R_\varphi g = \varphi^* g \in S^2_{(E,\sigma)}T^*M\,. \tag{6.2}$$

One can divide $S^2_{(E,\sigma)}T^*M$ by the action (6.2) to obtain a superphase space for General Relativity. If we describe $g_{\mu\nu}$ in terms of g_{ij}, π^{ij} on σ (on c) and N, N_k on M (on Ω) then we do not know whether the action (6.2) allows to change N, N_k in an arbitrary way. We know only that it is so in the infinitesimal case (Prop. 9). Therefore it is better to define a superphase space axiomatically.

Let the Cauchy data Cd be a subset of $\operatorname{den} S^2T(\sigma) \oplus S^2_+ T^*(\sigma)$ consisting of pairs (π^{ij}, g_{ij}) which fulfil constraint Equations (3.16b). We define an action of $\mathcal{O} \subset \operatorname{Diff}(M)$:

$$\begin{aligned}&\mathcal{O} \times \operatorname{den} S^2T(\sigma) \oplus S^2_+ T^*(\sigma) \supset \mathcal{O} \times Cd \ni (\varphi, \pi, g) \to R_\varphi(\pi, g)\\ &= ((\varphi^{-1})_* \pi, \varphi^* g) \subset Cd\,.\end{aligned} \tag{6.3}$$

[It follows from geometrical considerations that the couple $((\varphi^{-1})_*\pi, \varphi^* g)$ fulfils also (3.16b).]

A superphase space can be defined as a quotient space of Cd by the action (6.3).

Recently a similar object was proposed by Fischer and Marsden [13] as a possible choice of a superphase space for Einstein dynamics. Despite of a complicated structure of such an object (cf. [12, 13]) it is interesting to investigate a possibility of a formulation of dynamics in that space.

7. Proofs

A detailed analysis of the non-covariant Formula (2.6) shows that the transformation:

$$\begin{aligned}\xi &= ((\Gamma^0_{r0} - \Gamma^p_{rp})N^r/N^3 - \Gamma^0_{jr}(\bar{g}^{jr} + N^jN^r/N^2)N^{-1} + \Gamma^p_{0p}N^{-3})\sqrt{-g}\\ \xi^k &= ((\Gamma^0_{r0} - \Gamma^p_{rp})(-N^{-2}\bar{g}^{kr}) + 2\Gamma^0_{jr}\bar{g}^{kj}N^r/N^2)\sqrt{-g}\end{aligned} \tag{7.1}$$

together with (3.2), (3.3), (3.4), (3.10), (3.11) give

$$\begin{aligned}\psi &= \xi dN \wedge dx^1 \wedge dx^2 \wedge dx^3 + \xi^k dN_k \wedge dx^1 \wedge dx^2 \wedge dx^3\\ &\quad + (-\tfrac{1}{2}(\xi^s N^k + \xi^k N^s) - \tfrac{1}{2}\xi N \bar{g}^{ks} + \pi^{ks})dg_{ks} \wedge dx^1 \wedge dx^2 \wedge dx^3\\ &\quad + (\text{terms containing } dx^0)\end{aligned} \tag{7.2}$$

where

$$\xi = \sqrt{\bar{g}}N^{-2}\partial_k N^k\,;\quad \xi_k = -N^{-2}\sqrt{\bar{g}}\bar{g}^{kr}(\partial_r N - \bar{\Gamma}^j_{rj}N)\,. \tag{7.3}$$

The formal (non-covariant) expression (7.2) gives a covariant formula

$$\begin{aligned}\gamma = d\psi = & d\xi \wedge dN \wedge dx^1 \wedge dx^2 \wedge dx^3 + d\xi^k \wedge dN_k \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ & + (-(\xi^s dN^k + N^k d\xi^s) - \tfrac{1}{2}(N d\xi + \xi dN)\bar{g}^{ks} - \tfrac{1}{2}\xi N d\bar{g}^{ks} + d\pi^{ks}) \\ & \wedge dg_{ks} \wedge dx^1 \wedge dx^2 \wedge dx^3 + (\text{terms containing } dx^0)\end{aligned} \tag{7.4}$$

Proof of the Proposition 4. We shall prove the Proposition 4 in two steps. At first, we prove:

Lemma 1. *For* $\hat{Y}_1, \hat{Y}_2 \in \mathring{T}_{(\Omega, c)}(\mathscr{H})$, $\Gamma(\hat{Y}_1, \hat{Y}_2) = 0.$ (7.5)

Proof. Let Y_1, Y_2 be vector fields on of the type (4.4) fulfilling conditions (4.9). Then using (4.5), (4.6), (4.7), and (7.4) we obtain:

$$\Gamma(\hat{Y}_1, \hat{Y}_2) = \int_c (Y_1 \wedge Y_2) \lrcorner \gamma = \int_c (d\mu_1 - d\mu_2) \tag{7.6}$$

where

$$\begin{aligned}\mu_1 &= \sum_{r=1}^{3} (-1)^{r+1} (N^{-2} \sqrt{\bar{g}} \bar{g}^{kr} \underset{1}{\delta} N \underset{2}{\delta} N_k) dx^1 \wedge \ldots \underset{r}{\wedge} \ldots \wedge dx^3 \\ \mu_2 &= \sum_{r=1}^{3} (-1)^{r+1} (N^{-2} \sqrt{\bar{g}} \bar{g}^{kr} \underset{2}{\delta} N \underset{1}{\delta} N_k) dx^1 \wedge \ldots \underset{r}{\wedge} \ldots \wedge dx^3\end{aligned} \tag{7.7}$$

are 2-forms on the manifold $c(\sigma)$.

Using the boundary conditions for fields Y_1, Y_2 we obtain (7.5)

Let $\overset{1}{T}_\Omega(\mathscr{H})$ be defined by (5.2). We have the following:

Lemma 2. *If* $\hat{Y}_1 \in \mathring{T}_{(\Omega, c)}(\mathscr{H})$, $\hat{Y}_2 \in \overset{1}{T}_\Omega(\mathscr{H})$ *then* $\Gamma(\hat{Y}_1, \hat{Y}_2) = 0.$ (7.8)

Proof. Let Y_1, Y_2 be vector fields on Ω representing $\hat{Y}_1$, $\hat{Y}_2$. Using (4.9), (5.2), (4.5), (4.6), (7.3) and (7.4) we have:

$$\Gamma(\hat{Y}_1, \hat{Y}_2) = \int_c (Y_1 \wedge Y_2) \lrcorner \gamma = \int_c (d\eta_1 - d\eta_2) \tag{7.9}$$

where

$$\begin{aligned}\eta_1 &= \sum_{r=1}^{3} (-1)^{r+1} (N^{-2} \sqrt{\bar{g}} \bar{g}^{ku} N_u \bar{g}^{rs} \underset{1}{\delta} N \underset{2}{\delta} g_{ks}) dx^1 \wedge \ldots \underset{r}{\wedge} \ldots \wedge dx^3 \\ \eta_2 &= \sum_{r=1}^{3} (-1)^{r+1} ((2N)^{-1} \sqrt{\bar{g}} \bar{g}^{kr} \bar{g}^{ij} \underset{2}{\delta} g_{ij} \underset{1}{\delta} N_k) dx^1 \wedge \ldots \underset{r}{\wedge} \ldots \wedge dx^3\end{aligned} \tag{7.10}$$

are 2-forms on $c(\sigma)$.

Using the boundary conditions we have (7.8)

By (5.2′) we can split every $\hat{X} \in T_\Omega(\mathscr{H})$ into a sum $\hat{X} = \hat{X}_1 + \hat{X}_2$ where $\hat{X}_1 \in \mathring{T}_{(\Omega, c)}(\mathscr{H})$, $\hat{X}_2 \in \overset{1}{T}_\Omega(\mathscr{H})$. Therefore the Proposition 4 follows from Lemmas 1 and 2.

Proof of the Theorem 1. It follows from (5.2′) and the proposition 4 that we have to prove (4.10) only for $\hat{X}_1, \hat{X}_2 \in \overset{1}{T}_\Omega(\mathscr{H})$. Using (4.5), (4.6), (7.3), and (7.4) we obtain

$$\Gamma(\hat{X}_1, \hat{X}_2) = \int_c (\underset{1}{\delta} \pi^{ij} \underset{2}{\delta} g_{ij} - \underset{2}{\delta} \pi^{ij} \underset{1}{\delta} g_{ij}) dx^1 \wedge dx^2 \wedge dx^3 + \int_c (d\nu_1 - d\nu_2) \tag{7.11}$$

where

$$\begin{aligned} v_1 &= \sum_{r=1}^{3} (-1)^{r+1}((N^k/2N)\sqrt{\bar{g}}\bar{g}^{sr}\bar{g}^{ij}\underset{1}{\delta} g_{ij}\underset{2}{\delta} g_{sk})dx^1 \wedge \ldots \underset{r}{\wedge} \ldots \wedge dx^3 \\ v_2 &= \sum_{r=1}^{3} (-1)^{r+1}((N^k/2N\sqrt{\bar{g}}\bar{g}^{sr}\bar{g}^{ij}\underset{2}{\delta} g_{ij}\underset{1}{\delta} g_{sk})dx^1 \wedge \ldots \underset{r}{\wedge} \ldots \wedge dx^3 \end{aligned} \tag{7.12}$$

are 2-forms on $c(\sigma)$. The theorem 1 follows from the boundary conditions for fields X_1, X_2.

Proof of the Proposition 5. This propositions is a consequence of direct computations.

If we put (5.11) into (5.4) we obtain:

$$JA^*(u^j, \chi) = (\delta\pi^{ij}, \delta g_{ij}),$$

where

$$\begin{aligned} \delta\pi^{ij} &= -\tfrac{1}{2}(\pi^{ai}\bar{V}_a u^j + \pi^{aj}\bar{V}_a u^i - \bar{V}_a(\pi^{ij}u^a)) \\ &\quad + -(2/\sqrt{\bar{g}})(\pi^{is}g_{sk}\pi^{kj} - \tfrac{1}{2}\operatorname{tr}\pi\pi^{ij})\chi + \sqrt{\bar{g}}(\bar{V}^i\bar{V}^j\chi - \bar{g}^{ij}\bar{V}^k\bar{V}_k\chi) \\ &\quad + \sqrt{\bar{g}}(-\bar{R}^{ij}\chi + \bar{g}^{ij}(\bar{R} - 2\lambda)\chi) \\ \delta g_{ij} &= \tfrac{1}{2}(\bar{V}_j u_i + \bar{V}_i u_j) + (2/\sqrt{\bar{g}})(\pi_{ij} - \tfrac{1}{2}\operatorname{tr}\pi g_{ij})\chi\,. \end{aligned} \tag{7.13}$$

Using the formulas (3.16b) and

$$\begin{aligned} &\bar{V}_j\bar{R}^{ij} = \tfrac{1}{2}\bar{V}^i\bar{R} \\ &\bar{V}_j\bar{V}^j\bar{V}^i\chi - \bar{V}^i\bar{V}_j\bar{V}^j\chi = \bar{R}^{si}\bar{V}_s\chi \\ &(\bar{V}_i\bar{V}_j - \bar{V}_j\bar{V}_i)u^k = \bar{R}^k_{sij}u^s \\ &(\bar{V}_i\bar{V}_j - \bar{V}_j\bar{V}_i)u_k = -\bar{R}^s_{kij}u_s \\ &(\bar{V}_i\bar{V}_j - \bar{V}_j\bar{V}_i)v^{ks} = \bar{R}^k_{pij}v^{ps} + \bar{R}^s_{pij}v^{kp} \end{aligned} \tag{7.14}$$

we obtain

$$AJA^*(u^j, \chi) = 0\,. \tag{7.15}$$

Proof of the Proposition 8. We define two differential operators

$$A_1 : C^\infty(\operatorname{den} S^2T(\sigma) \oplus S^2T^*(\sigma)) \to C^\infty(T(\sigma))$$

$$A_2 : C^\infty(\operatorname{den} S^2T(\sigma) \oplus S^2T^*(\sigma)) \to C^\infty(\sigma)$$

$$A_1(\delta\pi^{ij}, \delta g_{ij}) = \bar{g}^{-1/2}(\bar{V}_j\delta\pi^{ij} + \delta\bar{\Gamma}^i_{ks}\pi^{ks}), \tag{7.16}$$

$$A_2(\delta\pi^{ij}, \delta g_{ij}) = \delta\bar{R} + \bar{g}^{-1}(\bar{R} - 2\lambda)\delta\bar{g} - \bar{g}^{-1}\delta(\pi^{ks}\pi_{ks} - \tfrac{1}{2}(\operatorname{tr}\pi)^2)\,. \tag{7.17}$$

The operator A_1 is a first order differential operator and A_2 is a second order differential operator [cf. (5.8)]. We have also $A = A_1 \oplus A_2$. Corresponding adjoint operators defined by means of the scalar products (5.3), (5.9) are:

$$A_1^* : C^\infty(T(\sigma)) \to C^\infty(\operatorname{den} S^2T(\sigma) \oplus S^2T^*(\sigma))$$

$$A_2^* : C^\infty(\sigma) \to C^\infty(\operatorname{den} S^2T(\sigma) \oplus S^2T^*(\sigma))$$

$$A_1^*(u^j) = (-\tfrac{1}{2}(\bar{V}^i u^j + \bar{V}^j u^i)\sqrt{\bar{g}}, -(2\sqrt{\bar{g}})^{-1}(\pi_{ai}\bar{V}^a u_j + \pi_{aj}\bar{V}^a u_i - \bar{V}_a(\pi_{ij}u^a)) \tag{7.16'}$$

$$\begin{aligned} A_2^*(\chi) &= ((-2(\pi^{ij} - \tfrac{1}{2}\operatorname{tr}\pi\bar{g}^{ij})\chi, -(2/\bar{g})(\pi_{is}\bar{g}^{sk}\pi_{kj} - \tfrac{1}{2}\operatorname{tr}\pi\pi_{ij}) \\ &\quad + \bar{V}_i\bar{V}_j\chi - g_{ij}\bar{V}_k\bar{V}^k\chi - \bar{R}_{ij}\chi + g_{ij}(\bar{R} - 2\lambda)\chi)\,. \end{aligned} \tag{7.17'}$$

It is easy to check that the operators A_1^*, A_2^* have injective symbols (for definition of the symbol see [21]). Therefore we have the orthogonal decompositions

$$C^\infty(\operatorname{den} S^2 T(\sigma) \oplus S^2 T^*(\sigma)) = \ker A_1 \oplus \operatorname{im} A_1^*, \tag{7.18a}$$

$$C^\infty(\operatorname{den} S^2 T(\sigma) \oplus S^2 T^*(\sigma)) = \ker A_2 \oplus \operatorname{im} A_2^*. \tag{7.18b}$$

Formulas (7.18) are proved in [4] for any differential operator with the injective symbol on a compact manifold.

We are going to prove that

$$C^\infty(\operatorname{den} S^2 T(\sigma) \oplus S^2 T^*(\sigma)) = (\ker A_1 \cap \ker A_2) \oplus (\operatorname{im} A_1^* + \operatorname{im} A_2^*) \tag{7.19}$$

($\operatorname{im} A_1^* + \operatorname{im} A_2^*$ is not a direct sum).

Let $\mathfrak{X} \in C^\infty(\operatorname{den} S^2 T(\sigma) \oplus S^2 T^*(\sigma))$ be orthogonal to $(\ker A_1 \cap \ker A_2) \oplus (\operatorname{im} A_1^* + \operatorname{im} A_2^*)$ then $\mathfrak{X}$ is orthogonal to $\ker A_1$, $\ker A_2$, $\operatorname{im} A_1^*$, $\operatorname{im} A_2^*$ and by virtue of (7.18) $\mathfrak{X} = 0$. Therefore $(\ker A_1 \cap \ker A_2) \oplus (\operatorname{im} A_1^* + \operatorname{im} A_2^*)$ is dense in $C^\infty(\operatorname{den} S^2 T(\sigma) \oplus S^2 T^*(\sigma))$. It remains to prove that $\operatorname{im} A_1^* + \operatorname{im} A_2^*$ is the closed subspace. We do not discuss that problem here. It will be done elsewhere[1].

Proof of Corollary 1. We have from Proposition 5

$$(\ker A \cap \ker AJ) \oplus \operatorname{im} JA^* \subset \ker A. \tag{7.20}$$

Let $\mathfrak{X} \in \ker A$. We have decomposition of $-J\mathfrak{X}$

$$-J\mathfrak{X} = y_1 + y_2 \tag{7.21}$$

where $y_1 \in \ker A$, $y_2 \in \operatorname{im} A^*$. Thus

$$\mathfrak{X} = Jy_1 + Jy_2 \tag{7.22}$$

But $Jy_2 \in \operatorname{im} JA^* \subset \ker A$ and therefore $Jy_1 \in \ker A$. Moreover $J(Jy_1) = -y_1 \in \ker A$.

Proof of Corollary 2. The first statement follows from (5.14), Proposition 5 and (5.6). The other is the consequence of decomposition (5.2′).

Proof of Proposition 9

Lemma 3. *Let in local coordinates* $\Omega = (x^\lambda, g_{\mu\nu}(x^\lambda), \Gamma^\tau_{\mu\nu}(x^\lambda))$ *and* $v = v^\mu \partial/\partial x^\mu$. *Then* $d\hat{R}_{\mathrm{id}}(\Omega)v = \hat{X}$, *where* $\hat{X}$ *is represented by a vector field* X *on* Ω:

$$X = \sum_{\mu \leqq \nu} (\nabla_\mu v_\nu + \nabla_\nu v_\mu)\, \partial/\partial g_{\mu\nu} + \sum_{\mu \leqq \nu} (\nabla_\mu \nabla_\nu v^\lambda + R^\lambda_{\nu\sigma\mu} v^\sigma)\, \partial/\partial \Gamma^\lambda_{\mu\nu}. \tag{7.23}$$

Proof. If $t \to \varphi_t$ is a one parameter family of diffeomorphisms, such that $\varphi_0 = \mathrm{id}$ and $d\varphi_0/dt = v$, then an infinitesimal change of $g_{\mu\nu}(x^\lambda)$ is given by the Killing formula [25]

$$\Delta_t g_{\mu\nu} = t\delta g_{\mu\nu} = t\mathscr{L}_v g_{\mu\nu} = t(\nabla_\mu v_\nu + \nabla_\nu v_u). \tag{7.24}$$

An infinitesimal change of $\Gamma^\lambda_{\mu\nu}$ is given by the formula

$$\Delta_t \Gamma^\lambda_{\mu\nu} = t\delta \Gamma^\lambda_{\mu\nu} = t\tfrac{1}{2} g^{\lambda\alpha}(\nabla_\mu \delta g_{\nu\alpha} + \nabla_\nu \delta g_{\mu\alpha} - \nabla_\alpha \delta g_{\mu\nu}). \tag{7.25}$$

---

[1] See Note added in proof

If we put (7.24) into (7.25) we obtain

$$\delta\Gamma^{\lambda}_{\mu\nu}=\nabla_\mu\nabla_\nu v^\lambda+R^{\lambda}_{\nu\sigma\mu}v^\sigma\,. \tag{7.26}$$

Thus (7.23) is proved.

To prove the Proposition 9 we have to express (7.23) in the ADMW coordinates. We are interested only in values of X on an admissible initial surface $c\subset\Omega$, which is the lift to $\mathscr{P}$ of a 3-dimensional space-like surface $\sigma\subset M$. We decompose the vector field v on into the tangent and the normal components. We obtain

$$v^\mu=\beta n^\mu+(0,\alpha^k),\quad \text{where}\quad n^\mu=(N^{-1},\,-N^s/N)$$

is a unit normal vector to σ. Then

$$\begin{aligned}\beta&=Nv^0, & \alpha^k&=v^k+v^0N^k\\ v^0&=\beta/N, & v^k&=\alpha^k-(\beta/N)N^k.\end{aligned} \tag{7.27}$$

α^k form components of a vector field tangent to σ and β is a scalar function on σ.

If we put (7.27) into (7.24) we obtain:

$$\delta g_{ij}=(\bar\nabla_i\alpha_j+\bar\nabla_j\alpha_i)+(2/\sqrt{\bar g})(\pi_{ij}-\tfrac{1}{2}g_{ij}\operatorname{tr}\pi)\beta \tag{7.28a}$$

where $\alpha_j=g_{js}\alpha^s=g_{js}v^s+g_{j0}v^0$.

According to (3.10)

$$\delta\pi^{ij}=\delta(\sqrt{-g}(\Gamma^0_{pq}\bar g^{ip}\bar g^{jq}-\bar g^{ij}\bar g^{pq}\Gamma^0_{pq}))\,. \tag{7.29}$$

Using (7.24), (7.25) and the equation (3.16a) we obtain:

$$\begin{aligned}\delta\pi^{ij}=&-(\pi^{bj}\bar\nabla_b\alpha^i+\pi^{bi}\bar\nabla_b\alpha^j-\bar\nabla_b(\pi^{ij}\alpha^b))\\ &+\sqrt{\bar g}(\bar\nabla^i\bar\nabla^j\beta-\bar g^{ij}\bar\nabla^r\bar\nabla_r\beta-\bar R^{ij}\beta+\bar g^{ij}(\bar R-2\lambda)\beta)\\ &+-(2/\sqrt{\bar g})(\pi^{im}g_{mn}\pi^{nj}-\tfrac{1}{2}\operatorname{tr}\pi\pi^{ij})\beta\,.\end{aligned} \tag{7.28b}$$

In a similar way we can obtain:

$$\begin{aligned}\delta N_k=\delta g_{0k}=&\,g_{ks}\partial_0\alpha^s+\alpha^p\bar\nabla_pN_k+N_s\bar\nabla_k\alpha^s+\bar\nabla_kN\beta-N\bar\nabla_k\beta\\ &+(2\beta/\sqrt{\bar g})\,N^s(\pi_{ks}-\tfrac{1}{2}g_{ks}\operatorname{tr}\pi)\end{aligned} \tag{7.30a}$$

$$\delta N=\partial_0\beta-N^k\bar\nabla_k\beta+\bar\nabla_kN\alpha^k\,. \tag{7.30b}$$

The terms $g_{ks}\partial_0\alpha^s$, $\partial_0\beta$ in (7.30a), (7.30b) show that $\delta N_k,\delta N$ can be obtained arbitrary in a neighbourhood of $c\subset\Omega$. We must choose an appropriate vector field v on M. On the other hand $(\delta\pi^{ij},\delta g_{ij})$ on c are determined by (7.28b) and (7.28a). Comparing these formulas with the definition of $\mathring{W}_\Omega$ we see that $\operatorname{im} d\hat{R}_{\mathrm{id}}(\Omega)=\mathring{W}_\Omega$.

References

1. Abraham, R.: Foundations of mechanics. New York: Benjamin 1967
2. Adler, R., Bazin, M., Schiffer, M.: Introduction to general relativity. New York: McGraw Hill 1965
3. Arnowitt, R., Deser, S., Misner, C. W.: The dynamics of general relativity. In: Gravitation—an introduction to current research (Witten, L., ed.). New York: John Wiley 1962

4. Berger, M., Ebin, D.: J. Diff. Geometry 3, 379—392 (1969)
5. Choquet-Bruhat, Y.: The Cauchy problem. In: Gravitation – an introduction to current research (Witten, L., ed.). New York: John Wiley 1962
6. Choquet-Bruhat, Y., Geroch, R.: Commun. math. Phys. 14, 329—335 (1969)
7. Dedecker, P.: Calcul des variations, formes differentieles et champ geodesiques. In: Coll. Intern. Geometrie Diff. Strasbourg: Publications CNRS 1953
8. De Witt, B.: Phys. Rev. 160, 1113—1148 (1967)
9. Dirac, P. A. M.: Proc. Roy. Soc. (London) A 246, 333—346 (1958)
10. Ebin, D., Marsden, J.: Ann. Math. 92, 102—163 (1970)
11. Fadeev, L.: Symplectic structure and quantization of the Einstein gravitation theory. In: Actes du Congres Int. des Math., Vol. 3, 35—39. Paris: Gauthier Villars 1970
12. Fischer, A.: The theory of superspaces. In: Relativity (Carmelli, M., Fickler, S., Witten, L., ed.). New York: Plenum Press 1970
13. Fischer, A., Marsden, J.: J. Math. Phys. 13, 546—568 (1972)
14. Fischer, A., Marsden, J.: Commun. math. Phys. 28, 1—38 (1972)
15. Gawędzki, K.: Reports Math. Phys. 3, 307—326 (1972)
16. Goldschmidt, H., Sternberg, S.: Ann. Instit. Fourier 23, 203—267 (1973)
17. Kijowski, J.: Commun. math. Phys. 30, 99—128 (1973)
18. Kijowski, J., Szczyrba, W.: A canonical structure of classical field theories (to appear in Commun. math. Phys.)
19. Kobayashi, S., Nomizu, K.: Foundations of sifferential geometry, Vol. 1, Vol. 2. New York: Interscience Publ. 1963/1969
20. Lichnerowicz, A.: Relativistic hydrodynamics and magnetohydrodynamics. New York: Benajmin 1967
21. Narasimhan, R.: Analysis on real and complex manifold. Paris: Masson and Cie 1968
22. Souriau, J. M.: Structure des systemes dynamiques. Paris: Dunod 1969
23. Szczyrba, W.: Lagrangian formalism in the classical field theory. Ann. Pol. Math. 32, 145—185 (1976)
24. Wheeler, J. A.: Geometrodynamics and the issue of the final state. In: Relativity, Groups and Topology (De Witt, B., De Witt, C., ed.). New Xork: Gordon and Breach 1964
25. Yano, K.: Integral formulas in Riemannian Geometry. New York: Marcel Dekker 1970

Communicated by J. Ehlers

Received January 16, 1975

Note Added in Proof

(i) Proposition 8 can be proved directly considering the properties of the operator $AA^{\star}$. It turns out that this operator is elliptic in the generalized sense of Douglis-Nirenberg (cf. Agmon, S., Douglis, A., Nirenberg, L.: Comm. Pure Appl. Math. 17, 35—92 (1964); Hörmander, L.: Linear partial differential operators, Chapter X. Berlin-Göttingen-Heidelberg: Springer 1963; Palais, R.: Seminar on the Atiyah-Singer index theorem, Chapter IV. Princeton 1965). Therefore the decomposition (5.14) follows directly by the arguments given in [4]. The complete, non-trivial proof of the ellipticity of $AA^{\star}$ will be published in the author's paper: On geometric structure of the set of solutions of Einstein equations (to appear in Dissertationes Mathematicae 1977).

(ii) Recently Moncrief in J. Math. Phys. 16, 1556—1560 (1975) gave the decomposition (5.15). However, it seems to us that his proof based strictly on the results of [4] is uncomplete by arguments presented above.

(iii) The generalization of the results of the present paper for the Einstein equations with presence of electromagnetic field will appear in Rept. Math. Phys. The first results concerning the case of a non-symmetric connection (the Einstein-Cartan theory) will be published soon in Bull. Poll. Acad. Sci.