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Taylor's Theorem for Analytic Functions of Operators

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Abstract. We discuss analytic functions on a Banach algebra into itself. In particular expressions for derivatives are given as well as convergent Taylor expansions.

Introduction

The problem of expansion of functions of non-commuting operators occurs in many branches of theoretical physics. Many formal schemes [1–5] have been used, but in very few cases [5] has convergence been established. We discuss a case for which convergence is established. Our approach follows in spirit the work [5] of Araki.

I. Analytic Functions of Operators and Derivatives

Let $F:\mathbb{C}\to\mathbb{C}$ be an analytic function in $G=\{z \mid |z|<\varrho\}$. In the domain G, F has a convergent power series expansion

$$F(z)=\sum_{n=0}^{\infty} c_n z^n . \tag{1}$$

The n^{th} derivative D^nF of F also has a convergent power series having the same domain of convergence as F.

Let $\mathcal{B}$ be a Banach algebra and denote by $\mathcal{L}=\mathcal{L}^1(\mathcal{B})$ the Banach algebra of bounded linear maps L of $\mathcal{B}$ into itself. The norm of $\mathcal{L}^1(\mathcal{B})$ is defined by $\|L\|=\sup\limits_{A\in\mathcal{B}}\frac{\|LA\|}{\|A\|}$. We then define the Banach algebras $\mathcal{L}^n(\mathcal{B})$ iteratively by $\mathcal{L}^1(\mathcal{L}^{n-1}(\mathcal{B}))$.

Definition 1. Let $\mathcal{B}$ be a Banach algebra and $A, B\in\mathcal{B}$. For $0\leqq\lambda\leqq 1$ let A_λ be the linear map from $\mathcal{B}$ into $\mathcal{B}$ defined by

$$A_\lambda B=AB-\lambda d_A B , \tag{2}$$

with

$$d_A B = [A, B] = AB - BA\,. \tag{3}$$

Lemma 1. *Let A_λ be defined as above. Then $\|A_\lambda\| \leqq \|A\|$.*

Proof.

$$\|A_\lambda\| = \sup_{B\in\mathscr{B}} \frac{\|A_\lambda B\|}{\|B\|} = \sup_{B\in\mathscr{B}} \frac{\|(1-\lambda)AB + \lambda BA\|}{\|B\|}$$
$$\leqq (1-\lambda)\|A\| + \lambda\|A\| = \|A\|\,.$$

Lemma 2. *For $A\in\mathscr{B}$ and n a positive integer, the following relations hold in $\mathscr{L}(\mathscr{B})$*

$$1)\quad A^n - (A-d_A)^n = n\int_0^1 d\lambda A_\lambda^{n-1} d_A\,, \tag{4}$$

$$2)\qquad (A-d_A)^n = \int_0^1 d\lambda A_\lambda^n - n\int_0^1 d\lambda\lambda A_\lambda^{n-1} d_A\,, \tag{5}$$

$$3)\qquad A^n = \int_0^1 d\lambda A_\lambda^n + n\int_0^1 d\lambda(1-\lambda) A_\lambda^{n-1} d_A\,. \tag{6}$$

Proof. First note that $Ad_A = d_A A$ as maps in $\mathscr{L}(\mathscr{B})$. Then

$$1)\quad n\int_0^1 d\lambda A_\lambda^{n-1} d_A = -\int_0^1 d\lambda \frac{d}{d\lambda} A_\lambda^n = A_0^n - A_1^n = A^n - (A-d_A)^n\,.$$

2) By partial integration we have

$$\int_0^1 d\lambda A_\lambda^n = \lambda A_\lambda^n \Big|_0^1 + \int_0^1 d\lambda\lambda n A_\lambda^{n-1} d_A$$
$$= (A-d_A)^n + n\int_0^1 d\lambda\lambda A_\lambda^{n-1} d_A\,.$$

3) Combine 1) and 2).

Lemma 3. $A^n - (A-d_A)^n = d_{A^n}\,,$ (7)

or, equivalently

$$(A-d_A)^n B = BA^n\,. \tag{8}$$

Proof. The lemma is true for $n=1$. By induction we then find that

$$A^{n+1}B - (A-d_A)^{n+1}B$$
$$= A^{n+1}B - (A-d_A)BA^n$$
$$= A^{n+1}B - ABA^n + [A, BA^n]$$
$$= A^{n+1}B - BA^{n+1} = d_{A^{n+1}}B\,.$$

An analytic function F with radius of cenvergence ϱ gives rise to a map

$$F:\mathscr{B}\rightarrow\mathscr{B}$$

by means of

$$F(A)=\sum_{n=0}^{\infty} c_n A^n . \tag{9}$$

This map is defined for all $A\in\mathscr{B}$ for which $\|A\|<\varrho$.

Lemma 4. *For F analytic and* $\|A\|<\varrho$ *we have*

$$F(A)-F(A-d_A)=d_{F(A)}, \tag{10}$$

or, equivalently for $B\in\mathscr{B}$,

$$F(A-d_A)B=BF(A). \tag{11}$$

Proof. This follows from Lemma 3 and the fact that $\|A-d_A\|\leqq\|A\|<\varrho$.

It may be noted that for $F=\exp$ we recover the well-known result

$$\exp(-d_A)\cdot B=\exp(-A)B\exp(A). \tag{12}$$

Let $A(t)\in\mathscr{B}$ be a differentiable path in $\mathscr{B}$ for $t\in I\subset R$ and for which $\|A(t)\|<\varrho$ and $\frac{d}{dt}A(t)\in\mathscr{B}$, $\forall t\in I$. Then $F(A(t))$ is a $\mathscr{B}$-valued function of t.

Theorem 1. *The function* $F(A(t))$ *is differentiable and its derivative is given by*

$$\frac{d}{dt}F(A(t))=\int_0^1 d\lambda DF(A_\lambda)\left(\frac{dA}{dt}\right). \tag{13}$$

Proof. It suffices to prove the statement for powers of $A(t)$, i.e. to show that

$$\frac{d}{dt}A(t)^n=\int_0^1 d\lambda n A_\lambda^{n-1}\left(\frac{dA}{dt}\right).$$

The statement is clearly valid for $n=1$, and by induction

$$\begin{aligned}\frac{d}{dt}A^{n+1}&=A^n\frac{dA}{dt}+\frac{dA^n}{dt}A\\ &=A^n\frac{dA}{dt}+\left\{n\int_0^1 d\lambda A_\lambda^{n-1}\left(\frac{dA}{dt}\right)\right\}A\\ &=A^n\frac{dA}{dt}+n\int_0^1 d\lambda A_\lambda^{n-1}\left(\frac{dA}{dt}A\right)\\ &=A^n\frac{dA}{dt}+n\int_0^1 d\lambda A_\lambda^{n-1}(A-d_A)\left(\frac{dA}{dt}\right).\end{aligned}$$

With the help of Equation (4) and Equation (5), the above expression becomes

$$\begin{aligned}\frac{d}{dt}A^{n+1} &= (A-d_A)^n\left(\frac{dA}{dt}\right) + n\int_0^1 d\lambda A_\lambda^{n-1}A\left(\frac{dA}{dt}\right)\\ &= \int_0^1 d\lambda A_\lambda^n\left(\frac{dA}{dt}\right) - n\int_0^1 d\lambda\lambda A_\lambda^{n-1}d_A\left(\frac{dA}{dt}\right)\\ &\quad + n\int_0^1 d\lambda A_\lambda^{n-1}A\left(\frac{dA}{dt}\right)\\ &= \int_0^1 d\lambda\left\{A_\lambda^n + nA_\lambda^{n-1}(A-\lambda d_A)\right\}\left(\frac{dA}{dt}\right)\\ &= (n+1)\int_0^1 d\lambda A_\lambda^n\left(\frac{dA}{dt}\right).\end{aligned}$$

For an analytic function F we obtain Equation (13), since for $\|A(t)\| < \varrho$ and $0 \leqq \lambda \leqq 1$ we have $\|A_\lambda(t)\| \leqq \|A(t)\| < \varrho$, and hence absolute convergence of the respective power series.

Corollary 1. *For the function $F(A(t))$ of Theorem 1 with $A(t)$ twice differentiable and $\frac{d^2A(t)}{dt^2} \in \mathscr{B}$, we have*

$$\begin{aligned}\frac{d^2F}{dt^2}(A(t)) &= \int_0^1 d\lambda DF(A_\lambda)\left(\frac{d^2A}{dt^2}\right)\\ &\quad + \int_0^1 d\lambda_1 \int_0^1 d\lambda_2 D^2F(A_{\lambda_1,\lambda_2})\left(\frac{dA_{\lambda_1}}{dt}, \frac{dA}{dt}\right)\end{aligned} \tag{14}$$

where it is implied that $D^2F(A_{\lambda_1,\lambda_2})$ acts on $\frac{dA_{\lambda_1}}{dt}$ and the result of the λ_2-integration acts then on $\frac{dA}{dt}$.

Proof. We have only to note that $\|A_{\lambda_1,\lambda_2}\| < \varrho$ and $\int_0^1 d\lambda_2 D^2F(A_{\lambda_1,\lambda_2}) \in \mathscr{L}^2(\mathscr{B})$. This map is applied to $\frac{dA_{\lambda_1}}{dt}$ and yields a result in $\mathscr{L}^1(\mathscr{B})$, which after λ_1-integration acts on $\frac{dA}{dt}$.

We now obtain a relation between the commutator of an analytic function $F(A)$ with an element of $\mathscr{B}$ and the commutator of A with the same element.

Lemma 5.

$$d_{F(A)} = \int_0^1 d\lambda DF(A_\lambda)d_A. \tag{15}$$

Proof. The extension of Lemma 2 to analytic functions yields

$$d_{F(A)} = F(A) - F(A-d_A) = \int_0^1 d\lambda DF(A_\lambda)d_A.$$

It may be noted that if we define the map

$$\frac{d_{F(A)}}{d_A} = \int_0^1 d\lambda DF(A_\lambda),$$

then the expression for the derivative [Eq. (13)] takes on the appearance of the chain-rule of elementary calculus, i.e.

$$\frac{dF(A)}{dt} = \frac{d_{F(A)}}{d_A}\frac{dA}{dt}.$$

Corollary 2. *For the special case that the tangent to the path $A(t)$ admits the following representation*

$$\frac{dA}{dt} = d_A H, \qquad H \in \mathscr{B}. \tag{16}$$

We obtain the Heisenberg equation

$$\frac{d}{dt}F(A(t)) = [F(A(t)), H]. \tag{17}$$

Proof.

$$\frac{d}{dt}F(A(t)) = \int_0^1 d\lambda DF(A_\lambda) d_A H$$
$$= d_{F(A)}H = [F(A(t)), H]$$

by Lemma 5.

II. Taylor's Theorem for Analytic Functions

We now apply our results to find the Taylor expansion of $F(A+\lambda B)$ in powers of λ.

Theorem 2.

$$F(A+\lambda B) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_0^1 d\lambda_1 \ldots \int_0^1 d\lambda_n D^n F(A_{\lambda_1\lambda_2\ldots\lambda_n}) \cdot (B_{\lambda_1\ldots\lambda_{n-1}}, B_{\lambda_1\ldots\lambda_{n-2}}, \ldots, B_{\lambda_1}, B) \tag{18}$$

with non-zero radius of convergence for $\|A+\lambda B\| < \varrho$.

Proof. Let $X(\lambda) = A + \lambda B$ then for $\|A+\lambda B\| < \varrho$, and because of Theorem 1 and the fact that $\frac{d^n X(\lambda)}{d\lambda^n} = 0$, $n \geqq 2$, the required higher derivatives can be obtained. Convergence is guaranteed from the facts that $\|A_{\lambda_1\ldots\lambda_n}\| \leqq \|A\|$, $\|B_{\lambda_1\ldots\lambda_n}\| \leqq \|B\|$, and that $\exists R > 0$ such that $F(z_1 + \lambda z_2)$ converges absolutely for $|z_1 + \lambda z_2| < \varrho$ and $|\lambda| < R$.

Formula 18 when applied to the function $\exp\{-it(H+\lambda V)\}$ yields the Feynman-Dyson series.

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