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Second Quantization of Classical Nonlinear Relativistic Field Theory*

Part II. Construction of Relativistic Interacting Local Quantum Field**

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Abstract. The construction of a relativistic interacting local quantum field is given in two steps: first the classical nonlinear relativistic field theory is written down in terms of Poisson brackets, with initial conditions as canonical variables: next a representation of Poisson bracket Lie algebra by means of linear operators in the topological vector space is given and an explicit form of a local interacting relativistic quantum field $\hat{\Phi}$ is obtained. The construction of asymptotic local relativistic fields $\hat{\Phi}_{\text{in}}$ and $\hat{\Phi}_{\text{out}}$ associated with $\hat{\Phi}$ is also given.

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I. Introduction

The construction of an interacting local quantum scalar field is given in two steps. First in the previous paper [1] (hereafter denoted as I) we have shown that the classical nonlinear relativistic field theory written down in terms of Poisson brackets, with initial conditions as canonical variables is a local field theory with local asymptotic fields: in particular we have

$$\{\Phi(x), \Phi(y)\}=0 \quad \text{if} \quad (x-y)^2<0 \tag{1.1}$$

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** In this work we consider the prequantized level of the theory only. However for the sake of simplicity we use adjective quantum instead of prequantum.

and

$$\Phi_{\substack{\text{in}\\\text{out}}}(x), \Phi_{\substack{\text{in}\\\text{out}}}(y) = -\Delta(x-y; m). \tag{1.2}$$

Next in the present paper we give the map $\Phi(x) \to \hat{\Phi}(x)$ which provides an operator representation of the Lie algebra of Poisson brackets in a topological vector space. Since the map $\Phi \to \hat{\Phi}$ conserves Lie brackets relations we obtain a local relativistic interacting quantum field. We show that the natural carrier spaces for $\hat{\Phi}$ are topological vector spaces $\mathscr{E}(\mathscr{F})$, $\mathscr{K}(\mathscr{F})$, and $\mathscr{D}(\mathscr{F})$, $\mathscr{F}$ – the Banach space of initial conditions. $\mathscr{E}(\mathscr{F})$ and $\mathscr{D}(\mathscr{F})$ are natural generalization of the Schwartz's space $\mathscr{E}(R^n)$ and $\mathscr{D}(R^n)$ respectively. The construction of asymptotic fields $\hat{\Phi}_{\text{in}}$ and $\hat{\Phi}_{\text{out}}$ associated with $\hat{\Phi}$ is given and it is shown that

$$\hat{\Phi}_{\text{in}}(t, \boldsymbol{x}) \xleftarrow[t \to -\infty]{} \hat{\Phi}(t, \boldsymbol{x}) \xrightarrow[t \to \infty]{} \hat{\Phi}_{\text{out}}(t, \boldsymbol{x})$$

in the strong topology of $\mathscr{E}(\mathscr{F})$, $\mathscr{K}(\mathscr{F})$ as well as $\mathscr{D}(\mathscr{F})$ space. It is also shown that

$$\hat{P}_\mu = \hat{P}_\mu^{\text{in}} = \hat{P}_\mu^{\text{out}}, \qquad \hat{M}_{\mu\nu} = \hat{M}_{\mu\nu}^{\text{in}} = \hat{M}_{\mu\nu}^{\text{out}}.$$

The construction of the quantum evolution and $\hat{S}$-operator in this theory will be considered in Part III of the present series.

This work represents a continuation of Segal's program of the construction of an interacting quantum field using the properties of the corresponding classical interacting field [2] (see also Streater [3]). Since this work is addressed to Quantum Field theorists the most of rather technical details concerning the convergence of solutions of certain linear and nonlinear partial differential equations are shifted to Appendix A.

II. Operator Representations of Lie Algebra of Poisson Brackets

Let $\mathscr{F}$ be the Banach space of initial conditions defined in Appendix A. Let $F(\mathfrak{z})$ be a functional over the space $\mathscr{F}$. We say that the functional F possesses a Frechet differential at a point $\mathfrak{z}$ if there exists a linear continuous map $DF[\mathfrak{z}](\mathfrak{z}_1)$ of the space $\mathscr{F}$ into R^1 such that

$$F(\mathfrak{z}+\mathfrak{z}_1) - F(\mathfrak{z}) = DF[\mathfrak{z}](\mathfrak{z}_1) + r(\mathfrak{z}; \mathfrak{z}_1)$$

where

$$\lim_{\mathfrak{z}_1 \to 0} \frac{|r(\mathfrak{z}; \mathfrak{z}_1)|}{\|\mathfrak{z}_1\|_F} = 0.$$

The value of $DF[\mathfrak{z}](\mathfrak{z}_1)$ on a given $\mathfrak{z}_1 \in \mathscr{F}$ is called the differential of the functional F and defines the Frechet derivative $\frac{\delta F}{\delta \mathfrak{z}}$

$$DF[\mathfrak{z}](\mathfrak{z}_1) = \left\langle \frac{\delta F}{\delta \mathfrak{z}}, \mathfrak{z}_1 \right\rangle = \left\langle \frac{\delta F}{\delta \varphi}, \varphi_1 \right\rangle + \left\langle \frac{\delta F}{\delta \pi}, \pi_1 \right\rangle. \tag{2.1}$$

Hence the Frechet derivative $\delta F/\delta \mathfrak{z}$ is in general an element of the dual space $\mathscr{F}'$ to $\mathscr{F}$.

The Poisson bracket $\{F, G\}$ of two functionals over the space $\mathscr{F}$ is formally defined by the formula

$$\{F, G\} = \int_{R^3} d^3z \left(\frac{\delta F}{\delta\varphi(z)}\frac{\delta G}{\delta\pi(z)} - \frac{\delta F}{\delta\pi(z)}\frac{\delta G}{\delta\varphi(z)}\right). \tag{2.2}$$

If σ_2 is the Pauli matrix $\sigma_2 = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}$ then $\sigma_2 \frac{\delta F}{\delta \mathfrak{z}} = \left(-\frac{\delta F}{\delta \pi}, \frac{\delta F}{\delta \varphi}\right)^1$ and the Poisson bracket may be written in the form

$$\{F, G\} = DG[\mathfrak{z}](\mathfrak{z}_F) = -DF[\mathfrak{z}](\mathfrak{z}_G) \tag{2.3}$$

where for a functional X we set $\mathfrak{z}_X = \sigma_2 \frac{\delta X}{\delta \mathfrak{z}}$. Hence we see that a Poisson bracket of two smooth functionals is well defined if either $\mathfrak{z}_F$ or $\mathfrak{z}_G$ is an element of the carrier Banach space $\mathscr{F}$.

It will be evident from the next considerations that in case of nonlinear field theory the most important role is played by a vector space Ω of functionals over the space $\mathscr{F}$, defined in the following manner.

Definition 1. A functional F over $\mathscr{F}$ belongs to Ω if

i) $F \in C^\infty(\mathscr{F})$

ii) $D^kF[\mathfrak{z}](\mathfrak{z}_1, \mathfrak{z}_2, \ldots, \mathfrak{z}_k)$ is bounded on bounded subsets of $\mathscr{F}^{k+1}$

iii) $D^k\left(\sigma_2 \frac{\delta F}{\delta \mathfrak{z}}\right)(\mathfrak{z}_1, \mathfrak{z}_2, \ldots, \mathfrak{z}_k) \in \mathscr{F}$, $\quad k = 0, 1, 2, \ldots$. □

If $F, G \in \Omega$ then $\{F, G\}$ is well defined and also belongs to Ω: indeed by virtue of (2.3) one obtains

$$\sigma_2 \frac{\delta}{\delta \mathfrak{z}}\{F, G\} = \sigma_2 \frac{\delta}{\delta \mathfrak{z}} DG[\mathfrak{z}](\mathfrak{z}_F) = D\sigma_2 \frac{\delta}{\delta \mathfrak{z}} G[\mathfrak{z}](\mathfrak{z}_F) + DG[\mathfrak{z}]\left(\sigma_2 \frac{\delta}{\delta \mathfrak{z}} \mathfrak{z}_F\right)$$
$$= D\sigma_2 \frac{\delta}{\delta \mathfrak{z}} G[\mathfrak{z}](\mathfrak{z}_F) - D\sigma_2 \frac{\delta}{\delta \mathfrak{z}} F[\mathfrak{z}](\mathfrak{z}_G)$$

which is an element of $\mathscr{F}$. Similarly for $k = 1, 2, \ldots$ we have

$$D^k\sigma_2 \frac{\delta}{\delta \mathfrak{z}}\{F, G\}[\mathfrak{z}](\mathfrak{z}_1, \ldots, \mathfrak{z}_k) = D^{k+1}\sigma_2 \frac{\delta}{\delta \mathfrak{z}} G[\mathfrak{z}](\mathfrak{z}_F, \mathfrak{z}_1, \ldots, \mathfrak{z}_k)$$
$$- D^{k+1}\sigma_2 \frac{\delta}{\delta \mathfrak{z}} F[\mathfrak{z}](\mathfrak{z}_G, \mathfrak{z}_1, \ldots, \mathfrak{z}_k)$$

which is also an element of $\mathscr{F}$. Therefore $\{F, G\} \in \Omega$. Similarly $\{\{F, G\}, H\}$ is in Ω if $F, G, H \in \Omega$. Consequently the vector space Ω is a Lie algebra under Poisson brackets.

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[1] Here $\frac{\delta F}{\delta \mathfrak{z}}$ and $\sigma_2 \frac{\delta F}{\delta \mathfrak{z}}$ should be written in the form of a column. However for the sake of simplicity of notation we write them in the form of row.

Moreover it follows from Definition 1 that if $F, G \in \Omega$ then $(F \cdot G)(\mathfrak{z})$ is in Ω. Consequently the space Ω represents also an algebra under point-wise multiplication.

We now construct an operator representation of Lie algebra of functionals from Ω. We begin with a construction of three carrier spaces. The first space will be a linear space of C^∞ functionals $\psi(\cdot)$ on $\mathscr{F}$ with the topology defined by the system of seminorms

$$\|\Psi\|_{B,m} = \sup_{\mathfrak{z}\in B} \sup_{\|\mathfrak{z}_i\|_F \leqq 1, i=1,\ldots,m} |D^m \Psi[\mathfrak{z}](\mathfrak{z}_1, \ldots, \mathfrak{z}_m)|, \tag{2.4}$$

where B is an arbitrary bounded subset of $\mathscr{F}$. Since this space resembles Schwartz space $\mathscr{E}(R^n)$, we shall denote it by the symbol $\mathscr{E}(\mathscr{F})$. The second space $\mathscr{D}(\mathscr{F})$ is the linear space of all C^∞ functionals on $\mathscr{F}$ with a bounded support: the topology will be defined by the system of seminorms

$$\|\Psi\|_{f,m} = \sup_{\mathfrak{z}} \sup_{\|\mathfrak{z}_i\|_F \leqq 1, i=1,\ldots,m} f(\mathfrak{z})\, |D^m \Psi[\mathfrak{z}](\mathfrak{z}_1, \ldots, \mathfrak{z}_m)|, \tag{2.5}$$

where $f \geqq 0$ is a continuous functional on the space $\mathscr{F}$.

The third space $\mathscr{K}(\mathscr{F})$ is the linear space $\Omega(\mathscr{F}) \subset \mathscr{E}(\mathscr{F})$ with a topology defined by seminorms,

$$\|\Psi\|_{\mathscr{K}_{B,m}} = \sup_{\mathfrak{z}\in B} \sup_{\|\mathfrak{z}_i\|_F \leqq 1, i=1,\ldots,m} \left\| \sigma_2 \frac{\delta}{\delta \mathfrak{z}} D^m \Psi[\mathfrak{z}](\mathfrak{z}_1, \ldots, \mathfrak{z}_m) \right\|_F .$$

We now give the representation of Lie algebra Ω in these spaces. We denote for the sake of simplicity by D_F the first order differential operator given by the formula

$$D_F = \int d^3z \left(\frac{\delta F}{\delta \varphi(z)} \frac{\delta}{\delta \pi(z)} - \frac{\delta F}{\delta \pi(z)} \frac{\delta}{\delta \varphi(z)} \right). \tag{2.6}$$

Theorem 1. *Let F be in Ω. Then the operator F associated with a given functional F by the formula*

$$\hat{F}[\mathfrak{z}] = F[\mathfrak{z}] - \tfrac{1}{2} DF[\mathfrak{z}](\mathfrak{z}) - iD_F \tag{2.7}$$

defines the continuous map of the spaces $\mathscr{E}(\mathscr{F})$, and $\mathscr{D}(\mathscr{F})$ into itself. If $F, G \in \Omega$ then for Ψ in $\mathscr{E}$, $\mathscr{K}$ or $\mathscr{D}$ we have

$$i[\hat{F}, \hat{G}]\Psi = \widehat{\{F, G\}}\Psi . \tag{2.8}$$

Proof. The operator $\hat{F}$ by virtue of (2.7) is the differential operator of the first order: the part of order zero is the multiplication by the functional $F - \frac{1}{2} DF$ from Ω which is C^∞ and bounded on bounded subsets: hence it maps the space $\mathscr{E}(\mathscr{F})$ into itself and defines in it the continuous mapping. It remains therefore to analyse the first order part of $\hat{F}$.

By virtue of (2.3) and Definition 1 $D_F \Psi = \{F, \Psi\} = D\Psi[\mathfrak{z}](\mathfrak{z}_F)$ is well defined. Differentiating $D_F \Psi$ m-times we get the sum of terms of the form

$$\int d^3z \left[D^k \left(\frac{\delta F}{\delta \varphi(z)} \right) \frac{\delta D^{m-k} \Psi}{\delta \pi(z)} - D^k \left(\frac{\delta F}{\delta \pi(z)} \right) \frac{\delta D^{m-k} \Psi}{\delta \varphi(z)} \right]. \tag{2.9}$$

By Definition 1, the vector $D^k \mathfrak{z}_F = \left(D^k\left(-\frac{\delta F}{\delta \pi}\right), D^k\left(\frac{\delta F}{\delta \varphi}\right)\right)$ is in the space $\mathscr{F}$: hence (2.9) is equal to

$$D^{m-k+1}\Psi[\mathfrak{z}](D^k \mathfrak{z}_F, \ldots). \tag{2.10}$$

By virtue of regularity of Ψ this derivative is well defined. Hence D_F and consequently also $\hat{F}$ maps $\mathscr{E}$ into $\mathscr{E}$.

The continuity of $\hat{F}$ in $\mathscr{E}(\mathscr{F})$ follows directly from Equation (2.10) and the assumed regularity of Ψ and F.

By the straightforward but tedious calculations one verifies that, on the algebraic level, we have

$$i[\hat{F}, \hat{G}] = \widehat{\{F, G\}}. \tag{2.11}$$

If F, G are in Ω then by (2.3) $\{F, G\}$ is also in Ω and both sides of (2.11) are well defined on $\mathscr{E}(\mathscr{F})$. Consequently, the equality (2.8) holds on $\mathscr{E}(\mathscr{F})$.

It is evident that all above considerations remain true also for the spaces $\mathscr{K}(\mathscr{F})$ and $\mathscr{D}(\mathscr{F})$. $\square$

One may introduce a discrete set of canonical variables $q_l, p_l, l = 1, 2, \ldots$ by writing $\varphi(x) = \sum_l h_l(x) q_l$ and $\pi(x) = \sum_l h_l(x) p_l$ where $\{h_l(x)\}_1^\infty$ is an orthonormal system in L^2 (R^3) which satisfies regularity conditions imposed on elements of $\mathscr{F}$ space. In this case, using the definition of the variational derivative, one may write in a more transparent form the formula (2.7)

$$\hat{F} = F - \frac{1}{2}\sum_l \left(q_l \frac{\partial F}{\partial q_l} + p_l \frac{\partial F}{\partial p_l}\right) - i\sum_l \left(\frac{\partial F}{\partial q_l}\frac{\partial}{\partial p_l} - \frac{\partial F}{\partial p_l}\frac{\partial}{\partial q_l}\right). \tag{2.12}$$

One readily verifies using (2.7) that if $F \in \Omega$ and $\varrho(\cdot)$ is C^∞ then

$$\varrho(\hat{F}) = \varrho(F) + \varrho'(F)[\hat{F} - F]. \tag{2.13}$$

The formula (2.13) implies that $\widehat{F^n} \neq (\hat{F})^n$ in general. Hence the quantization formula (2.7) applied for a product Φ^n of fields gives some "renormalization" counter terms.

It should be stressed that in general counter terms are necessary for a proper product of quantities which have a distributional character. Consequently, the present quantization of field theory may be more effective than conventional quantization schemes, since a renormalization is built in the theory from the beginning.

III. Construction of Interacting Local Quantum Field

Let $\Phi[x|\varphi, \pi]$ be a solution of the dynamical equation

$$(\Box + m^2)\Phi(x) = \lambda\Phi^3(x), \quad \lambda < 0, \quad x = (t, \mathbf{x}) \in R^4 \tag{3.1}$$

defined by the initial conditions $\mathfrak{z} = (\varphi, \pi) \in \mathscr{F}$. We showed in I that Φ is a local relativistic field, with respect to the Poisson bracket Lie algebra, having the local relativistic asymptotic fields Φ_{in} and Φ_{out}. We also showed that the free field $\Phi_\tau(t, \mathbf{x}|\varphi, \pi)$ defined by the initial condition $\Phi_\tau(\tau, \mathbf{x}) = \Phi(\tau, \mathbf{x})$ and $\Pi_\tau(\tau, \mathbf{x}) = \Pi(\tau, \mathbf{x})$ is a local relativistic field.

We begin the construction of a quantum field $\hat{\Phi}(t, \mathbf{x})$ by quantizing first the free field $\Phi_\tau(t, \mathbf{x})$. Let $\hat{\Phi}_\tau(t, \alpha)$ denote the operator field obtained from

$$\Phi_\tau[t, \alpha|\varphi, \pi] = \int d^3\mathbf{x}\alpha(\mathbf{x})\Phi_\tau[t, \mathbf{x}|\varphi, \pi]$$

by formula (2.7). Then we have

Theorem 2. *The operator field $\hat{\Phi}_\tau(t, \alpha)$, for any $\tau \in (-\infty, \infty)$ and $\alpha \in S(R^3)$ is the continuous mapping of the spaces $\mathscr{E}(\mathscr{F})$, $\mathscr{K}(\mathscr{F})$, and $\mathscr{D}(\mathscr{F})$ into itself and satisfies on each of these spaces the commutation relations:*

$$[\hat{\Phi}_\tau(t, \alpha), \hat{\Phi}_\tau(r, \beta)] = i \int d^3\mathbf{x} d^3\mathbf{y} \alpha(\mathbf{x}) \Delta(t-r, \mathbf{x}-\mathbf{y}) \beta(\mathbf{y}) . \tag{3.2}$$

The field $\hat{\Phi}_\tau(t, \alpha)$ is the strongly continuous function of τ and t.

Proof. Let $\mathfrak{z} = (\varphi, \pi)$ be an element of $\mathscr{F}$. Then by Lemma 4 of Appendix A the smeared out field $\Phi[t, \alpha|\mathfrak{z}]$ is an element of the space Ω. Consequently also the functional $\Phi_\tau[t, \alpha|\mathfrak{z}]$ is in the space Ω. Hence by Theorem 1 the operator field $\hat{\Phi}_\tau(t, \alpha)$ is a continuous mapping of the spaces $\mathscr{E}$, $\mathscr{K}$, and $\mathscr{D}$ into itself.

By virtue of Equation (2.14) of I we have

$$\{\Phi_\tau(t, \alpha), \Phi_\tau(r, \beta)\} = -\int d^3\mathbf{x} d^3\mathbf{y} \alpha(\mathbf{x}) \Delta(t-r, \mathbf{x}-\mathbf{y}) \beta(\mathbf{y})$$

hence by virtue of Equation (2.8) we obtain Equation (3.2).

We now prove the continuity of $\hat{\Phi}_\tau$ in τ. For the first order part of $\hat{\Phi}_\tau$ we have

$$(D_{\Phi_{\tau_2}} - D_{\Phi_{\tau_1}})\Psi[\mathfrak{z}] = D\Psi[\mathfrak{z}](\mathfrak{z}_{\Phi_{\tau_2}} - \mathfrak{z}_{\Phi_{\tau_1}}) .$$

Hence we obtain the strong convergence if this expression tends to zero in the topology of $\mathscr{E}(\mathscr{F})$ when $|\tau_2 - \tau_1| \to 0$. The last fact directly follows from (A.16) and (A.17) where we proved that

$$\|D^k(u^0_{\tau_2} - u^0_{\tau_1})\|_F \to 0 \quad \text{when} \quad |\tau_2 - \tau_1| \to 0$$

where u^0_τ is the solution of the free Klein-Gordon equation with the initial conditions at $t=0$ equal to $\mathfrak{z}_{\Phi_\tau}$.

To show the continuity for the zeroth order part of $\hat{\Phi}_\tau$ we note first that

$$|D^m(\Phi_{\tau_1}(t, \alpha) - \Phi_{\tau_2}(t, \alpha))| \leqq c_1 \|D^m(\Phi_{\tau_1} - \Phi_{\tau_2})\|_E \|\alpha\|_E .$$

Hence it is sufficient to show that $\|D^m(\Phi_{\tau_1} - \Phi_{\tau_2})\|_E \to 0$ when $|\tau_1 - \tau_2| \to 0$. Now in Part I Equation (2.29) we have given an evaluation of $\|D(\Phi_{\tau_1} - \Phi_{\tau_2})\|_E$, whose proof can be lifted also to the evaluation of $\|D^m(\Phi_{\tau_1} - \Phi_{\tau_2})\|_{\hat{L}}$. This implies the continuity of the zeroth order part of $\hat{\Phi}_\tau$ in $\mathscr{E}(\mathscr{F})$. Hence $\hat{\Phi}_\tau$ is strongly continuous in $\mathscr{E}(\mathscr{F})$ in τ. The strong continuity of $\Phi_\tau(t, \alpha)$ in t follows from the fact that the solution Φ_τ of free Klein-Gordon equation, generated by the initial data $\Phi(\tau, \cdot)$, $\Pi(\tau, \cdot)$ from $\mathscr{F}$ space is continuous in t.

The proof of the strong continuity of $\hat{\Phi}_\tau(t, \alpha)$ in τ and t in the spaces $\mathscr{K}$ and $\mathscr{D}$ is similar. □

Remark 1. For simplicity of notation in the following we shall write formulae (3.2) and similar formula in the unsmeared form:

$$[\hat{\Phi}_\tau(x), \hat{\Phi}_\tau(y)] = i\Delta(x-y) . \tag{3.3}$$

The operator field $\hat{\Phi}_\tau$ plays the basic role in the determination of the quantum evolution operation $\hat{U}(\tau, \tau_0)$ and the quantum scattering operator $\hat{S}$. These problems will be considered in Part III.

We now describe the quantum interacting field $\hat{\Phi}(t, \alpha)$ associated with the classical field $\Phi(t, \alpha|\varphi, \pi)$ by formula (2.7). We first recall that the function $\Delta^\lambda[x, y|\Phi]$ denotes the Green function of the linear equation

$$(\Box + m^2)u(x) = V(x)u(x)\,, \qquad V = 3\lambda\Phi^2 \tag{3.4}$$

satisfying for $t_x = t_y = r$ the initial conditions $\Delta^\lambda[r, \boldsymbol{x}, r, \boldsymbol{y}|\Phi] = 0$, $(\partial_t \Delta^\lambda)[r, \boldsymbol{x}, r, \boldsymbol{y}|\Phi] = -\delta^3(\boldsymbol{x} - \boldsymbol{y})$. This function can be written as the following series, which after smearing with a test function $\beta(\boldsymbol{y}) \in S(R^3)$ is convergent in the energy norm.

$$\Delta^\lambda[x, y|\Phi] = \Delta(x-y) + \sum_{n=1}^{\infty} (3\lambda)^n \int\!\!\int_{y^0}^{x^0} \ldots \int\!\!\int_{y^0_{n-1}}^{x^0_{n-1}} \Delta(x - x_1)\Phi^2(x_1)\Delta(x_1 - x_2) \ldots \Phi^2(x_n)\Delta(x_n - y)d^4x_1 \ldots d^4x_n\,. \tag{3.5}$$

Theorem 3. *The operator field $\hat{\Phi}(t, \alpha)$ is the continuous mapping of the spaces $\mathscr{E}(\mathscr{F})$, $\mathscr{K}(\mathscr{F})$, and $\mathscr{D}(\mathscr{F})$ into itself and satisfies in the distribution sense on each of these spaces the commutation relations*

$$[\hat{\Phi}(x)\,,\ \hat{\Phi}(y)] = i\widehat{\Delta^\lambda[x, y|\Phi]}\,. \tag{3.6}$$

The map $t \to \hat{\Phi}(t, \alpha)$ is strongly continuous.

Proof. By virtue of Lemma 1 and 2 of Appendix A the functional $\Phi[t, \alpha|\mathfrak{z}]$ belongs to the space Ω. This, by virtue of the equality

$$\{\Phi(x)\,,\ \Phi(y)\} = -\Delta^\lambda[x, y|\Phi]$$

derived in Section 3, of I and Theorem 1 implies the first part of Theorem 3. The last assertion follows from Theorem 2. □

Corollary 1. *The field $\hat{\Phi}(x)$ is local* i.e.

$$[\hat{\Phi}(x)\,,\ \hat{\Phi}(y)] = 0 \quad \text{if} \quad (x-y)^2 < 0 \tag{3.7}$$

and satisfies on $\mathscr{E}$, $\mathscr{K}$, and $\mathscr{D}$ the canonical commutation relations

$$\begin{aligned} &[\hat{\Phi}(t, \boldsymbol{x})\,,\ \hat{\Pi}(t, \boldsymbol{y})] = i\delta^3(\boldsymbol{x} - \boldsymbol{y})\,, \\ &[\hat{\Phi}(t, \boldsymbol{x})\,,\ \hat{\Phi}(t, \boldsymbol{y})] = [\hat{\Pi}(t, \boldsymbol{x})\,,\ \hat{\Pi}(t, \boldsymbol{y})] = 0\,. \end{aligned} \tag{3.8}$$

Proof. If $(x-y)^2 < 0$ then by formula (3.5) $\Delta^\lambda[x, y|\Phi] = 0$. Similarly, if $t_x = t_y$ then $\partial_t \Delta^\lambda[x, y|\Phi] = \delta^3(\boldsymbol{x} - \boldsymbol{y})$. □

The formulae (3.8, 3.6, and 3.5) show that the interacting field has the same distributional character as the free field. Indeed if the map $x \to \hat{\Phi}(x)\Psi$, $\Psi \in \mathscr{E}$ would represent a continuous or smooth map from R^4 into $\mathscr{E}$ then the matrix elements $\langle \Psi', [\hat{\Phi}(x), \hat{\Phi}(y)]\Psi \rangle$, $\Psi' \in \mathscr{E}'$ would represent a continuous or smooth function respectively of the variables x and y. The form (3.5) for the commutator function shows that matrix elements represent a distribution of $S'(R^4)$ type. Consequently the quantum field $\hat{\Phi}(x)$ represent an operator valued distribution of S' type. Let us note, however, that by Theorem 3, $\hat{\Phi}(t, \alpha)\Psi$, $\alpha \in S(R^3)$ is the continuous function of t.

It follows from Remark 1 to Lemma 5 of Appendix A that the regularity properties of Φ, Φ_τ, Φ_{in}, and Φ_{out} fields will not change if we take initial conditions $\mathfrak{z}_{in}=(\varphi_{in}, \pi_{in})$ at $t_0=-\infty$. This implies that all assertions of Theorems 2 and 3 remain true also for this case.

We now find an equation of motion for the quantum field $\hat{\Phi}(t, \boldsymbol{x})$. Acting on the field $\hat{\Phi}(t, \boldsymbol{x})$ by the operator $\square+m^2$ and using Equation (3.1) and (2.7) one finds that $\hat{\Phi}(t, \boldsymbol{x})$ satisfies the following dynamical equation

$$(\square+m^2)\hat{\Phi}(t, \boldsymbol{x})=\lambda\widehat{\Phi^3}(t, \boldsymbol{x}). \tag{3.9}$$

By virtue of Equation (2.13), the interaction term in Equation (3.9) is automatically renormalized: consequently, Equation (3.9) represents a meaningful equality on the space $\mathscr{K}^{\cdot}$. It should be stressed, however, that the dynamical Equation (3.9) losts its primary meaning as a tool for description of a dynamics of interacting quantum fields: in fact, the quantum interacting field is not obtained by a solution of Equation (3.9) but is constructed independently from the classical solution $\Phi(t, \boldsymbol{x})$ by formula (2.7).

It is instructive to apply the present quantization method in case of the free field equation $(\square+m^2)\Phi_0(x)=0$. In this case the solution $\Phi_0[x|\varphi, \pi]$ is given by the formula

$$\Phi_0[t, \boldsymbol{x}|\varphi, \pi]=-\int \Delta(t, \boldsymbol{x}-\boldsymbol{y})\pi(\boldsymbol{y})d^3\boldsymbol{y}+\int \partial_t\Delta(t, \boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})d^3\boldsymbol{y}.$$

Applying the formula (2.7) one obtains the quantum field $\hat{\Phi}_0(x)$ which satisfies the following commutation relations

$$[\hat{\Phi}_0(x), \hat{\Phi}_0(y)]=i\Delta(x-y).$$

Calculating in the standard manner the creation and annihilation operators one easily verifies that the equation $\hat{a}\psi_0=0$ is satisfied by the Poincare invariant functional $\psi_0(\varphi, \pi)=1$ and that the n-particle states are represented by polynomials in canonical variables. The Fock space H is a subspace of $\mathscr{E}(\mathscr{F})$ and the obtained realization is identical with the conventional Bargmann-Segal representation.

IV. Asymptotic Condition

Let $\hat{\Phi}_{in}(t, \alpha)$ and $\hat{\Phi}_{out}(t, \alpha)$ be the operator fields obtained from classical solutions $\Phi_{in}[t, \alpha|\varphi; \pi]$ and $\Phi_{out}[t, \alpha|\varphi; \pi]$ respectively by formula (2.7). Then we have

Theorem 4. *The operator fields $\hat{\Phi}_{in}(t, \alpha)$ and $\hat{\Phi}_{out}(t, \alpha)$, $\alpha\in S(R^3)$ represent the continuous mappings of the spaces $\mathscr{E}(\mathscr{F})$, $\mathscr{K}(\mathscr{F})$, and $\mathscr{D}(\mathscr{F})$ into itself and satisfy on each of these spaces the commutation relations*

$$[\hat{\Phi}_{\substack{in\\out}}(x), \hat{\Phi}_{\substack{in\\out}}(y)]=i\Delta(x-y). \tag{4.1}$$

Proof. The field $\Phi_{out}[t, \alpha|\varphi, \pi]$ belongs to the space Ω by Lemma 4 of Appendix A. Consequently, by virtue of Theorem 1 and the equality $\{\Phi_{out}(x), \Phi_{out}(y)\}=-\Delta(x-y)$ the operator $\hat{\Phi}_{out}$ satisfies all assertions of Theorem 4. The same considerations hold also for Φ_{in} field. $\square$

We now show that the fields $\hat{\Phi}_{\text{in}}(t, \alpha)$ and $\hat{\Phi}_{\text{out}}(t, \alpha)$ are asymptotic to the interacting field $\hat{\Phi}(t, \alpha)$. Because $\hat{\Phi}_\tau(t, \alpha)$ coincides for $t=\tau$ with $\hat{\Phi}(t, \alpha)$ it is in fact sufficient to show the asymptotic condition for $\hat{\Phi}_\tau(t, \alpha)$ field.

Theorem 5. *For every Ψ in $\mathscr{E}(\mathscr{F})$, $\mathscr{K}(\mathscr{F})$ or $\mathscr{D}(\mathscr{F})$ in the topology of these spaces we have*

$$\lim_{\tau \to \mp\infty} \hat{\Phi}_\tau(t, \alpha)\Psi = \hat{\Phi}_{\substack{\text{in}\\ \text{out}}}(t, \alpha)\Psi . \tag{4.2}$$

Proof. It follows from the formula (2.7) and formulae (2.33 and 2.34) of I and regularity of Φ_τ, Φ_{in} and Φ_{out} fields that we have convergence of (4.2) in $\mathscr{E}$, $\mathscr{K}$, and $\mathscr{D}$ for the zeroth order part of the operator $\hat{\Phi}_\tau$. Now, the first order part of $\hat{\Phi}_\tau$, by virtue of (2.3) and the equality $\left(-\frac{\delta\Phi_\tau}{\delta\pi}, \frac{\delta\Phi_\tau}{\delta\varphi}\right)=(u_\tau, \partial_t u_\tau)=\mathfrak{z}_{\Phi_\tau}$ we have

$$D_{\Phi_\tau}\Psi = D\Psi[\mathfrak{z}]\,(\mathfrak{z}_{\Phi_\tau}) . \tag{4.3}$$

For higher order derivatives we have the analogous formula [cf. Eqs. (2.9) and (2.10)]: *hence for the convergence of first order part of operator $\hat{\Phi}_\tau$ it is sufficient* to show the convergence

$$(D^m u_\tau(0,\cdot)\,,\ \partial_t D^m u_\tau(0,\cdot)) \to (D^m u_{\substack{\text{in}\\ \text{out}}}(0,\cdot)\,,\ \partial_t D^m u_{\substack{\text{in}\\ \text{out}}}(0,\cdot)) \tag{4.4}$$

in the F-norm: more precisely, we need to show the convergence of (4.4) as the multilinear operations with values in $\mathscr{F}$. This convergence is shown in Appendix A Equation (A.18). □

V. Relativistic Covariance

Let $\varphi_{\text{in}}(\boldsymbol{x})=\Phi_{\text{in}}(0, \boldsymbol{x})$ and $\pi_{\text{in}}(x)=\Pi_{\text{in}}(0, \boldsymbol{x})$ be initial conditions for classical free field $\Phi_{\text{in}}(x)$. Let $\Phi_{\text{in}}(x)$ represent initial conditions at $t=-\infty$ for the interacting field $\Phi(x)$ which satisfies Equation (3.1). The map $(a, \Lambda) \to U_{(a, \Lambda)}$ in the Banach space $\mathscr{F}$ given by the formula

$$U_{(a, \Lambda)}\Phi_{\text{in}}(x) = \Phi_{\text{in}}(\Lambda^{-1}(x-a)) \tag{5.0}$$

defines the continuous representation of the Poincaré group in the space $\mathscr{F}$. The elements

$$(U_{(a, \Lambda)}\Phi_{\text{in}})\,(0, \boldsymbol{x}) \quad \text{and} \quad (U_{(a, \Lambda)}\Pi_{\text{in}})\,(0, \boldsymbol{x}) \tag{5.1}$$

define the element $\mathfrak{z}_{\text{in}}=(\varphi_{\text{in}}, \pi_{\text{in}})$ after the transformation. We shall denote the transformed element (5.1) by the symbol $U_{(a, \Lambda)}\mathfrak{z}_{\text{in}}$.

The map $(a, \Lambda) \to \hat{U}_{(a, \Lambda)}$ in the space $\mathscr{E}(\mathscr{F})$ given by the formula

$$(\hat{U}_{(a, \Lambda)}\Psi)\,(\mathfrak{z}_{\text{in}}) = \Psi(U^{-1}_{(a, \Lambda)}\mathfrak{z}_{\text{in}}) \tag{5.2}$$

defines the continuous representation of the Poincaré group in $\mathscr{E}(\mathscr{F})$. Similarly, by formula (5.2) one defines the action of the Poincaré group in the spaces $\mathscr{K}(\mathscr{F})$ and $\mathscr{D}(\mathscr{F})$.

Remark 1. The global action of the Poincaré group in case when the canonical variables φ and π are taken at a finite time t_0 can be also easily calculated with the help of Trotter product formula: (cf. Part III, Appendix). We shall not need however this formula in the present work.

We now show the covariance property of the quantum field $\hat{\Phi}(x)$.

Proposition 6. *The field $\hat{\Phi}(x)$ has the following transformation properties relative to the representation $(u, \Lambda) \to \hat{U}_{(a,\Lambda)}$ of the Poincaré group*

$$(\hat{U}_{(a,\Lambda)}\hat{\Phi}(x)\hat{U}^{-1}_{(a,\Lambda)}\Psi)(\mathfrak{z}_{\text{in}}) = \hat{\Phi}(\Lambda x + a)\Psi(\mathfrak{z}_{\text{in}}) . \tag{5.3}$$

Proof. The classical solution $\Phi[x|\mathfrak{z}_{\text{in}}]$ of Equation (3.1), by virtue of formula (5.2) of I satisfies the identity: $\Phi[x|U^{-1}_{(a,\Lambda)}\mathfrak{z}_{\text{in}}] = \Phi[\Lambda x + a|\mathfrak{z}_{\text{in}}]$. This implies by virtue of the quantization formula (2.7), the equality

$$\hat{\Phi}[x|U^{-1}_{(a,\Lambda)}\mathfrak{z}_{\text{in}}] = \hat{\Phi}[\Lambda x + a|\mathfrak{z}_{\text{in}}] .$$

Hence by virtue of Equation (4.3) for every $\Psi(\mathfrak{z}_{\text{in}})$ in $\mathscr{E}(\mathscr{F})$, $\mathscr{K}(\mathscr{F})$ or $\mathscr{D}(\mathscr{F})$ we have

$$\begin{aligned}(\hat{U}_{(a,\Lambda)}\hat{\Phi}(x)\hat{U}^{-1}_{(a,\Lambda)}\Psi)(\mathfrak{z}_{\text{in}}) &= (\hat{\Phi}(x)\hat{U}^{-1}_{(a,\Lambda)}\Psi)(U^{-1}_{(a,\Lambda)}\mathfrak{z}_{\text{in}}) \\ &= \hat{\Phi}[x|U^{-1}_{(a,\Lambda)}\mathfrak{z}_{\text{in}}](\hat{U}^{-1}_{(a,\Lambda)}\Psi)(U^{-1}_{(a,\Lambda)}\mathfrak{z}_{\text{in}}) \\ &= \hat{\Phi}(\Lambda x + a)\Psi(\mathfrak{z}_{\text{in}}) . \quad \square\end{aligned}$$

It is noteworthy that generators $\hat{P}_\mu$ and $\hat{M}_{\mu\nu}$, $\mu,\nu = 0, 1, 2, 3$ can be defined only on $\mathscr{K}(\mathscr{F})$ carrier space. Indeed, since by virtue of Equation (4.5) of I they are bilinear in $\mathfrak{z}_{\text{in}} = (\varphi_{\text{in}}, \pi_{\text{in}})$ variables we have by Equations (2.3) and (2.7)

$$\hat{X}\Psi = -i\{X, \Psi\} = -iDX(\mathfrak{z}_\Psi) = iD\Psi(\mathfrak{z}_X) \tag{5.4}$$

where X is any classical generator of the Poincaré group.

The element $\mathfrak{z}_X = \left(-\frac{\delta X}{\delta \pi_{\text{in}}}, \frac{\delta X}{\delta \varphi_{\text{in}}}\right)$ is not in general in $\mathscr{F}$: for instance if $X = P_0$ then

$$\frac{\delta P_0}{\delta \pi_{\text{in}}(z)} = \pi_{\text{in}}(z) \quad \text{and} \quad \frac{\delta P_0}{\delta \varphi_{\text{in}}(z)} = m^2 \varphi_{\text{in}}(z) + \Delta \varphi_{\text{in}}(z) \tag{5.5}$$

and $\mathfrak{z}_{P_0}$ is not in $\mathscr{F}$ for an arbitrary $\mathfrak{z}_{\text{in}}$ in $\mathscr{F}$. Hence $\hat{X}\Psi$ is an element of the carrier space only when $\mathfrak{z}_\Psi \in \mathscr{F}$. This holds only if we take $\mathscr{K}(\mathscr{F})$ as the carrier space. In this case all fields $\hat{\Phi}$, $\hat{\Phi}_\tau$, $\hat{\Phi}_{\text{in}}$, and $\hat{\Phi}_{\text{out}}$ as well as all generators $\hat{P}_\mu$ and $\hat{M}_{\mu\nu}$ are defined as continuous mappings of $\mathscr{K}$ into $\mathscr{E}$.

Because in the classical field theory we have $P_\mu = P^{\text{in}}_\mu = P^{\text{out}}_\mu$ and $M_{\mu\nu} = M^{\text{in}}_{\mu\nu} = M^{\text{out}}_{\mu\nu}$ by virtue of (2.7) we obtain

$$\hat{P}_\mu = \hat{P}^{\text{in}}_\mu = \hat{P}^{\text{out}}_\mu , \qquad \hat{M}_{\mu\nu} = \hat{M}^{\text{in}}_{\mu\nu} = \hat{M}^{\text{out}}_{\mu\nu} . \tag{5.6}$$

By Equation (5.6) the quantum generators $\hat{P}_\mu$ and $\hat{M}_{\mu\nu}$ are represented by the first order differential operator only. Consequently, the vacuum state Ψ_0 defined by the formula

$$\hat{P}_\mu \Psi_0 = 0 , \qquad \hat{M}_{\mu\nu}\Psi_0 = 0 \tag{5.7}$$

is given in $\mathscr{K}(\mathscr{F})$ by the functional $\Psi_0(\varphi, \pi) = 1$. Hence by (5.7) the interacting and the asymptotic quantum fields have the same vacuum Ψ_0 in $\mathscr{K}(\mathscr{F})$. The elements of the Wightman domain given by the formula

$$\Psi(f_1, \ldots, f_n) = \prod_{i=1}^{n} \hat{\Phi}(f_i)\Psi_0 \tag{5.8}$$

by virtue of Equation (2.7) are represented by the sums of products of Frechet derivatives of the classical field Φ. Hence the Wightman domain as well as the Fock space H_{in} of $\hat{\Phi}_{\text{in}}$ field are subspaces of the carrier space $\mathscr{K}$.

VI. Discussion

A. We have constructed here the relativistic local quantum field $\hat{\Phi}(x)$, possessing the local relativistic asymptotic fields $\hat{\Phi}_{\text{in}}(x)$ and $\hat{\Phi}_{\text{out}}(x)$. The explicit form of the quantum scattering operator $\hat{S}$ which transform $\hat{\Phi}_{\text{in}}$ into $\hat{\Phi}_{\text{out}}$ field can be also given: this problem is considered in Part III of our work. The problem of unitary of the quantum scattering operator reduces to the problem of a construction of an invariant measure in the space of initial conditions.

B. The formula (3.9) shows that the present quantization method provides a certain normal ordering which is given by the formulae

$$N(\hat{\Phi}^n)(x) = \widehat{\Phi^n}(x) . \tag{6.1}$$

Using the formula (2.13) we find that

$$\widehat{\Phi^n}(x) = (\mathrm{I} - n)\Phi^n(x) + n\Phi^{n-1}(x)\hat{\Phi}(x) . \tag{6.2}$$

The powers $N(\hat{\Phi}^n)(x)$ are local respect to the quantum field $\hat{\Phi}(x)$ and all other powers $N(\hat{\Phi}^m)\, n, m = 1, 2, 3, \ldots$. Indeed, using the formula (2.11) we obtain

$$[N(\hat{\Phi}^n)(x),\ N(\hat{\Phi}^m)(y)] = i\overline{nm\Phi^{n-1}(x)\Phi^{m-1}(y)\Delta^{\lambda}}[x, y|\Phi] . \tag{6.3}$$

Hence if x and y are space-like separated then, by virtue of (3.5), $\Delta^{\lambda}[x, y|\Phi] = 0$ and we have:

$$[N(\hat{\Phi}^n)(x),\ N(\hat{\Phi}^m)(y)] = 0 . \tag{6.4}$$

The normally ordered powers $N(\hat{\Phi}^n)$ are also Poincaré covariant: indeed using Equations (5.2), (5.3), and (6.1) we obtain

$$\hat{U}_{(a,\Lambda)} N(\hat{\Phi}^n)(x) \hat{U}^{-1}_{(a,\Lambda)} = N(\hat{\Phi}^n)(\Lambda x + a) . \tag{6.5}$$

The formula (6.4) and (6.5) show that the normal ordering (6.1) satisfies the most important requirements which are usually imposed on normal ordering in the axiomatic quantum field theory. One can easily verify using results of Lemma 2 of Appendix A that the formula (6.1–6.3) after smearing with a test function $\beta(\boldsymbol{x}) \in S(R^3)$ provides well defined operators from the carrier space $\mathscr{K}$ into $\mathscr{K}$.

C. The canonical quantization discussed in the present paper may be extended to a certain class of nonpolynomial interactions $F(\Phi)$. The extension of the present results may be proved by using, in the proofs of Theorems 2–6 the corresponding results for a classical nonlinear relativistic wave equation with a nonpolynomial nonlinear term (cf. 5, App. A).

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Appendix A

In this appendix we shall prove some technical lemmas and inequalities used previously. We first recall the definition of the space $\mathscr{F}$: we say that an element $\mathfrak{z}=(\varphi,\pi)\in\mathscr{F}$ iff $\Phi_0\in\mathscr{F}$ where $\Phi_0(t,\boldsymbol{x})$ is the solution of Klein-Gordon equation with the initial conditions $\Phi_0(0,\boldsymbol{x})=\varphi(\boldsymbol{x})$, $\partial_t\Phi_0(0,\boldsymbol{x})=\pi(\boldsymbol{x})$: in turn an element $\Phi_0\in\mathscr{F}$ iff it has finite F-norm given by

$$\|\Phi_0\|_F^2=\sup_t\|\Phi_0(t,\cdot)\|_E^2+\sup_x|\Phi_0(x)|^2+\int_{-\infty}^{\infty}\sup_{\boldsymbol{x}}|\Phi_0(t,\boldsymbol{x})|^2dt$$

where $\|\cdot\|_E$ is the energy norm.

Lemma 1. *A mapping $\Phi[\cdot|\mathfrak{z}]:\mathscr{F}\ni\mathfrak{z}\to X_F$ where X_F is a Banach space of functions defined on R^4 and having finite F-norm (a completion of $C_0^\infty(R^4)$ in F-norm), is infinitely differentiable in Frechet sense, a differential of m-th order satisfies an equation*

$$(\square+m^2)D^m\Phi(x)-3\lambda\Phi^2(x)D^m\Phi(x)+\sum_{\substack{j+k+l=m\\ j,k\leqq l<m}}c_{j,k,l}D^j\Phi(x)D^k\Phi(x)D^l\Phi(x)=0 \tag{A.1}$$

and an inequality

$$\sup_{\mathfrak{z}\in B}\ \sup_{\|\mathfrak{z}_i\|_F\leqq 1,\,i=1,\ldots,m}\|D^m\Phi[\cdot|\mathfrak{z}](\mathfrak{z}_1,\ldots,\mathfrak{z}_m)\|_F<\infty \tag{A.2}$$

for every bounded subset B of the space $\mathscr{F}$.

Proof. It will be clear from the following considerations that the inequality is satisfied uniformly with respect to $\mathfrak{z}$ from a bounded set B and $\mathfrak{z}_i$ satisfying $\|\mathfrak{z}_i\|_F\leqq 1$, so we shall omit in denotations the functional dependence on $\mathfrak{z}$, $\mathfrak{z}_i$ and the symbols $\sup_{\mathfrak{z}\in B}$, $\sup_{\|\mathfrak{z}_i\|_F\leqq 1,\,i=1,\ldots,m}$. The proof of (A.1). (A.2) and all the further results will be based on the four fundamental inequalities. We shall now write and prove them. The first one is:

$$\begin{aligned}&\left|\iint_\tau^{t-T}\Delta(t-s,\boldsymbol{x}-\boldsymbol{y})u(s,\boldsymbol{y})v(s,\boldsymbol{y})w(s,\boldsymbol{y})dsd^3\boldsymbol{y}\right|^2\\&\leqq c(T)\left(\sup_{\tilde S}\int_{\tilde S}u^2d\tilde S\right)\int_\tau^{t-T}(\|v(s)\|_\infty^2\|w(s)\|_E^2+\|v(s)\|_E^2\|w(s)\|_\infty^2)(t-s)^{-3/2}ds\end{aligned} \tag{A.3}$$

where $\tau\leqq t-T<t$; u,v,w are arbitrary, sufficiently regular, functions, and the supremum in the first bracket is taken with respect to all the backward cones with vertices lying in slab $\tau\leqq s\leqq t$ (more exactly we take the intersections of the cones with the slab).

This inequality is proved in [5] p. 14 as Corollary to Lemma 7. We shall denote as in [5]

$$\langle u\rangle^2=\sup_{\tilde S}\int_{\tilde S}u^2d\tilde S\,.$$

The second inequality is:

$$\left|\iint\limits_{t-T}^{t-t_0} \Delta(t-s, \boldsymbol{x}-\boldsymbol{y})u(s,\boldsymbol{y})v(s,\boldsymbol{y})w(s,\boldsymbol{y})ds d^3\boldsymbol{y}\right|^2 \leqq a(T)\left(\sup_{\tilde{S}}\int_{\tilde{S}} u^2 d\tilde{S}\right)\left(\int_{t-T}^{t-t_0} \|v(s)\|_\infty^2 \|w(s)\|_\infty^2 ds\right), \quad 0 \leqq t_0 < T \tag{A.4}$$

where the meaning of the expressions occuring in it is the same as in (A.3).

We shall now prove (A.4). We use a representation

$$\iint\limits_{t-T}^{t-t_0} \Delta(t-s, \boldsymbol{x}-\boldsymbol{y})uvw ds d^3\boldsymbol{y} = \frac{1}{4\pi}\int_{t-T}^{t-t_0}\int_{|\boldsymbol{x}-\boldsymbol{y}|=t-s} uvw d\tilde{S}\frac{ds}{t-s} + \int_{t-T}^{t-t_0}\int_{|\boldsymbol{x}-\boldsymbol{y}|\leqq t-s} k(\mu)uvw d^3\boldsymbol{y} ds = I_s + I_c$$

$\mu^2 = (t-s)^2 - (\boldsymbol{x}-\boldsymbol{y})^2$, and $k(\mu)$ is a bounded function.

We have:

$$I_s^2 \leqq \left(\int_{t-T}^{t-t_0}\int u^2 d\tilde{S}_s ds\right)\left(\int_{t-T}^{t-t_0}\int v^2 w^2 d\tilde{S}_s \frac{ds}{4\pi(t-s)^2}\right) \leqq \left(\int_{\tilde{S}} u^2 d\tilde{S}\right)\left(\int_{t-T}^{t-t_0} \|v(s)\|_\infty^2 \|w(s)\|_\infty^2 ds\right)$$

$$I_c^2 \leqq c_1\left(\int_{t-T}^{t-t_0}\int_{|\boldsymbol{x}-\boldsymbol{y}|\leqq t-s} u^2 d^3\boldsymbol{y} ds\right)\left(\frac{4}{3}\pi T^3 \int_{t-T}^{t-t_0} \|v(s)\|_\infty^2 \|w(s)\|_\infty^2\right)$$

but

$$\int_{t-T}^{t-t_0}\int_{|\boldsymbol{x}-\boldsymbol{y}|\leqq t-s} u^2 d^3\boldsymbol{y} ds = \int_{t-T}^{t-t_0}\int_0^{t-s}\int_{\tilde{S}_{s,\varrho}} u^2 d\tilde{S}_{s,}\, ds d\varrho$$

and introducing the new variables $\xi = (t-s)+\varrho$, $\eta = (t-s)-\varrho$ instead of ϱ, s we obtain

$$\int_{t-T}^{t-t_0}\int_{|\boldsymbol{x}-\boldsymbol{y}|\leqq t-s} u^2 d^3\boldsymbol{y} ds \leqq \frac{1}{2}\int_0^T\int_\eta^{2T-\eta}\int_{\tilde{S}_{\xi,\eta}} u^2 d\tilde{S}_{\xi,\eta} d\xi d\eta \leqq T\sup_{\eta\in[0,T]}\int_{\tilde{S}_\eta} u^2 d\tilde{S} \leqq T\sup_{\tilde{S}}\int_{\tilde{S}} u^2 d\tilde{S}$$

and this implies (A.4).

The third inequality is a simple generalization of the inequality proved in Appendix II, Part I: if u is a regular solution of an equation

$$(\Box + m^2 + V(x))u(x) = f(x) \tag{A.5}$$

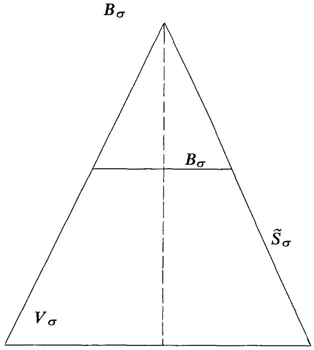
in a domain bounded by a cone $\tilde{S}$ and hyperplane $t=0$ then

$$\varrho(\tau) \leqq \exp\left[\frac{2}{m}\int_0^\tau \sup_{\boldsymbol{x}\in B_\sigma}|V(\sigma,\boldsymbol{x})| d\sigma\right] 2\left\{\varrho(0) + \left(\int_0^\tau \|f(\sigma,\cdot)\|_2^* d\sigma\right)^2\right\} \tag{A.6}$$

where

$$\varrho(\tau)=\sup_{\sigma\in[0,\tau]}\left\{\int_{B_\sigma}\left[\frac{1}{2}(\partial_t u)^2+\frac{1}{2}|\nabla_{\mathbf{x}}u|^2+\frac{1}{2}m^2u^2\right]d^3\mathbf{x}\right.$$
$$\left.+\int_{\tilde S_\sigma}\left[\frac{1}{2}\sum_{j=1}^{3}(\cos(n,x_j)\partial_t u-\cos(n,t)\partial_{x_j}u)^2+\frac{1}{2}m^2\cos^2(n,t)u^2\right]\frac{d\tilde S}{\cos(n,t)}\right\}$$

$\|f(\sigma,\cdot)\|_2^{*2}=\int_{B_\sigma}|f(\sigma,\mathbf{x})|^2d\mathbf{x}$, and $\tilde S_\sigma$, B_σ — as on the picture



Proof of (A.6): Multiply the Equation (A.5) by $\partial_t u$:

$$\partial_t[\tfrac{1}{2}(\partial_t u)^2+\tfrac{1}{2}|\nabla_{\mathbf{x}}u|^2+\tfrac{1}{2}m^2u^2]-\nabla_{\mathbf{x}}\cdot(\nabla_{\mathbf{x}}u\partial_t u)+Vu\partial_t u=f\partial_t u\,,$$

integrating over V_τ:

$$E_\tau-E_0+\int_{\tilde S_\tau}\mathscr{H}_0(x)\cos(n,t)d\tilde S-\int_{\tilde S_\tau}\sum_{j=1}^{3}\cos(n,x_j)\partial_{x_j}u\partial_t u d\tilde S$$
$$+\int_{V_\tau}Vu\partial_t u dx=\int_{V_\tau}f\partial_t u dx$$

where

$$E_\tau=\int_{B_\tau}\mathscr{H}_0(x)d\mathbf{x}\,,\qquad \mathscr{H}_0(x)=\frac{1}{2}(\partial_t u)^2+\frac{1}{2}|\nabla_{\mathbf{x}}u|^2+\frac{1}{2}m^2u^2\,,$$

and transforming the left-hand side of the equality we obtain:

$$E_\tau-E_0+\int_{\tilde S_\tau}\left[\frac{1}{2}\sum_{j=1}^{3}(\cos(n,x_j)\partial_t u-\cos(n,t)\partial_{x_j}u)^2+\frac{1}{2}m^2\cos^2(n,t)u^2\right]\frac{d\tilde S}{\cos(n,t)}$$
$$=-\int_{V_\tau}Vu\partial_t u dx+\int_{V_\tau}f\partial_t u dx\leqq\frac{1}{m}\int_0^\tau\sup_{\mathbf{x}\in B_\sigma}|V(\sigma,\mathbf{x})|E_\sigma d\sigma+\sqrt{2}\int_0^\tau\|f(\sigma,\cdot)\|_2^*\sqrt{E_\sigma}d\sigma$$

hence

$$\varrho(\tau)\leqq\varrho(0)+\frac{1}{m}\int_0^\tau\sup_{\mathbf{x}\in B_\sigma}|V(\sigma,\mathbf{x})|\varrho(\sigma)d\sigma+\left(\int_0^\tau\|f(\sigma,\cdot)\|_2^*d\sigma\right)^2+\frac{1}{2}\varrho(\tau)$$

and

$$\varrho(\tau)\leqq 2\left[\varrho(0)+\left(\int_0^\tau\|f(\sigma,\cdot)\|_2^*d\sigma\right)^2\right]+\frac{2}{m}\int_0^\tau\sup_{\mathbf{x}\in B_\sigma}|V(\sigma,\mathbf{x})|\varrho(\sigma)d\sigma\,.$$

Now, reasoning as in the end of the proof of Lemma 1 in Appendix II, Part I, we get the inequality (A.6). When the integrals $\int_0^\infty \|V(t,\cdot)\|_\infty dt$ and $\int_0^\infty \|f(t,\cdot)\|_2 dt$ are convergent we have the estimate:

$$\sup_{\tilde{S}} \int_{\tilde{S}} u^2 d\tilde{S} + \sup_{t\geqq 0} \|u(t,\cdot)\|_E^2 \leqq c_0 \exp\left[\frac{2}{m}\int_0^\infty \|V(t,\cdot)\|_\infty dt\right] \cdot \left\{\|u(0,\cdot)\|_E^2 + \left(\int_0^\infty \|f(t,\cdot)\|_2 dt\right)^2\right\}. \tag{A.7}$$

The fourth inequality is a slightly modified Lemma 8 in [5]: let u, v, w be any sufficiently regular functions and

$$\Psi(x) = \int_a^b \int \Delta(x-y)u(y)v(y)w(y)d^4y$$

a, b-real numbers, then

$$\begin{aligned}
\|\Psi\|_F &\leqq c_1 \left\{\langle u\rangle \left[\sup_{a\leqq s\leqq b} \|v(s)\|_E \left(\int_a^b \|w(s)\|_\infty^2 ds\right)^{1/2} + \sup_{a\leqq s\leqq b} \|w(s)\|_E \left(\int_a^b \|v(s)\|_\infty^2 ds\right)^{1/2}\right]\right.\\
&\quad + \sup_{a\leqq s\leqq b} \|v(s)\|_\infty \left(\int_a^b \|w(s)\|_\infty^2 ds\right)^{1/2}\\
&\quad \left. + \sup_{a\leqq s\leqq b} \|u(s)\|_E \left(\int_a^b \|v(s)\|_\infty^2 ds\right)^{1/2} \left(\int_a^b \|w(s)\|_\infty^2 ds\right)^{1/2}\right\}\\
&\leqq c_1 \left\{\langle u\rangle \left[\|v\|_F^{[a,b]} \left(\int_a^b \|w(s)\|_\infty^2 ds\right)^{1/2} + \|w\|_F^{[a,b]} \left(\int_a^b \|v(s)\|_\infty^2 ds\right)^{1/2}\right]\right.\\
&\quad \left. + \sup_{a\leqq s\leqq b} \|u(s)\|_E \left(\int_a^b \|v(s)\|_\infty^2 ds\right)^{1/2} \left(\int_a^b \|w(s)\|_\infty^2 ds\right)^{1/2}\right\}\\
&\leqq c_1 \left(\langle u\rangle + \sup_{a\leqq s\leqq b} \|u(s)\|_E\right) (\|v\|_F^{[a,b]}[w]^{[a,b]} + \|w\|_F^{[a,b]}[v]^{[a,b]})
\end{aligned} \tag{A.8}$$

with a constant c_1 independent of u, v, w, a, b; here

$$[v]^{[a,b]} = \left(\int_a^b \|v(s)\|_\infty^2 ds\right)^{1/2}$$

and the meaning of $\|v\|_F^{[a,b]}$ is clear.

Proof of (A.8): At first we shall estimate $|\Psi(x)|^2$ by the help of the inequalities (A.3) and (A.4). We have to consider three cases: $t\leqq a<b$, $a<t<b$, $a<b\leqq t$, but the inequalities for the first two cases are the simple consequences of this for the last by the use of a time reflection and a division of the interval of integration $[a,b]$ into two intervals $[a,t]$ and $[t,b]$. Thus we consider the case $a<b\leqq t$. From (A.3) and (A.4) we have:

$$\begin{aligned}
|\Psi(x)|^2 &= \left|\int_a^{t-1} \int \Delta(x-y)uvwd^4y + \int_{t-1}^b \int \Delta(x-y)uvwd^4y\right|^2\\
&\leqq c\langle u\rangle^2 \left[\int_a^{t-1} (\|v(s)\|_\infty^2 \|w(s)\|_E^2 + \|v(s)\|_E^2 \|w(s)\|_\infty^2) \frac{ds}{(t-s)^{3/2}}\right.\\
&\quad \left. + \int_{t-1}^b \|v(s)\|_\infty^2 \|w(s)\|_\infty^2\right].
\end{aligned}$$

Of course, if $t-1>b$ then the second integral does not occur and in the first we have b instead of $t-1$ as the upper limit of integration. Thus

$$\sup_{x,t\geqq b}|\Psi(x)|^2\leqq c\langle u\rangle^2\Bigg[\sup_{a\leqq s\leqq b}\|w(s)\|_E^2\int_a^b\|v(s)\|_\infty^2 ds+\sup_{a\leqq s\leqq b}\|v(s)\|_E^2\int_a^b\|w(s)\|_\infty^2 ds + \sup_{a\leqq s\leqq b}\|v(s)\|_\infty^2\int_a^b\|w(s)\|_\infty^2 ds\Bigg]$$

because $(t-s)^{-3/2}\leqq 1$ in the first integral, and $\int_b^\infty \sup_x |\Psi(t,x)|^2 dt$ is estimated by the same expression, since on the right hand side we have

$$c\langle u\rangle^2\left[\int_b^{b+1}\int_a^{t-1}(\dots)\frac{ds}{(t-s)^{3/2}}dt+\int_{b+1}^{\infty}\int_a^{b}(\dots)\frac{ds}{(t-s)^{3/2}}dt+\int_b^{b+1}\int_{t-1}^{b}(\dots)dsdt\right]$$

$$\leqq c\langle u\rangle^2\left[\int_a^{b-1}(\dots)\int_b^\infty\frac{dt}{(t-s)^{3/2}}ds+\int_{b-1}^{b}(\dots)\int_{s+1}^{\infty}\frac{dt}{(t-s)^{3/2}}ds+\int_a^b\dots ds\right]$$

$$\leqq c\int_1^\infty\frac{dt}{t^{3/2}}\langle u\rangle^2\left[\int_a^b(\dots)ds+\int_a^b\dots ds\right].$$

Finally

$$\sup_{t\geqq b}\|\Psi(t,\cdot)\|_E^2\leqq\left(\int_a^b\|u(s)v(s)w(s)\|_2 ds\right)^2$$

$$\leqq\frac{1}{m^2}\sup\|u(s)\|_E^2\left(\int_a^b\|v(s)\|_\infty^2 ds\right)\left(\int_a^b\|w(s)\|_\infty^2 ds\right).$$

The same inequalities hold for the remaining cases, so (A.8) is proved.

Now we are ready to prove Lemma 1. The proof will be inductive, the induction with respect to m. For $m=0$ it is proved in [5] and let us consider the general case. We assume that Φ has the Frechet derivatives to m-th order satisfying the Equations (A.1). An expression $\Delta_s D^m\Phi(x)=s^{-1}(D^m\Phi[x|\mathfrak{z}+s\mathfrak{z}']-D^m\Phi[x|\mathfrak{z}]$ satisfies an equation

$$(\Box+m^2)\Delta_s D^m\Phi(x)+\sum_{\substack{j+k+l=m\\ j,k\leqq l<m}}c_{j,k,l}(\Delta_s D^j\Phi(x)D^k\Phi[x|\mathfrak{z}+s\mathfrak{z}']\,D^l\Phi[x|\mathfrak{z}+s\mathfrak{z}'] + D^j\Phi(x)\Delta_s D^k\Phi(x)D^l\Phi[x|\mathfrak{z}+s\mathfrak{z}'] + D^j\Phi(x)D^k\Phi(x)\Delta_s D^l\Phi(x))=0 \tag{A.9}$$

and let us define Φ_1 as a solution of an equation

$$(\Box+m^2)\Phi_1(x)-3\lambda\Phi^2(x)\Phi_1(x)-6\lambda\Phi(x)D'\Phi(x)D^m\Phi(x) + \sum_{\substack{j+k+l=m\\ j,k\leqq l<m}}c_{j,k,l}D'(D^j\Phi(x)D^k\Phi(x)D^l\Phi(x))=0 \tag{A.10}$$

satisfying the initial conditions

$$(\Phi_1(0,\cdot),\ \partial_t\Phi_1(0,\cdot))=\begin{cases}\mathfrak{z}' & \text{if } m=0\\ (0,0) & \text{if } m>0.\end{cases}$$

Taking the difference of (A.9) and (A.10) we obtain an equation for $\Delta_s D^m \Phi(x) - \Phi_1(x)$. It follows from the considerations similar to these presented below that Φ_1 has a finite F-norm so the difference $\Delta_s D^m \Phi(x) - \Phi_1(x)$ has a finite F-norm. We shall estimate it now. From the equation satisfied by this difference we obtain an integral representation of the form:

$$\begin{aligned}\Delta_s D^m\Phi(x) - \Phi_1(x) = &- \iint_0^t d^4y \Delta(x-y) \,[3\lambda\Phi^2(y)\,(\Delta_s D^m\Phi(y) - \Phi_1(y)) \\ &- 3\lambda(\Delta_s\Phi(y)\Phi[y|\mathfrak{z}+s\mathfrak{z}']\, D^m\Phi[y|\mathfrak{z}+s\mathfrak{z}'] - D'\Phi(y)\Phi(y)D^m\Phi(y)) \\ &- 3\lambda(\Phi(y)\Delta_s\Phi(y)D^m\Phi[y|\mathfrak{z}+s\mathfrak{z}'] - \Phi(y)D'\Phi(y)D^m\Phi(y)) \\ &+ \sum_{\substack{j+k+l=m \\ j,k\leq l<m}} c_{j,k,l}(\Delta_s D^j\Phi(y)D^k\Phi[y|\mathfrak{z}+s\mathfrak{z}']\, D^l\Phi[y|\mathfrak{z}+s\mathfrak{z}'] \\ &+ D^j\Phi(y)\Delta_s D^k\Phi(y)D^l\Phi[y|\mathfrak{z}+s\mathfrak{z}'] + D^j\Phi(y)D^k\Phi(y)\Delta_s D^l\Phi[y|\mathfrak{z}+s\mathfrak{z}'] \\ &- D'D^j\Phi(y)D^k\Phi(y)D^l\Phi(y) - D^j\Phi(y)D'D^k\Phi(y)D^l\Phi(y) - D^j\Phi(y)D^k\Phi(y)D'D^l\Phi(y)]\,.\end{aligned}$$

Applying the inequality (A.8) to this representation we obtain

$$\|\Delta_s D^m\Phi - \Phi_1\|_F \leq c\left[\left(\langle \Delta_s D^m\Phi - \Phi_1\rangle + \sup_t \|\Delta_s D^m\Phi - \Phi_1\|_E\right) \|\Phi\|_F^2\right.$$

+ sum of the terms with a typical representative of the form

$$\left(\langle D^p\Phi\rangle + \sup_t \|D^p\Phi\|_E\right) \|\Delta_s D^q\Phi - D'D^q\Phi\|_F \|D^r\Phi\|_F$$

or

$$\left(\langle D^p\Phi\rangle + \sup_t \|D^p\Phi\|_E\right) \|D^q\Phi[\cdot|\mathfrak{z}+s\mathfrak{z}'] - D^q\Phi[\cdot|\mathfrak{z}+s\mathfrak{z}']\|_F \|D^r\Phi\|_F\,,$$

with

$$p, q, r \leq m\,, \qquad q < m$$

in the first form, and functional arguments

$$\mathfrak{z}+s\mathfrak{z}' \quad \text{or} \quad \mathfrak{z}\Big]\,.$$

Now it follows from the induction hypotheses and from the inequality (A.7) that

$$\langle D^p\Phi\rangle + \sup_t \|D^p\Phi\|_E\,, \qquad \|D^r\Phi\|_F$$

are bounded, and

$$\|\Delta_s D^q\Phi - D'D^q\Phi\|_F \to 0\,, \qquad \|D^q\Phi[\cdot|\mathfrak{z}+s\mathfrak{z}'] - D^q\Phi[\cdot|\mathfrak{z}]\|_F \to 0 \quad \text{as} \quad s\to 0$$

so the sum on the right hand side of the above inequality tends to zero. Thus it remains to consider an expression

$$\langle \Delta_s D^m\Phi - \Phi_1\rangle + \sup_t \|\Delta_s D^m\Phi - \Phi_1\|_E\,.$$

The difference $\Delta_s D^m\Phi - \Phi_1$ satisfies an equation of the form (A.5), hence applying the inequality (A.7) we can estimate this expression by

$$c_1 \exp\left(c_2 \int_{-\infty}^{\infty} \sup_{\boldsymbol{x}} \Phi^2(t, \boldsymbol{x}) dt\right)$$

times the sum of the terms of the form

$$\left(\int_{-\infty}^{\infty} \sup_x |D^p\Phi(t,x)|^2 dt\right)^{1/2} \left(\int_{-\infty}^{\infty} \sup_x |D^r\Phi(t,x)|^2 dt\right)^{1/2} \Big[\sup_t \|\Delta_s D^q\Phi - D' D^q\Phi\|_E + \sup_t \|D^q\Phi[t,\cdot|\mathfrak{z}+s\mathfrak{z}'] - D^q\Phi[t,\cdot|\mathfrak{z}]\|_E\Big]$$

which can be estimated by

$$\|D^p\Phi\|_F \|D^r\Phi\|_F (\|\Delta_s D^q\Phi - D'D^q\Phi\|_F + \|D^q\Phi[\cdot|\mathfrak{z}+s\mathfrak{z}'] - D^q\Phi[\cdot|\mathfrak{z}]\|_F)\,.$$

The last expression tends to zero as $s\to 0$ thus we have proved that

$$\|\Delta_s D^m\Phi - \Phi_1\|_F \to 0 \quad \text{as} \quad s\to 0\,.$$

This ends the proof of Lemma 1.

Let us now remind the definition of the function u_τ. It was defined as a solution of the equation

$$(\Box + m^2 - 3\lambda\Phi^2(x))u_\tau(x) = 0$$

with the Cauchy data

$$(u_\tau(\tau,\cdot),\ \partial_t u_\tau(\tau,\cdot)) = (\alpha(\tau,\cdot),\ \partial_t\alpha(\tau,\cdot))$$

where α is a solution of Klein-Gordan equation from $\mathscr{F}$.

Lemma 2. *The function $u_\tau[\cdot|\mathfrak{z}]$ determines a C^∞ mapping – in the Frechet sense – from $\mathscr{F}$ into X_F, a differential of m-th order satisfies an equation*

$$\begin{gathered}(\Box + m^2 - 3\lambda\Phi^2(x))D^m u_\tau(x) + \sum_{\substack{j+k+l=m\\ l<m}} d_{j,k,l} D^j\Phi(x) D^l u_\tau(x) = 0\\ (D^m u_\tau(\tau,\cdot),\ \partial_t D^m u_\tau(\tau,\cdot)) = \begin{cases}(0,0) & \text{for} \quad m>0\\ (\alpha(\tau,\cdot),\ \partial_t\alpha(\tau,\cdot)) & \text{for} \quad m=0\end{cases}\end{gathered} \tag{A.11}$$

and an inequality

$$\sup_{\mathfrak{z}\in B} \sup_{\|\mathfrak{z}_i\|_F \leqq 1,\, i=1,\dots,m} \|D^m u_\tau[\cdot|\mathfrak{z}](\mathfrak{z}_1,\dots,\mathfrak{z}_m)\|_F < \infty \tag{A.12}$$

for every bounded subset B of the space $\mathscr{F}$.

The proof of Lemma 2 is identical to the proof of Lemma 1 because the equations which occur here are of the same form as equations occuring in Lemma 1, so we omit it.

Now we are going to prove the very important fact: the convergence of $D^m u_\tau(\cdot)$ in the space $\mathscr{F}$ when $\tau\to\mp\infty$. More exactly, we shall prove

Lemma 3.

$$\sup_{\mathfrak{z}\in B} \sup_{\|\mathfrak{z}_i\|_F \leqq 1,\, i=1,\dots,m} \|D^m u_{\tau_1}[\cdot|\mathfrak{z}](\mathfrak{z}_1,\dots,\mathfrak{z}_m) - D^m u_{\tau_2}[\cdot|\mathfrak{z}](\mathfrak{z}_1,\dots,\mathfrak{z}_m)\|_F \to 0 \quad \text{as} \quad \tau_1,\tau_2\to\mp\infty \tag{A.13}$$

where B – a bounded set in the space $\mathscr{F}$.

Proof. From the Equation (A.11) we obtain an integral representation

$$D^m u_\tau(x) = \delta_{m,0}\alpha(x) - 3\lambda \iint_\tau^t \Delta(x-y)\Phi^2(y) D^m u_\tau(y) d^4y$$
$$+ \sum_{\substack{j+k+l=m \\ l<m}} d_{j,k,l} \iint_\tau^t \Delta(x-y) D^j\Phi(y) D^k\Phi(y) D^l u_\tau(y) d^4y\,. \tag{A.14}$$

Similarly as in the above considerations the dependence on $\mathfrak{z}$ and $\mathfrak{z}_i$ will be obvious and we shall omit it. The integral representation (A.14) gives us

$$D^m u_{\tau_1}(x) - D^m u_{\tau_2}(x)$$
$$= \sum_{j+k+l=m} d_{j,k,l} \iint_{\tau_2}^t \Delta(x-y) D^j\Phi(y) D^k\Phi(y)\,(D^l u_{\tau_1}(y) - D^l u_{\tau_2}(y)) d^4y$$
$$+ \sum_{j+k+l=m} d_{j,k,l} \iint_{\tau_1}^{\tau_2} \Delta(x-y) D^j\Phi(y) D^k\Phi(y) D^l u_{\tau_1}(y) d^4y \tag{A.15}$$

and applying the inequality (A.8) we get

$$\|D^m u_{\tau_1} - D^m u_{\tau_2}\|_F \leqq c \sum_{j+k+l=m} \left\{ \left(\langle D^l u_{\tau_1} - D^l u_{\tau_2} \rangle \right.\right.$$
$$\left. + \sup_t \|D^l u_{\tau_1}(t,\cdot) - D^l u_{\tau_2}(t,\cdot)\|_E \right) \|D^j\Phi\|_F \|D^k\Phi\|_F + \left(\langle D^l u_{\tau_1} \rangle \right.$$
$$\left.\left. + \sup_t \|D^l u_{\tau_1}(t,\cdot)\|_E \right) (\|D^j\Phi\|_F [D^k\Phi]^{[\tau_1,\tau_2]} + \|D^k\Phi\|_F [D^j\Phi]^{[\tau_1,\tau_2]}) \right\}. \tag{A.16}$$

Now, similarly as in the proof of Lemma 1, we use (A.6) and obtain

$$\langle D^l u_{\tau_1} \rangle + \sup_t \|D^l u_{\tau_1}(t,\cdot)\|_E \leqq c_2 \exp\left(c_3 \int_{-\infty}^{\infty} \|\Phi(t)\|_\infty^2 dt \right)$$
$$\cdot \left[\|\alpha\|_E + \sum_{\substack{j+k+i=l \\ i<l}} [D^j\Phi]\,[D^k\Phi] \sup_t \|D^i u_{\tau_1}(t,\cdot)\|_E \right]$$

– from (A.1) and (A.2) it follows that the right hand side of the inequality is a finite expression, and

$$\langle D^l u_{\tau_1} - D^l u_{\tau_2} \rangle + \sup_t \|D^l u_{\tau_1}(t,\cdot) - D^l u_{\tau_2}(t,\cdot)\|_E$$
$$\leqq c_2 \exp\left(c_3 \int_{-\infty}^{\infty} \|\Phi(t)\|_\infty^2 dt \right) \sum_{\substack{i+j+k=l \\ i<l}} [D^j\Phi]\,[D^k\Phi] \sup_t \|D^i u_{\tau_1}(t,\cdot) - D^i u_{\tau_2}(t,\cdot)\|_E\,. \tag{A.17}$$

By induction we can estimate the left hand side of this inequality by a finite expression multiplied by $\sup_t \|u_{\tau_1}(t,\cdot) - u_{\tau_2}(t,\cdot)\|_E$.

Now, by virtue of Equation (2.39) of I we have

$$\sup_t \|u_{\tau_1}(t,\cdot) - u_{\tau_2}(t,\cdot)\|_E \to 0 \quad \text{as} \quad \tau_1, \tau_2 \to \mp\infty\,.$$

Because we have also

$$[D^k\Phi]^{[\tau_1,\tau_2]} = \left| \int_{\tau_1}^{\tau_2} \|D^k\Phi(t)\|_\infty^2 dt \right|^{1/2} \to 0 \quad \text{as} \quad \tau_1, \tau_2 \to \mp\infty$$

thus the inequality (A.16) implies the condition (A.13). Strictly speaking, what we need it is not the condition (A.13) but a condition

$$\sup_{\mathfrak{z},\mathfrak{z}i} \|D^m u^0_{\tau_1} - D^m u^0_{\tau_2}\|_F \to 0 \quad \text{as} \quad \tau_1, \tau_2 \to \mp\infty \tag{A.18}$$

where $D^m u^0_\tau(x)$ is a solution of Klein-Gordan equation with the Cauchy data at $t=0$ equal to $(D^m u_\tau(0,\cdot), (\partial_t D^m u_\tau)(0,\cdot))$. But using the Equation (A.11) again we obtain another integral representation:

$$D^m u_\tau(x) = D^m u^0_\tau(x) + \sum_{j+k+l=m} d_{j,k,l} \iint_0^t \Delta(x-y) D^j \Phi(y) D^k \Phi(y) D^l u_\tau(y) d^4 y \tag{A.19}$$

and from this and (A.8) we obtain

$$\|D^m u^0_{\tau_1} - D^m u^0_{\tau_2}\|_F \leqq D^m u_{\tau_1} - D^m u_{\tau_2}\|_F + c \sum_{j+k+l=m} \Big(\langle D^l u_{\tau_1} - D^l u_{\tau_2} \rangle + \sup_t \|D^l u_{\tau_1}(t,\cdot) - D^l u_{\tau_2}(t,\cdot)\|_E \Big) \|D^j \Phi\|_F \|D^k \Phi\|_F .$$

Using (A.17) we get finally an equality

$$\|D^m u^0_{\tau_1} - D^m u^0_{\tau_2}\|_F \leqq M \sum_{l=0}^m \|D^l u_{\tau_1} - D^l u_{\tau_2}\|_F$$

and this together with (A.13) imply (A.18). The representation (A.19) gives, by the same considerations, that the function $D^m u^0_\tau(\cdot)$ belongs to the space $\mathscr{F}$.

Lemma 4. *The functionals $\Phi_\tau(t,\alpha)$, $\Phi_{\text{in}}(t,\alpha)$, $\Phi_{\text{out}}(t,\alpha)$ belong to the space Ω.*

Proof. At first we shall consider $\Phi_\tau(t,\alpha)$. It is clear from Lemma 1 that it is infinitely differentiable in the Frechet sense on the space $\mathscr{F}$ and the all differentials are bounded on the bounded subsets of $\mathscr{F}$. For a functional derivative $\delta\Phi_\tau(t,\alpha)/\delta\mathfrak{z}(\boldsymbol{x})$ we have

$$\sigma_2 \frac{\delta \Phi_\tau(t,\alpha)}{\delta \mathfrak{z}(\boldsymbol{x})} = (u_\tau(0,\boldsymbol{x}),\ \partial_t u_\tau(0,\boldsymbol{x}))$$

thus, by Lemma 2 and 3,

$$\sigma_2 \frac{\delta D^m \Phi_\tau(t,\alpha)}{\delta \mathfrak{z}(\boldsymbol{x})} = (D^m u_\tau(0,\boldsymbol{x}),\ \partial_t D^m u_\tau(0,\boldsymbol{x}))$$

and this element has a finite F-norm, equal to $\|D^m u^0_\tau\|_F$. It is bounded on the bounded subsets of $\mathscr{F}$; hence $\Phi_\tau(t,\alpha) \in \Omega$. We have proved in Part I that

$$\Phi_\tau(t,\alpha) \to \Phi_{\substack{\text{in}\\ \text{out}}}(t,\alpha) \quad \text{as} \quad \tau \to \mp\infty .$$

From Lemma 3 it follows that $\Phi_\tau(t,\alpha)$ is convergent in the topology of the space Ω, because

$$\left\| \sigma_2 \frac{\delta}{\delta \mathfrak{z}(\cdot)} D^m \Phi_{\tau_1}(t,\alpha) - \sigma_2 \frac{\delta}{\delta \mathfrak{z}(\cdot)} D^m \Phi_{\tau_2}(t,\alpha) \right\|_F = \|D^m u^0_{\tau_1} - D^m u^0_{\tau_2}\|_F$$

so the functionals $\Phi_{\substack{\text{in}\\ \text{out}}}(t,\alpha)$ belong to Ω also. We have used in the paper the functionals depending on $\mathfrak{z}_{\text{in}}$ not on $\mathfrak{z}$, so we must show that the results of Appendix A are valid for the dependence on $\mathfrak{z}_{\text{in}}$ also. It suffices to show

Lemma 5. *The mappings* $W^{\pm}: \mathfrak{z}_{\substack{\text{in}\\ \text{out}}} \to \mathfrak{z}$ *are the* C^{∞}*-homeomorphisms of the space* $\mathscr{F}$ *onto itself, mapping bounded sets onto bounded sets, similarly the inverse mappings.*

Proof. In the paper of Moravetz-Strauss it was shown that the mappings are the homeomorphisms of the space $\mathscr{F}$ onto itself, transforming bounded sets onto bounded sets, and there was essentially proven the regularity of $W^{\pm}$ ([5], Corollary, p. 25). The regularity of the inverse transformations was proved in Lemma 4. □

To show that, for example, (A.2) remains valid for the dependence on $\mathfrak{z}_{\text{in}}$, let us notice that (A.2) is equivalent with the statement that a mapping

$$\mathscr{F} \ni \mathfrak{z} \to \Phi[\cdot|\mathfrak{z}] \in X_F$$

(a Banach space of functions with finite F-norm) is of C^{∞} class and bounded on bounded subsets of $\mathscr{F}$. But from the above lemma it follows that the same remains valid for a mapping

$$\mathscr{F} \ni \mathfrak{z}_{\text{in}} \to \Phi[\cdot|W^{\pm}\mathfrak{z}_{\text{in}}] \in X_F ,$$

thus (A.2) holds for $\Phi[\cdot|W^{\pm}\mathfrak{z}_{\text{in}}]$. Similar situation is for the other inequalities.

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