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Remarks on the Wilson-Zimmermann Expansion and Some Properties of the m-Point Distribution*

Oluwole Adetunji

Department of Physics, University of Manitoba, Winnipeg R3T 2N2, Canada

Abstract. This paper contains a few simple remarks on a paper by S. Schlieder and E. Seiler. For the special class of local fields treated by these authors we arrive at the same necessary condition for the existence of the Wilson-Zimmermann expansion (considered both as an operator expansion and as an expansion in bilinear forms) of the product of n real scalar fields under the assumption that the singularities occurring as $x_j \to x_{j+1}$; $j=1, 2, \ldots, n-1$, do not influence each other as long as these limits are simultaneously taken.

1. Introduction

In this paper the connection between the Wilson-Zimmermann expansion of the product of n local fields and some properties of the m-point distribution will be discussed. This discussion is restricted to the special class of Wightman fields discussed by E. Seiler and S. Schlieder [1] (see also Ref. [2]).

If the Wilson-Zimmermann expansion exists then the singularities arising in the $2n$-point distributions for the set $\{x_j \to x_{j+1}; j=n+1, n+2, \ldots, 2n-1\}$ must control the singularities of the m-point distribution ($m>2n$) for the set $\{x_j \to x_{j+1};$ $j=k, k+1, \ldots, k+n-2$; with $k \leqq m-n+1\}$. In this paper simple generalizations of one or two lemmas and of the theorem in Ref. [1] will be given. Let us start by considering the Wilson-Zimmermann expansion of the product of n real scalar fields. If $\xi' = (\xi'_1, \ldots, \xi'_{n-1})$ and $x = \frac{1}{n} \sum_{i=1}^{n} x_i$ where $\xi'_j = (x_{j+1} - x_j)/2\varrho$, $j=1, 2, \ldots, n-1$, and $\varrho > 0$, the Wilson-Zimmermann asymptotic expansion can be written [3, 4] in the following manner

$$(\Phi, A(x+\varrho\alpha_1) \ldots A(x+\varrho\alpha_n)\Psi)$$
$$= \sum_{j=1}^{k} f_j(\varrho)(\Phi, C_j(x, \xi')\Psi) + R_{k+1}(\Phi, \Psi; x, \xi') \tag{1.1}$$

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where

$$\lim_{\varrho\downarrow 0}(f_{k+1}(\varrho))^{-1}R_{k+1}(\Phi,\Psi;x,\xi')=0$$

$$\lim_{\varrho\downarrow 0}\frac{f_{j+1}(\varrho)}{f_j(\varrho)}=0$$

and $\Phi, \Psi\in\mathscr{H}$, the Hilbert space of the theory. In Equation (1)

$$\alpha_j=\frac{1}{n}\sum_{i=1}^{j-1} i\xi_i'-\frac{1}{n}\sum_{i=j}^{n-1}(n-i)\xi_i'.$$

Setting $\chi_i=\varrho\alpha_i$, $i=1,2,\ldots,n$, an equivalent form of this expression is

$$\begin{aligned}&A(x+\chi_1)A(x+\chi_2)\ldots A(x+\chi_n)\\&=\sum_{j=1}^{k}s_j(\chi_1,\ldots,\chi_n)B_j(x)+R_{k+1}(x;\chi_1\ldots\chi_n).\end{aligned}\tag{1.2}$$

The fields under consideration are assumed to satisfy the Gårding-Wightman axioms [5, 6].

The following is the major assumption of this paper. When the Wilson-Zimmermann expansion exists as an operator product [see conditions (A) below] the singularities occurring in the vacuum expectation values, i.e. in the Wightman distributions

$$\begin{aligned}(\Omega, A(x_1)A(x_2)\ldots A(x_m)\Omega)&=\mathscr{W}_m(x_1,\ldots,x_m)\\&=W_m(\xi_1,\ldots,\xi_j,\xi_{j+1},\ldots,\xi_{j+n-2},\ldots,\xi_{m-1})\end{aligned}\tag{1.3}$$

for $(\xi_j,\xi_{j+1},\ldots,\xi_{j+n-2})\rightarrow 0$, where $\xi_j=x_{j+1}-x_j$, do not influence one another provided that the limits are simultaneously taken. As a consequence of this assumption and the essential independence of the singularities of $W_m(\xi_1,\xi_2,\ldots,\xi_{m-1})$ for $(\xi_j,\xi_{j+1},\ldots,\xi_{j+n-2})\rightarrow 0$, $j\leqq m-n+1$, it can be concluded that the singularities arising in the $2n$-point distribution, as a consequence of taking the limits $(\xi_n,\xi_{n+1},\ldots,\xi_{2n-1})\rightarrow 0$, must control the singularities of the m-point distribution ($m>2n$) for $(\xi_j,\ldots,\xi_{j+n-2})\rightarrow 0$, $j\leqq m-n+1$, and that the singularities in Equation (2) are already contained in the singularities of $W_{2n}(\xi_1,\ldots,\xi_{2n-1})$ for $(\xi_{n+1},\ldots,\xi_{2n-1})\rightarrow 0$.

The notations to be used will now be stated. These are the same as in Refs. [1], [5], and [6]. With Ω being the invariant cyclic vacuum, the subspace, D, the Gårding domain, of the Hilbert Space $\mathscr{H}$ is given by

$$D=\left\{\Phi\in\mathscr{H}:\ \Phi=\sum_{k=0}^{n}\Phi_k,\ \text{where}\ \Phi_k=\int A(x_1)\ldots A(x_k)\Omega f_k(\underline{x})d\underline{x}\quad n<\infty\right\}.\tag{1.4}$$

In Equation (4) $f_k \in \mathcal{S}(\mathbb{R}^{4k})$, the space of rapidly decreasing test functions over $\mathbb{R}^{4k}$.

$$\zeta, z \in \mathbb{C}^4; x, y, \xi, \eta \in \mathbb{R}^4$$

$$\underset{\sim}{\zeta} = (\zeta_1, \ldots, \zeta_{m-1}); \; \underset{\sim}{\eta} = (\eta_1, \ldots, \eta_{m-1}); \; \underset{\sim}{\xi} = (\xi_1, \ldots, \xi_{m-1})$$

$$\underset{\sim}{\zeta}_{(j,j+n-2)} = (\zeta_1, \ldots, \zeta_{j-1}, \zeta_{j+n-1}, \ldots, \zeta_{m-1}) \quad j \leqq m-n+1$$

$$V^{\pm} = \{\eta : \eta^2 > 0, \eta_0 \gtrless 0\}$$

$$\mathcal{T}^{\pm} = \{\zeta : \zeta = \xi + i\eta, \eta \in V^{\pm}\}$$

$$V^{\pm}_{m-1} = \{\underset{\sim}{\eta} : \underset{\sim}{\eta} = (\eta_1, \ldots, \eta_{m-1}), \; \eta_i \in V^{\pm}, \; i = 1, 2, \ldots, m-1\}$$

$$\mathcal{T}^{\pm}_{m-1} = \{\underset{\sim}{\zeta} : \underset{\sim}{\zeta} = (\zeta_1, \ldots, \zeta_{m-1}), \; \zeta_i = \xi_i + i\eta_i, \; \eta_i \in V^{\pm} \, i = 1, 2, \ldots, m-1\}.$$

We shall also be employing the subspaces D_l of D given by

$$D_l = \left\{ \Phi \in \mathcal{H} : \Phi = \sum_{k=0}^{n} \Phi_k \quad n \leqq l \right\}$$

where the vector-valued distribution Φ_k is given in Equation (4). $\mathcal{H}_l$ denotes the closure of D_l and P_l is the corresponding projection operator

$$\lim_{\varrho \downarrow 0} \equiv \lim_{\varrho \to 0+}$$

$\mathcal{T}'_{m-1}$ is the extended tube.

S_{m-1} is the symmetric group of $m-1$ elements.

For $\pi \in S_{m-1}$, the permuted forward-backward tube is given by

$$\mathcal{T}^{\pm\pi}_{m-1} = \{\underset{\sim}{\zeta} \in \mathbb{C}^{4(m-1)} : \underset{\sim}{\zeta}_\pi \in \mathcal{T}^{\pm}_{m-1}\}.$$

The vector-valued holomorphic function $\Phi_n(z_1, \ldots, z_n)$, the Fourier-Laplace transform of the vector-valued distribution $\tilde{A}(p_1) \ldots \tilde{A}(p_n)\Omega$, is given by

$$\Phi_n(z_1, \ldots, z_n) = \int \exp i \left(\sum_{j=1}^{n} p_j z_j \right) \tilde{A}(p_1) \ldots \tilde{A}(p_n) \Omega \, dp_1 \ldots dp_n. \tag{1.5}$$

2. Necessary Conditions for the Existence of the Wilson-Zimmermann Expansion

It will be shown that a necessary condition for the existence of the Wilson-Zimmermann expansion is that the singularities occurring for $\xi_j, \xi_{j+1}, \ldots, \xi_{j+n-2} \to 0$ simultaneously ($j \leqq m-n+1$) in

$$W_m(\xi_1, \xi_2, \ldots, \xi_j, \ldots, \xi_{j+n-2}, \ldots, \xi_{m-1})$$

have their counterparts in

$$W_{2n}(\xi_1, \ldots, \xi_{2n-1}) \quad \text{for} \quad \xi_{n+1}, \xi_{n+2}, \ldots, \xi_{2n-1} \to 0.$$

(A) The Wilson-Zimmermann expansion is assumed to be valid in the following form

$$A(x+\chi_1)A(x+\chi_2)\dots A(x+\chi_n)$$
$$=\sum_{j=1}^{k} s_j(\chi_1,\dots,\chi_n)B_j(x)+R_{k+1}(x;\chi_1\dots\chi_n) \tag{2.1}$$

and satisfies

$$\lim_{\chi_1\dots\chi_n\downarrow 0}\frac{1}{s_l(\chi_1\dots\chi_n)}\left[A(x+\chi_1)\dots A(x+\chi_n)-\sum_{j=1}^{l-1} s_j(\chi_1\dots\chi_n)B_j(x)\right]=B_l(x). \tag{2.2}$$

In Equations (2.1) and (2.2), since only $n-1$ of the $\chi_1\dots\chi_n$ are independent, we can also write the equations with n replaced by $n-1$. In these equations the limits are assumed to be taken simultaneously and in a fixed direction of each of the χ_i's, $i=1,2,\dots,n-1$. If we denote the left hand side of Equation (2.2) by $C_l(x;\chi_1\dots\chi_{n-1})$, then for $f,g_j\in\mathscr{S}(\mathbb{R}^4)$, $g_j\to\delta$ (the Dirac distribution) the following is assumed to hold

$$\lim_{j\to\infty}(\Phi,\int(C_l(x;\chi_1\dots\chi_{n-1})-B_l(x))\,f(x)g_j(\chi_1)g_j(\chi_2)\dots g_j(\chi_{n-1})$$
$$dx\,d\chi_1\dots d\chi_{n-1}\,\Psi)=0 \tag{2.3}$$

with $\Phi\in\mathscr{H}$ and $\Psi\in D$. Moreover, $B_l(x)$ is assumed to be relatively local to $A(y)$ in the weak sense [1].

The necessary condition for the existence of the Wilson-Zimmermann expansion will now be established.

Lemma 1. *If the Wilson-Zimmermann expansion exists and satisfies the conditions collected under (A), then to each term $s_j(\chi_1\dots\chi_{n-1})B_j(x)$ corresponds a singularity of the 2n-point distribution $W_{2n}(\xi_1,\xi_2,\dots,\xi_{2n-1})$ characterised by $s_j(\xi_{n+1}\dots\xi_{2n-1})$ for $\xi_{n+1},\dots,\xi_{2n-1}\to 0$ simultaneously.*

The proof is by contradiction and is analogous to the $n=2$ case [1].

Let us now consider the case in which the Wilson-Zimmermann expansion is assumed to exist as a sum of bilinear terms [1].

(B) This means that Equations (2.1) and (2.2) are valid but that instead of Equation (2.3) we have

$$\lim_{j\to\infty}(\Phi,\int(C_l(x;\chi_1\dots\chi_{n-1})-B_l(x))\,f(x)g_j(\chi_1)\dots g_j(\chi_{n-1})$$
$$dx\,d\chi_1\dots d\chi_{n-1}\,\Psi)=0 \tag{2.4}$$

for Φ, $\Psi\in D$.

The lemma corresponding to Lemma 2, Section 2 of Ref. [1] will now be stated. Since the proof of this lemma is completely analogous to that of Ref. [1], it will not be repeated. Henceforth it is assumed that

$$(A(x_1)\dots A(x_n)\Omega,\,B_l(g)A(y_1)\dots A(y_k)\Omega)\in\mathscr{S}(\mathbb{R}^{4(n+k)})'$$

where $\mathscr{S}(\mathbb{R}^{4(n+k)})'$ is the space of tempered distributions over $\mathbb{R}^{4(n+k)}$, the dual of $\mathscr{S}(\mathbb{R}^{4(n+k)})$.

Lemma 2. *If the Wilson-Zimmermann expansion exists and satisfies (B), then the $2n$-point distribution $W_{2n}(\xi_1, \ldots, \xi_{2n-1})$ has a part which for $\xi_1, \xi_2, \ldots, \xi_{n-1}, \xi_{n+1}, \xi_{n+2}, \ldots, \xi_{2n-1} \to 0$ simultaneouly becomes as singular as $|s_j(\chi_1, \ldots, \chi_{n-1})|^2$, i.e. $\|C_1(x; \chi_1 \ldots \chi_{n-1})\Omega\|^2 \to 0$ as $\chi_1 \ldots \chi_{n-1} \to 0$.*

3. Connection between the Singularities of the $2n$- and m-Point Distributions in Special Cases

The theorem generalising the theorem in Ref. [1] can be formulated as follows:

Theorem 1. *Suppose the function $r(\zeta_{n+1}, \ldots, \zeta_{2n-1})$ exists and satisfies the following properties*

a) *$r(\zeta_{n+1}, \zeta_{n+2}, \ldots, \zeta_{2n-1})$ is holomorphic in $\mathscr{T}^+_{n-1}$.*

b) *$r(\zeta_{n+1}, \ldots, \zeta_{2n-1}) = r(-\zeta_{n+1}, \ldots, -\zeta_{2n-1})$.*

c) *$r(\zeta_{n+1}, \ldots, \zeta_{2n-1})$ is invariant under the homogeneous Lorentz group.*

d) *$W_{2n}(\zeta_1, \zeta_2, \ldots, \zeta_{n+1}, \ldots, \zeta_{2n-1}) r(\zeta_{n+1}, \ldots, \zeta_{2n-1})$ has a (locally unique) analytic continuation into $(\zeta_1, \zeta_2, \ldots, \zeta_n)$, $(\zeta_{n+1}, \ldots, \zeta_{2n-1}) \in U_R(0)$. Here $U_R(0)$ is the bounded open polydisc $\subset \mathbb{R}^{4(n-1)}$ centered at 0, i.e. $U_R(0) = \{(\xi_{n+1}, \ldots, \xi_{2n-1}): |\xi_j| < R_j, j = n+1, \ldots, 2n-1, R = (R_{n+1}, \ldots, R_{2n-1})\}$. It is independent of $(\zeta_1, \ldots, \zeta_n)$.*

e) *There exists an integer $N > 0$ such that for every compact subset $K \subset V^+$, $\underset{\sim}{\xi} \in U_R$, $\eta_j / \sqrt{\eta_j^2} \in K$, $n+1 \leqq j \leqq 2n-1$ r satisfies the inequality*

$$|r(\underset{\sim}{\xi} + i\underset{\sim}{\eta})| \leqq C_k(\underset{\sim}{\eta}^2)^{-N} \qquad (0 < \underset{\sim}{\eta}^2 \leqq 1)$$

where C_K is a constant depending upon K.

Then $W_m(\underset{\sim}{\xi}) r(\zeta_j, \zeta_{j+1}, \ldots, \zeta_{j+n-2})$, $j \leqq m-n+1$, (where $(n+1, \ldots, 2n-1)$ is relabelled as $(j, j+1, \ldots, j+n-2)$), $m > 2n$, has a (locally unique) analytic continuation to the points $(\zeta_j, \zeta_{j+1}, \ldots, \zeta_{j+n-2}) = 0$, $\underset{\sim}{\xi}_{(j,j+n-2)} \in \mathscr{T}'_{m-n}$.

As in the $n=2$ case, the proof of Theorem 1 can be carried out in three steps each of which is formulated as a lemma (Lemmas 3–5). The proofs of these lemmas are similar to the $n=2$ case and hence only Lemma 3 will be proved.

Lemma 3. *Under the assumptions of Theorem 1, for every $\Psi \in \mathscr{H}$ the function $(\Psi, \Phi_n(z, z+\zeta_1, z+\zeta_1+\zeta_2, \ldots, z+\zeta_1+\ldots+\zeta_{n-1})) r(\zeta_1, \ldots, \zeta_{n-1})$ has a (locally unique) analytic continuation to the points $z \in \mathscr{T}^+$, $(\zeta_1, \ldots, \zeta_{n-1}) \in U_R(0)$ where in this case $R = (R_1, \ldots, R_{n-1})$.*

Proof. We shall be using the edge of the wedge theorem. For fixed $z \in \mathscr{T}^+$ and $\Psi \in \mathscr{H}$ let us start by constructing the two functions

$$F_{1,z} = (\Psi, \Phi_n(z, z+\zeta_1, z+\zeta_1+\zeta_2, \ldots, z+\zeta_1+\ldots+\zeta_{n-1})) r(\zeta_1, \ldots, \zeta_{n-1}) \tag{3.1}$$

$$F_{2,z} = (\Psi, \Phi_n(z+\zeta_1+\ldots+\zeta_{n-1}, \ldots, z+\zeta_1+\zeta_2, z+\zeta_1, z)) r(\zeta_1, \ldots, \zeta_{n-1}). \tag{3.2}$$

Now $F_{1,z}$ is holomorphic when $(\zeta_1, \ldots, \zeta_{n-1}) \in \mathscr{T}^+_{n-1}$; $F_{2,z}$ is holomorphic when $(\zeta_1, \ldots, \zeta_{n-1}) \in \mathscr{T}^-_{n-1}$, $z+\zeta_1+\ldots+\zeta_{n-1} \in \mathscr{T}^+$. If we can show that $F_{1,z}(\underset{\sim}{\xi}+i0) = F_{2,z}(\underset{\sim}{\xi}-i0)$ as elements of $\mathscr{D}(U_R)'$ (the space of distributions over U_R) where $\underset{\sim}{\xi} = (\xi_1, \ldots, \xi_{n-1})$, then by the edge of the wedge theorem there exists a function F_z

which is holomorphic in the sets $\{(\zeta_1, \ldots, \zeta_{n-1}) : (\zeta_1, \ldots, \zeta_{n-1}) \in \mathscr{T}_{n-1}^+\}$, $\{(\zeta_1, \ldots, \zeta_{n-1}) : (-\zeta_1, \ldots, -\zeta_{n-1}) \in \mathscr{T}_{n-1}^+,\ z+\zeta_1+\ldots+\zeta_{n-1} \in \mathscr{T}^+\}$ and $(\zeta : \zeta \in U_R\}$ coinciding with $F_{1,z}$ and $F_{2,z}$ respectively in their domains of definition.

For any function $g \in \mathscr{D}(U_R)$, the space of test functions with compact support in $U_R(0)$, for $\eta \in V_{n-1}^+$ and for $\{(\zeta_1, \ldots, \zeta_{n-1}) : (\zeta_1, \ldots, \zeta_{n-1}) \in \mathscr{T}_{n-1}^-,\ z+\zeta_1+\ldots+\zeta_{n-1} \in \mathscr{T}^+\}$, we define

$$\begin{aligned} \Phi_\eta = & \int \Phi_n(z, z+\xi_1+i\eta_1, \ldots, z+\xi_1+i\eta_1+\ldots+\xi_{n-1}+i\eta_{n-1}) \\ & \cdot r(\xi+i\eta)g(\xi)\,d\xi \end{aligned} \tag{3.3}$$

$$\begin{aligned} \Phi_{-\eta} = & \int \Phi_n(z+\xi_1+i\eta_1+\ldots+\xi_{n-1}+i\eta_{n-1}, \ldots, z+\xi_1+i\eta_1, z) \\ & \cdot r(\xi-i\eta)g(\xi)\,d\xi \end{aligned} \tag{3.4}$$

$$\Phi_+ = \operatorname*{s-lim}_{\substack{(\eta_1,\ldots,\eta_{n-1})\downarrow 0 \\ (\eta_1,\ldots,\eta_{n-1}) \in V_{n-1}^+}} \Phi_\eta \tag{3.5}$$

$$\Phi_- = \operatorname*{s-lim}_{\substack{(-\eta_1,\ldots,-\eta_{n-1})\downarrow 0 \\ (\eta_1\ldots\eta_{n-1}) \in V_{n-1}^-}} \Phi_{-\eta}\,. \tag{3.6}$$

These limits have been shown to exist in Appendix 2 of Ref. [1] on account of assumption (e) (of Theorem 1) which is a sufficient condition for the existence of these limits as distributions. All we have to do is to verify that

$$\left|\int F_{1,z}(\xi+i0)g(\xi)\,d\xi - \int F_{2,z}(\xi-i0)g(\xi)\,d\xi\right| \leqq 0 \tag{3.7}$$

for this will imply the equality we are after. But the left hand side of Equation (3.7) equals

$$|(\Psi, \Phi_+ - \Phi_-)| \leqq \|\Psi\|\ \|\Phi_+ - \Phi_-\|$$

by Schwartz inequality.

Showing that $\|\Phi_+ - \Phi_-\|^2 = 0$ follows the same procedure as in the $n=2$ case. It then follows that

$$F_{1,z}(\xi+i0) = F_{2,z}(\xi-i0) \qquad \xi \in U_R(0)\,.$$

The conclusion of the lemma follows from the edge of the wedge theorem and the generalised Hartogs' theorem. □

Lemma 4. *Under the assumption of Theorem 1*

$$W_m(\zeta_1, \ldots, \zeta_{m-1})\,r(\zeta_{m-n+1}, \zeta_{m-n+2}, \ldots, \zeta_{m-1})$$

has a (locally unique) analytic continuation to the points $(\zeta_1, \ldots, \zeta_{m-n}) \in \mathscr{T}'_{m-n}$, $(\zeta_{m-n+1}, \ldots, \zeta_{m-1}) = 0$.

Let us note that in Lemma 5 below, the elements said to belong to $\mathscr{T}_{m-1}^+$ in the definition of $\mathscr{T}_{m-1}^{+\pi}$ correspond to the permutation $(z_1, z_2, \ldots, z_{j-1}, z_{j+n-1}, z_j, z_{j+1}, \ldots, z_{j+n-2}, z_{j+n}, \ldots z_m)$ of $(z_1, z_2, \ldots, z_{m-1}, z_m)$.

Lemma 5. *Let $F(\zeta_1, \ldots, \zeta_{m-1})$ be holomorphic in $\mathscr{F} = \mathscr{T}^+_{m-1} \cup \mathscr{T}^{+\pi}_{m-1}$ where*

$$\mathscr{T}^{+\pi}_{m-1} = \left\{\xi \in \mathbb{C}^{4(m-1)} : \left(\zeta_1, \ldots, \zeta_{j-2}, \sum_{k=j-1}^{j+n-2} \zeta_k, -\sum_{k=j}^{j+n-2} \zeta_k, \zeta_j, \zeta_{j+1}, \ldots, \zeta_{j+n-3}, \right.\right.$$
$$\left.\left. \zeta_{j+n-1} + \zeta_{j+n-2}, \zeta_{j+n}, \ldots, \zeta_{m-1}\right) \in \mathscr{T}^+_{m-1}\right\},$$

and in the points $(\zeta_j, \zeta_{j+1}, \ldots, \zeta_{j+n-2}) = 0$, $j \leqq m-n+1$, $\xi_{(j,j+n-2)} \in \mathscr{G} \subset \mathscr{T}^+_{m-n}$ ($\mathscr{G}$ open and not empty). Assume that the boundary values $F(\zeta_1, \ldots, \zeta_{j-1}, \xi_j \pm i0, \ldots, \xi_{j+n-2} \pm i0, \zeta_{j+n-1}, \ldots, \zeta_{m-1})$ exist as distributions in the variables $(\xi_j, \ldots, \xi_{j+n-2})$ depending holomorphically on $\xi_{(j,j+n-2)} \in \mathscr{T}^+_{m-n}$. Then F has a (locally unique) analytic continuation to the points $(\zeta_j, \zeta_{j+1}, \ldots, \zeta_{j+n-2}) \in V_\varrho(0)$, $\xi_{(j,j+n-2)} \in \mathscr{T}^+_{m-n}$, where $V_\varrho(0)$ is some real neighborhood of 0 in $\mathbb{C}^{4(n-1)}$.

Once the theorem has been proved, we can in analogy with Ref. [1] conclude by obvious modifications of Lemmas 1 and 2, Section 4 of Ref. [1], that the condition said to be necessary for the existence of the Wilson-Zimmermann expansion is indeed sufficient not only when this expansion is in terms of bilinear forms but also when it is an operator expansion in a modified sense.

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References

1. Schlieder, S., Seiler, E.: Commun. math. Phys. **31**, 137 (1973)
2. Kühn, J., Seiler, E.: Commun. math. Phys. **33**, 253 (1973)
3. Otterson, P.: Nuovo Cimento **18** A, 683 (1973)
4. Adetunji, O.: Nuovo Cimento **29** A, 174 (1975)
5. Streater, R., Wightman, A.S.: PCT, spin and statistics and all that. New York: W.A. Benjamin Inc. 1964
6. Jost, R.: The general theory of quantized fields. Providence: AMS 1965

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