Commun. math. Phys. 48, 191—194 (1976)

Communications in Mathematical Physics

Parity Operator and Quantization of δ-Functions

A. Grossmann

Centre de Physique Théorique, C.N.R.S., F-13274 Marseille Cedex 2, France

Abstract. In the Weyl quantization scheme, the δ-function at the origin of phase space corresponds to the parity operator. The quantization of a function $f(v)$ on phase space is the operator $\int f(v/2)W(v)dvM$, where M is the parity and $W(v)$ the Weyl operator.

Introduction

We are concerned here with the elementary problem of writing down an operator $Q(f)$ which quantizes a function f on (flat) phase space. The existing solutions [1] (see also [2]) all involve, to the best of our knowledge, the performing of Fourier transforms. By contrast, our equation (10 bis) picks up local contribution from the classical function and also exhibits a rather unexpected role played by the parity operator.

1. Displaced Parity Operators

Let E be the phase space for $\nu<\infty$ degrees of freedom, i.e. a 2ν-dimensional vector space over $\mathbb{R}$, with a symplectic form $\sigma(v, a)\cdot(a, v\in E)$. Let $v\rightarrow W(v)\,(v\in E)$ be a Weyl system over E, i.e. a strongly continuous family of unitary operators acting irreducibly on a separable Hilbert space $\mathscr{H}$ and satisfying

$$W(a)W(v)=e^{a}(v)W(a+v)\,. \tag{1}$$

We have introduced the abbreviation

$$e^{a}(v)=e^{2i\pi\sigma(a,\,v)}\,. \tag{2}$$

The family $W'(v)=W(-v)$ also satisfies (1). By the uniqueness theorem of von Neumann, there exists in $\mathscr{H}$ a unitary operator M, determined up to a phase, and such that $W(v)M=MW(-v)$ for every $v\in E$. Since M^2 commutes with the irreducible family of operators $W(v)$, it is a number of modulus 1, which can be adjusted to 1 by a multiplication of a suitable number $e^{i\theta}$ to M. Then $M=M^*$ and M is determined up to a sign.

For every $v \in E$, define $M(v) = W(v)M = MW(-v) = W(v/2)MW(-v/2)$. Every $M(v)$ is both unitary and self-adjoint.

The spectrum of every $M(v)$ consists of the numbers ± 1. The corresponding eigenspaces are ranges of the projection operators $\frac{1}{2}\, W(a/2)\,(1 \pm M)\, W(-a/2)$.

The operators M, W satisfy the relations

$$\begin{aligned} W(a)\, W(b) &= e^a(b)\, W(a+b) \\ M(a)\, M(b) &= e^b(a)\, W(a-b) \\ W(a)\, M(b) &= e^a(b)\, M(a+b) \\ M(a)\, W(b) &= e^b(a)\, M(a-b) \end{aligned} \tag{3}$$

$(a, b \in E)$.

Notice that $M(a)\, M(-a) = W(2a)$.

2. Fourier Transform of the Family $W(a)$

The (symplectic) Fourier transform of a function f on E is defined by

$$\tilde{f}(v) = \int e^v(v_1) f(v_1) dv_1 \,. \tag{4}$$

The invariant measure dv_1 on E shall be normalized by the requirement that $F^2 = 1$. This, together with (2) allows us to write formulae where v, the number of degrees of freedom, does not appear explicitly.

Theorem. *The sign of M can be chosen so that, for every $a \in E$ and all Φ, Ψ in a dense linear subset $\mathcal{D} \subset \mathcal{H}$, one has*

$$\int e^a(v)\, (\Phi, W(v)\Psi) dv = (\Phi, M(a)\Psi); \tag{5}$$

the integral on the l.h.s. of (5) *is absolutely convergent. In shorthand,*

$$\int e^a(v) W(v) dv = W(a) M \,. \quad \square \tag{6}$$

Proof of (5). We shall work in a special representation space $\mathcal{H}$ which will simplify calculations, and will be used in forthcoming papers. Consider in $L^2(E; dv)$ the unitary operators $T^a\!:(T^a\Phi)(v) = \Phi(v-a)$ and $E^a\!:(E^a\Phi)(v) = e^{2i\pi\sigma(a,v)}\Phi(v)$. We have $T^a E^b = e^a(b) E^b T^a$, $FT^a = E^{-a}F$, $FE^a = T^{-a}F$, $MT^a = T^{-a}M$, $ME^a = E^{-a}M$. Here F is defined by (4) and $(M\Phi)(v) = \Phi(-v)$.

The operators

$$W^{\mathrm{reg}}(a) = T^{-a} E^{-a} \tag{7}$$

satisfy (1) but act reducibly on $L^2(E; dv)$.

A closed invariant irreducible subspace of $L^2(E; dv)$ may be constructed with the help of a σ-allowed complex structure on E, i.e. an $\mathbb{R}$-linear map J satisfying $J^2 = -1$, $\sigma(Ja, Jv) = \sigma(a, v)$ $(a, v \in E)$ and $\sigma(a, Ja) > 0$ $(a \in E,\ a \neq 0)$. Define $s(a, v) = \sigma(a, Jv)$ and $h(a, v) = s(a, v) + i\,\sigma(a, v)$. Now introduce $\mathcal{H}$ as the set of continuously differentiable functions in $L^2(E; dv)$ that satisfy the modified Cauchy-Riemann equations:

$$(\nabla^a \Phi)(v) + 2\pi s(a, v)\Phi(v) = i\,[(\nabla^{Ja}\Phi)(v) + 2\pi s(Ja, v)\Phi(v)]$$

for all $a \in E$. Here

$$(\nabla^a \Phi)(v) = \left(\frac{d}{d\lambda} \Phi(v + \lambda a)\right)_{\lambda=0} .$$

The scalar product in $\mathscr{H}$ is $\int \bar{\Phi}(v)\Psi(v)dv$.

Let $\Omega(v) = e^{-\pi s(v,v)}$. Then $\mathscr{H}$ consists exactly of the functions $\Phi(v) = \Omega(v)\varphi(v)$ where φ belongs to the holomorphic representation space of Bargmann.

The operators $W(a) = T^{-a}E^{-a}$ act irreducibly on $\mathscr{H}$.

Consider in $\mathscr{H}$ the family of coherent states

$$\Omega^a = W(a)\Omega = T^{-a}E^{-a}\Omega .$$

One has

$$W(a)\Omega^b = e^a(b)\Omega^{a+b}, \qquad M\Omega^b = \Omega^{-b} .$$

Furthermore, the complex conjugate of $\Omega^a(v)$ is $\Omega^v(a)$, since

$$\Omega^a(v) = \Omega(a)\Omega(v)e^{-2\pi h(a,v)} .$$

We have

$$F\Omega^a = FT^{-a}E^{-a}\Omega = T^aE^aF\Omega = \Omega^{-a} \tag{8}$$

since $F\Omega = \Omega$.

The linear span of the Ω^a is dense in $\mathscr{H}$; so we have proved that $F\Phi = M\Phi$ for every $\varphi \in \mathscr{H}$. One obtains next from (8)

$$\int e^a(v)\Omega^v dv = \Omega^a \tag{9}$$

(in the sense, say, of pointwise convergence). Finally, (9) gives, for all $a, c \in E$

$$\int e^a(b)W(b)db\Omega^c = \int e^a(b)e^b(c)\Omega^{b+c}db = e^c(a)\Omega^{a-c} = W(a)M\Omega^c$$

which proves (5) on the dense set of finite linear combinations of coherent states.

3. Weyl Quantization of δ-Functions

Given a function or distribution f on E, the Weyl quantization procedure consists in associating to it the operator $Q(f)$ in $\mathscr{H}$, formally defined by

$$Q(f) = \int \tilde{f}(v)W(-v/2)dv . \tag{10}$$

One has $Q(1) = 1$, and

$$Q(T^af) = W(a)Q(f)W^{-1}(a)$$

in agreement with the interpretation of $W(a)$ as displacement operator.

We are interested in the quantization of δ_a, the δ-function located at the point a of phase space. The operator $Q(\delta_a)$ is formally given by

$$Q(\delta_a) = \int e^{-a}(v)W(-v/2)dv . \tag{11}$$

It is claimed that

$$Q(\delta_a)=2^{2\nu}W(a)MW(-a)=2^{2\nu}W(2a)M\,. \tag{12}$$

The assertion (12) follows immediately from (5).

The expression (10) can now be supplemented by

$$Q(f)=2^{2\nu}\int f(v)M(2v)dv=\int f(v/2)W(v)dvM \tag{10. bis}$$

involving the classical function $f(v)$ itself rather than its symplectic Fourier transform. This is often rather useful: consider e.g. the equation $Q(h)=Q(f)Q(g)$. Simultaneous use of (10) and of (10. bis) allows us to relate in a simple way the supports of $f, g, h, \tilde{f}, \tilde{g}, \tilde{h}$ which all have physical significance.

It is instructive to compute

$$Q(\delta_v)\Omega^b=2^{2\nu}e^{-2v}(b)\Omega^{2v-b}\,.$$

Notice that $2v-b$ is obtained from b through reflection at the point v. Consequently the expectation value $(\Omega^b, Q(\delta_v)\Omega^b)$ is peaked at $b=v$, as it should physically.

As a final exercise, we look at (10. bis) in the x-representation, with $\nu=1$ and $\hbar=1$. One has then

$$(W(x,p)\psi)(x')=e^{-\frac{1}{2}ixp}e^{ipx'}\psi(x'-x)$$

$$(M\psi)(x)=\psi(-x)$$

$$dv=(4\pi)^{-1}dxdp$$

$$\sigma(v,v')=(4\pi)^{-1}(px'-xp')\,.$$

If $f(v)=f(x,p)$ depends only on x, a trivial explicitation of (10. bis) gives $(Q(f)\psi)(x')=f(x')\psi(x')$.

References

1. Pool, J. C. T.: J. Math. Phys. **7**, 66 (1966)
2. Grossmann, A., Loupias, G., Stein, E. M.: Ann. Inst. Fourier **18**, 343 (1968)

Communicated by H. Araki

Received October 29, 1975