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Global Properties of Radial Wave Functions in Schwarzschild's Space-Time

I. The Regular Singular Points

S. Persides

University of Thessaloniki, Thessaloniki, Greece

Abstract. The radial factor $R(x, x_s)$ of a scalar field in Schwarzschild's space-time satisfies a second order ordinary differential equation with two regular singular points at $x=0$ and $x=x_s$ and one irregular singular point at $x=\infty$. The analytical properties of four solutions $\mathscr{R}_1$, $\mathscr{R}_2$, $\mathscr{R}_3$, and $\mathscr{R}_4$ (defined by their power series expansions about $x=0$ and $x=x_s$) with respect to x_s are studied. An analytical continuation is given for each solution outside its circle of convergence. Relations to the flat-space solutions are established. Finally the coefficients relating linearly any three of these solutions are determined and studied as functions of the parameter x_s.

1. Introduction

Physical phenomena around black holes are being studied recently with increasing interest and success. The most rigorous of these studies refer to weak fields imposed on a known curved background, usually the Schwarzschild or Kerr space-time. The pattern followed in solving such problems is familiar. A numerical or approximate study [1–3] indicates the answers and then analytical methods are used to establish rigorously the results. Combinations of numerical and analytical methods [4–9] have been used in cases where some of the answers can be established rigorously and some cannot.

The obstacles preventing a rigorous analytical treatment of perturbation phenomena in curved space-times are related directly to the procedure followed in such studies. Since after linearization the essence of the problem is contained in a second order linear partial differential equation [10–12], we have to separate the partial differential equation into ordinary differential equations using the method of separation of variables and then try to solve the ordinary differential equations. The separation in Schwarzschild's space-time is simple in all cases of scalar, electromagnetic and gravitational radiation. In Kerr's space-time Carter [13, 14] and Teukolsky [15, 16] have succeeded in separating the original partial differential equation into ordinary differential equations with independent variables the angles θ and φ and the radial coordinate r, the time t having been taken out through a Fourier transformation. Thus further studies [17–19] have been made possible, but a major obstacle prevents a completely analytical treatment of wave fields around black holes. In both Schwarzschild's and Kerr's space-times the (ordinary) radial differential equation cannot be solved. This difficulty has been encountered repeatedly in many works [20–23] and has forced the use of numerical techniques [9, 1] or an effective potential [24, 4, 5, 2, 6, 7].

For the time-independent scalar, electromagnetic or gravitational perturbations the radial equation has three regular singular points [25–29] and it can be easily reduced to the hypergeometric differential equation with well known solutions. For time-dependent perturbations in both Schwarzschild's and Kerr's space-times the radial equation has two regular singular points and one irregular singular point [21, 23]. Its solutions are very little known and we need to study them further. We want to know how they behave as functions of the independent variable r and other parameters, how they are connected to the solutions of the corresponding case in flat space and how their behavior at a certain point relates to the behavior at another point. The discovery and proof of such properties is expected to be much more difficult and complicated than for the Bessel functions which represent the special case in flat space-time.

Previous works [20, 21, 23] have shed some light on the nature of the radial wave functions, but all have run into difficulties which sooner or later make proofs based on numerical calculations necessary. In this paper we focus our attention on the radial equation obtained after separating the scalar wave equation in Schwarzschild's space-time. Our objective is to establish through analytical methods certain properties of the radial wave functions. More specifically we are interested in the answers to the following three questions which are raised in the studies of physical phenomena around black holes:

(a) How can a radial function (a solution of the radial equation) defined by a power series expansion about a regular singular point be continued outside its circle of convergence?

(b) How are the solutions of the radial equation related to the spherical Bessel functions?

(c) Any three radial functions satisfy a linear equation with coefficients depending on the Schwarzschild radius r_s [see (18) below]. Can we find explicit expressions of these coefficients?

In Section 2 we present the flat-space case in a way which will be useful for reference later in the paper and we formulate the problem for the curved-space case. In Section 3 we examine the solutions as defined by power series about the regular singularity at $x=0$ and in Section 4 about $x=x_s$. In both cases analytical continuations of the solutions are given outside their circle of convergence. In Section 5 we determine the radial function that reduces to $n_l(x)$. In Section 6 we calculate the coefficients $K_{ij}(x_s)$ (which relate linearly any three radial functions) as functions of x_s. Some closing remarks are presented in Section 7. Theorems and properties needed in intermediate steps but not related directly to the radial functions are presented independently in the appendices.

The point $x=\infty$ is an irregular singular point of the radial equation and the method to be followed is somewhat different. For this reason we will present the corresponding results for the solutions defined by their asymptotic expansions about $x=\infty$ in a subsequent paper.

In the presentation of properties, theorems and other results we have given emphasis on the method followed and the final result while minimizing the intermediate and some times long calculations. Although the notation is explained where it is used for the first time in the paper we assume in general the following: The statement $f(y)=o[g(y)]$ as $y \to y_o$ means that $\lim[f(y)/g(y)]=0$ as $y \to y_0$. All quantities with negative subscripts, i.e. a_{-1}, β_{-1}, Φ_{-1} are identically zero by definition. In all sums the index takes all the integer values from 0 to ∞, unless it is noted specifically otherwise. P_l and Q_l are the Legendre functions of the first and second kind with argument $1-2y$. The asterisk on a quantity means that wherever the imaginary unit i appears in S, it has been replaced by $-i$ in S^*. Thus if i.e. $S=1+iz$ (z complex in general), then $S^*=1-iz$.

2. Formulation of the Problem

2.1 The Flat-Space Case

In flat space-time the development in time of a scalar or electromagnetic field is determined by the wave equation

$$\Box \Psi \equiv g^{\mu\nu}\Psi_{;\mu\nu}=0\,, \tag{1}$$

where $g^{\mu\nu}$ is the contravariant form of the metric tensor. If we consider a separable solution of (1)

$$\Psi=R(r)Y(\theta,\varphi)e^{-i\omega t}\,, \tag{2}$$

then the radial factor $R(r)$ satisfies the equation [30]

$$x^2d^2R/dx^2+2xdR/dx+[x^2-l(l+1)]R=0\,, \tag{3}$$

where $x=kr$ with $k=w/c$ (c is the velocity of light). If we set

$$R(x)=e^{ix}F(x) \tag{4}$$

then $F(x)$ satisfies the equation

$$x^2d^2F/dx^2+(2ix^2+2x)dF/dx+[2ix-l(l+1)]F=0\,. \tag{5}$$

The differential equations (3) and (5) have one regular singular point at $x=0$ and one irregular singular point at $x=\infty$. A pair of linearly independent solutions, the spherical Bessel functions, can be written in a form appropriate for this paper as

$$\begin{aligned} j_l(x)&=\sum_n((-1)^l2^l(l+n)!/n!(2l+2n+1)!)x^{l+2n}\\ &=e^{ix}\sum_n((-i)^n2^{l+n}(l+n)!/n!(2l+2n+1)!)x^{l+n} \end{aligned} \tag{6}$$

for $|x|<+\infty$ and

$$n_l(x)=e^{ix}\sum_{n=0}^{l}((-i)^{n+2}(2l-n)!/2^{l-n}n!(l-n)!)x^{n-l-1} \tag{7}$$

for $0<|x|<+\infty$. Although $x=\infty$ is an irregular singular point, we can express the solutions about $x=\infty$ in a closed form as

$$h_l^{(1)}(x)\equiv h_l(x)=j_l(x)+in_l(x)=e^{ix}\sum_{n=0}^{l}(i^{n-l-1}(l+n)!/2^n n!(l-n)!)x^{-n-1} \tag{8}$$

and

$$h_l^{(2)}(x)=h_l^*(x)\,, \tag{9}$$

where the asterisk denotes the expression obtained from (8) after replacement of i by $-i$. The functions $h_l^{(1)}(x)$ and $h_l^{(2)}(x)$ are the spherical Hankel functions and their expansions (8) and (9) converge obviously for all $x\neq 0$ including the point $x=\infty$. The functions $j_l(x)$ and $h_l(x)$ are called *interior* and *exterior* solutions respectively, since they satisfy the appropriate boundary conditions [30] at $x=0$ and $x=\infty$. They will be related to the solutions of the radial equation in Schwarzschild's space-time.

2.2 The Curved-Space Case

In Schwarzschild's space-time the metric tensor

$$g_{\mu\nu}=\mathrm{diag}[(1-r_s/r)c^2,\ -(1-r_s/r)^{-1},\ -r^2,\ -r^2\sin^2\vartheta] \tag{10}$$

introduces a parameter $r_s=2GMc^{-2}$, the Schwarzschild radius. The radial factor R of a scalar field ψ satisfying (2) obeys the differential equation

$$x(x-x_s)^2d^2R/dx^2+(x-x_s)(2x-x_s)dR/dx+[x^3-l(l+1)(x-x_s)]R=0\,, \tag{11}$$

where $x=kr$ and $x_s=kr_s$. Generalizing transformation (4) to (A is independent of x)

$$R=A\exp[ix+ix_s\ln(x-x_s)]\cdot F \tag{12}$$

we obtain for F the differential equation [21]

$$x(x-x_s)d^2F/dx^2+(2ix^2+2x-x_s)dF/dx+[2ix-l(l+1)]F=0\,. \tag{13}$$

Obviously (3) and (5) are special cases of (11) and (13) for $x_s=0$.

If we change the independent variable to $y=x/x_s$, (11) and (13) become

$$y(y-1)^2d^2R/dy^2+(y-1)(2y-1)dR/dy+[y^3x_s^2-l(l+1)(y-1)]R=0 \tag{14}$$

and

$$y(y-1)d^2F/dy^2+(2ix_sy^2+2y-1)dF/dy+[2ix_sy-l(l+1)]F=0\,. \tag{15}$$

Thus a radial function R can be considered and discussed as a solution $R=R(x,x_s)$ of (11) or as a solution $R=R[y,x_s]$ of (14). We will call such a procedure an *R-description* of the radial function. Alternatively a radial function can be considered and discussed through (12) with $F=F(x,x_s)$ a solution of (13) or $F=F[y,x_s]$ a solution of (15). We will call such a procedure an *F-description* of the radial function. As we will see later for some radial functions both descriptions are simple, while for others one is complicated. We also have an F-description if i in the exponent of (12) is replaced by $-i$.

The point $x=0$ is a regular singular point of (11). Consequently we can find two solutions $\mathscr{R}_1$ and $\mathscr{R}_2$ as power series of x converging up to the next singular point. Similarly we can find two other solutions $\mathscr{R}_3$ and $\mathscr{R}_4$ as power series of $x-x_s$ converging around the other regular singular point $x=x_s$ of (11). Finally we can find two more solutions $\mathscr{R}_5$ and $\mathscr{R}_6$ as asymptotic expansions around the irregular singular point $x=\infty$. Each $\mathscr{R}_i$ $(i=1,2,3,4,5,6)$ depends on x and x_s or on y and x_s. We express this fact by writting

$$\mathscr{R}_i \equiv \mathscr{R}_i(x, x_s) \equiv \mathscr{R}_i[y, x_s]\,. \tag{16}$$

If $W[\mathscr{R}_i, \mathscr{R}_j]$ is the Wronskian of any two solutions of (11), it can be easily shown that

$$W[\mathscr{R}_i, \mathscr{R}_j] \equiv \mathscr{R}_i d\mathscr{R}_j/dx - \mathscr{R}_j d\mathscr{R}_i/dx = K_{ij}/x(x-x_s) \tag{17}$$

and

$$K_{ij}\mathscr{R}_k + K_{jk}\mathscr{R}_i + K_{ki}\mathscr{R}_j = 0\,, \tag{18}$$

where $K_{ij} = K_{ij}(x_s)$ depend on x_s but not on x.

We are ready now to formulate in a precise form the questions to be answered in this paper as follows:

(a) What can be said about $\mathscr{R}_i$ as a function of (the complex in general) x_s?

(b) Can we give an analytic continuation of $\mathscr{R}_i$ outside the circle of convergence of the defining series?

(c) To which solution of the flat-space case each $\mathscr{R}_i$ reduces when x_s (or r_s) goes to zero?

(d) Can we give an explicit expression for $K_{ij}(x_s)$?

(e) What can be said about the limits $\lim K_{ij}$ when x_s (or r_s) goes to zero?

The answers for indices i and j equal to 1, 2, 3, 4 will be given in the following sections. The answers for indices 5 and 6 will be given in another paper, since a slightly different procedure has to be followed.

In the following sections we will use a theorem which can be obtained easily from well known theorems [31] on the behavior of solutions of a differential equation with respect to a parameter. We state the theorem without proof in a form appropriate for our case.

Theorem. *Let the differential equation*

$$d^2R/dz^2 + p(z,s)dR/dz + q(z,s)R = 0 \tag{19}$$

have isolated singularities at z_ν $(\nu=1,2,\ldots,N)$ *and coefficients* $p(z,s)$ *and* $q(z,s)$ *analytic in z for every* $z \neq z_\nu$ *and linear in s. Let* D_z *be a finite* $(z=\infty \notin D_z)$ *region of the complex z-plane such that* $z_\nu \notin D_z$ *and let also* D_s *be a finite* $(s=\infty \notin D_s)$ *region of the complex s-plane. If for a solution* $R(z,s)$ *of* (19) *the quantities* $R(z_0, s)$ *and* $[dR(z,s)/dz]_{z=z_0}$ *are analytic functions of s on* D_s *for a fixed* $z_0 \in D_z$, *then* $R(z,s)$ *is an analytic function of s on* D_s *for every* $z \in D_z$.

The usefulness of the above theorem is due to the fact that if we divide (14) and (15) by $y(y-1)^2$ and $y(y-1)$ the resulting equations have coefficients p and q analytic in y (except at the points $y=0$, $y=1$ and $y=\infty$) and linear in x_s^2 and x_s respectively. Thus they satisfy the conditions of the theorem.

3. The Solution about $x=0$

3.1 The R-Description

The radial function $\mathscr{R}_1$ is defined [21] as that solution of (11) which for $|x|<|x_s|$ is given by the absolutely convergent power series

$$\mathscr{R}_1 \equiv \mathscr{R}_1(x, x_s) = \sum_n a'_n x^n \tag{20}$$

with $a'_0 = 1$ and

$$n^2 x_s^2 a'_n + [l(l+1)-(n-1)(2n-1)] x_s a'_{n-1} + [(n-1)(n-2)-l(l+1)] a'_{n-2} + a'_{n-4} = 0\,. \tag{21}$$

Note that in evaluating the successive a'_n's we divide at each step by x_s^2. Also note that when $x_s \to 0$, the circle of convergence of the series (20) vanishes. Thus $\mathscr{R}_1$ as given by (20) cannot be studied as a function of x_s near $x_s = 0$.

If we set $y = x/x_s$ in (20) and (21) we obtain for $|y|<1$

$$\mathscr{R}_1 \equiv \mathscr{R}_1[y, x_s] = \sum_n \alpha'_n y^n \tag{22}$$

with $\alpha'_0 = 1$, $\alpha'_n = a'_n x_s^n$ and

$$n^2 \alpha'_n + [l(l+1)-(n-1)(2n-1)] \alpha'_{n-1} + [(n-1)(n-2)-l(l+1)] \alpha'_{n-2} + x_s^2 \alpha'_{n-4} = 0\,. \tag{23}$$

Note now that the coefficient of α'_n in (23) is n^2 and that the circle of convergence has radius 1. We can prove that $\mathscr{R}_1[y, x_s]$ *is an entire function of* x_s^2 *for every finite* $y \neq 1$. The proof of this property follows a combined application of the theorem of Section 2.2 and Theorem A1 in Appendix A. Since the series (22) satisfies the conditions of Theorem A1, it converges uniformly for some y near $y=0$. But α'_n is an entire function of x_s^2. Consequently $\sum_n \alpha'_n y^n$ is an entire function of x_s^2 for some y. The same is true for the series $\sum_n n\alpha'_n y^n$. From the theorem of Section 2.2 we conclude that $\mathscr{R}_1[y, x_s]$ is an entire function of x_s^2 for every y different from the singularities of (11). For $y=0$ the property is obvious.

Since $\mathscr{R}_1[y, x_s]$ is an entire function of x_s^2 we can write

$$\mathscr{R}_1[y, x_s] = \sum_n \Phi'_n(y) x_s^{2n}\,. \tag{24}$$

This is an analytic continuation of $\mathscr{R}_1$ outside the circle of convergence of the series (20) or (22).

Substituting expression (24) into (14) we find that the sequence $\Phi'_n(y)$ satisfies the differential equation (B 1). Moreover, from (22) we have the relations

$$\mathscr{R}_1[0, x_s] = 1\,, \qquad d\mathscr{R}_1[y, x_s]/dy|_{y=0} = -l(l+1)\,, \tag{25}$$

which imply that $\Phi'_n(y)$ satisfies the conditions (B2). Thus the functions $\Phi'_n(y)$ have the properties proved in Theorem B1. The expansion (24) can now be written

$$x_s^l \mathscr{R}_1(x, x_s) = \sum_n \Phi'_n(x/x_s) \cdot (x/x_s)^{-l-2n} \cdot x^{l+2n} \tag{26}$$

and as $x_s \to 0$ with $x = \text{const.}$ we obtain using (B5)

$$\lim_{x_s \to 0} [x_s^l \mathscr{R}_1(x, x_s)] = \sum_n \varphi'_n x^{l+2n}. \tag{27}$$

From (B6) or (B12) we find easily that the ratio of two consecutive terms of the series in (27) equals to the ratio of two consecutive terms of the expansion of $j_l(x)$ in (6). Consequently the quantity $\sum_n \varphi'_n x^{l+2n}$ is proportional to $j_l(x)$ with proportionality factor $\varphi'_0[2^l l!/(2l+1)!]^{-1}$. With φ'_0 from (B6) and setting

$$A_l = (-2)^l (l!)^3/(2l)!(2l+1)! \tag{28}$$

we have from (27)

$$\lim_{x_s \to 0} [A_l x_s^l \mathscr{R}_1(x, x_s)] = j_l(x). \tag{29}$$

This equation relates the (analytic at $x=0$) solution $\mathscr{R}_1(x, x_s)$ in the case of Schwarzschild's space-time to the solution $j_l(x)$ in the case of flat space-time.

The radial function $\mathscr{R}_2$ is defined [21] as that solution of the differential equation (11) which for $0 < |x| < |x_s|$ is given by the expression

$$\mathscr{R}_2 \equiv \mathscr{R}_2(x, x_s) = \mathscr{R}_1(x, x_s) \ln x + \sum_n b'_n x^n \tag{30}$$

with $b'_0 = 1$ and

$$\begin{aligned} n^2 x_s^2 b'_n &+ [l(l+1) - (n-1)(2n-1)] x_s b'_{n-1} + [(n-1)(n-2) - l(l+1)] b'_{n-2} + b'_{n-4} \\ &+ 2n x_s^2 a'_n - (4n-3) x_s a'_{n-1} + (2n-3) a'_{n-2} = 0. \end{aligned} \tag{31}$$

If we set $y = x/x_s$ we obtain

$$\mathscr{R}_2 \equiv \mathscr{R}_2[y, x_s] = \mathscr{R}_1[y, x_s] \ln x_s + \mathscr{R}_1[y, x_s] \ln y + \sum_n \beta'_n y^n \tag{32}$$

with $\beta'_0 = 1$, $\beta'_n = b'_n x_s^n$ and

$$\begin{aligned} n^2 \beta'_n &+ [l(l+1) - (n-1)(2n-1)] \beta'_{n-1} + [(n-1)(n-2) - l(l+1)] \beta'_{n-2} + x_s^2 \beta'_{n-4} \\ &+ 2n\alpha'_n - (4n-3)\alpha'_{n-1} + (2n-3)\alpha'_{n-2} = 0. \end{aligned} \tag{33}$$

As a consequence of (32) the function

$$\mathscr{R}'_2[y, x_s] = \mathscr{R}_1[y, x_s] \ln y + \sum_n \beta'_n y^n \tag{34}$$

is a linear combination of $\mathscr{R}_1[y, x_s]$ and $\mathscr{R}_2[y, x_s]$. Hence $\mathscr{R}'_2[y, x_s]$ is a solution of (14).

According to Theorem A2 the series $\sum_n \beta'_n y^n$ and $\sum_n n\beta'_n y^n$ converge uniformly for some y. Since β'_n are entire functions of x_s^2, the above series are entire functions of x_s^2 for some y. Thus from the theorem of Section 2.2 we conclude that $\mathscr{R}'_2[y, x_s]$ *is an entire function of* x_s^2 *for every* y *different from* 0, 1 *and* ∞.

We can write now

$$\mathscr{R}'_2[y, x_s] = \sum_n \Psi'_n(y) x_s^{2n}. \tag{35}$$

Setting this expression into (14) we find that the sequence $\Psi'_n(y)$ satisfies the differential equations (B1). Moreover, since from (34)

$$[\mathscr{R}'_2 - \ln y]_{y=0} = 1\,, \qquad [d\mathscr{R}'_2/dy - 1/y + l(l+1)\ln y]_{y=0} = 1\,, \tag{36}$$

the functions $\Psi'_n(y)$ satisfy also the conditions (B13) and, consequently, they have the properties proved in Theorem B2. Thus using (B16) we have from (35)

$$x_s^l \mathscr{R}'_2(x, x_s) = \sum_n \Psi'_n(x/x_s)\cdot(x/x_s)^{-l-2n}\cdot x^{l+2n} \tag{37}$$

and

$$\lim_{x_s\to 0} [x_s^l \mathscr{R}'_2(x, x_s)] = \sum_n \psi'_n x^{l+2n}\,. \tag{38}$$

Combining this relation with (32) and (34) we find

$$\lim_{x_s\to 0} [A_l(x_s^l/\ln x_s)\mathscr{R}_2(x, x_s)] = j_l(x)\,. \tag{39}$$

This equation relates the (nonanalytic at $x=0$) solution $\mathscr{R}_2(x, x_s)$ to the spherical Bessel function $j_l(x)$.

3.2 The F-Description

Instead of studying R about $x=0$ we can study F about $x=0$. If transformation (12) is taken to be

$$\mathscr{R}_1(x, x_s) = \exp(ix + ix_s \ln((x-x_s)/x_s))\mathscr{F}_1(x, x_s)\,, \tag{40}$$

then we find that for $|x| < |x_s|$

$$\mathscr{F}_1 \equiv \mathscr{F}_1(x, x_s) = \sum_n a_n x^n \tag{41}$$

with $a_0 = 1$ and

$$n^2 x_s a_n + (l-n+1)(l+n)a_{n-1} - (n-1)2ia_{n-2} = 0\,. \tag{42}$$

Setting $y = x/x_s$ we obtain

$$\mathscr{F}_1 \equiv \mathscr{F}_1[y, x_s] = \sum_n \alpha_n y^n \tag{43}$$

with $\alpha_0 = 1$, $\alpha_n = a_n x_s^n$ and

$$n^2\alpha_n + (l-n+1)(l+n)\alpha_{n-1} - (n-1)2ix_s\alpha_{n-2} = 0\,. \tag{44}$$

As defined by (43) $\mathscr{F}_1[y, x_s]$ is a solution of the differential equation (15) and satisfies the conditions of Theorem A1. Consequently, *$\mathscr{F}_1[y, x_s]$ is an entire function of x_s for $y \neq 1$ and $y \neq \infty$.*

If we set

$$\mathscr{F}_1[y, x_s] = \sum_n \Phi_n(y) x_s^n\,, \tag{45}$$

then the sequence $\Phi_n(y)$ satisfies the differential equations (B19). Furthermore, since from (43)

$$\mathscr{F}_1[0, x_s] = 1\,, \qquad d\mathscr{F}_1[y, x_s]/dy|_{y=0} = -l(l+1)\,, \tag{46}$$

the functions $\Phi_n(y)$ satisfy the relations (B20) and, consequently, they have the properties proved in Theorem B3. From (45) we have

$$x_s^l \mathscr{F}_1(x, x_s) = \sum_n \Phi_n(x/x_s) \cdot (x/x_s)^{-l-n} \cdot x^{l+n} \tag{47}$$

and because of (B27)

$$\lim_{x_s \to 0} [x_s^l \mathscr{F}_1(x, x_s)] = \sum_n \varphi_n x^{l+n} . \tag{48}$$

We observe now that the ratio $(\varphi_n x^{l+n})/(\varphi_{n-1} x^{l+n-1})$ is equal to the ratio of two consecutive terms of $e^{-ix} j_l(x)$ as given by expansion (6). Hence after multiplying by $2^l l! [\varphi_0 (2l+1)!]^{-1}$ we find from (48)

$$\lim_{x_s \to 0} [A_l x_s^l \mathscr{F}_1(x, x_s)] = e^{-ix} j_l(x) . \tag{49}$$

This equation relates the curved-space solution $\mathscr{F}_1(x, x_s)$ to the flat-space solution $e^{-ix} j_l(x)$. If we multiply the quantity inside the brackets by $\exp(ix + ix_s \ln((x-x_s)/x_s))$ we rediscover (29).

For the nonanalytic at $x=0$ solution we write

$$\mathscr{R}_2(x, x_s) = \exp(ix + ix_s \ln((x-x_s)/x_s)) \mathscr{F}_2(x, x_s) \tag{50}$$

and we find for $0 < |x| < |x_s|$

$$\mathscr{F}_2 \equiv \mathscr{F}_2(x, x_s) = \mathscr{F}_1(x, x_s) \ln x + \sum_n b_n x^n \tag{51}$$

with $b_0 = 1$ and

$$n^2 x_s b_n + (l-n+1)(l+n) b_{n-1} - (n-1) 2i b_{n-2} + 2n x_s a_n + (1-2n) a_{n-1} - 2i a_{n-2} = 0. \tag{52}$$

Setting $y = x/x_s$ we obtain

$$\mathscr{F}_2 \equiv \mathscr{F}_2[y, x_s] = \mathscr{F}_1[y, x_s] \ln x_s + \mathscr{F}_1[y, x_s] \ln y + \sum_n \beta_n y^n \tag{53}$$

with $\beta_0 = 1$, $\beta_n = b_n x_s^n$ and

$$n^2 \beta_n + (l-n+1)(l+n) \beta_{n-1} - (n-1) 2i x_s \beta_{n-2} + 2n \alpha_n + (1-2n) \alpha_{n-1} - 2i x_s \alpha_{n-2} = 0. \tag{54}$$

Since $\mathscr{F}_1[y, x_s]$ and $\mathscr{F}_2[y, x_s]$ are solutions of (15), we conclude from (53) that the function

$$\mathscr{F}_2'[y, x_s] = \mathscr{F}_1[y, x_s] \ln y + \sum_n \beta_n y^n \tag{55}$$

is a solution of (15). Using Theorem A2 we can prove as for $\mathscr{R}_2'[y, x_s]$ that $\mathscr{F}_2'[y, x_s]$ *is an entire function of* x_s *for every* y *different from* 0, 1 *and* ∞.

We write now

$$\mathscr{F}_2'[y, x_s] = \sum_n \Psi_n(y) x_s^n . \tag{56}$$

Substituting in (15) we find that the sequence $\Psi_n(y)$ satisfies the differential equations (B19). Furthermore, since from (55)

$$[\mathscr{F}_2' - \ln y]_{y=0} = 1 , \quad [d\mathscr{F}_2'/dy - 1/y + l(l+1) \ln y]_{y=0} = 1 , \tag{57}$$

the functions $\Psi_n(y)$ satisfy the recurrence relations (B36) and hence. Theorem B4. From (56) we have

$$x_s^l \mathscr{F}_2'(x, x_s) = \sum_n \Psi_n(x/x_s) \cdot (x/x_s)^{-l-n} x^{l+n} \tag{58}$$

and using (B42) we obtain

$$\lim_{x_s \to 0} [x_s^l \mathscr{F}_2'(x, x_s)] = \sum_n \psi_n x^{l+n} . \tag{59}$$

Finally because of (53) and (55) we find

$$\lim_{x_s \to 0} [A_l(x_s^l/\ln x_s) \mathscr{F}_2(x, x_s)] = e^{-ix} j_l(x) . \tag{60}$$

This equation relates the curved-space solution $\mathscr{F}_2(x, x_s)$ to the flat-space solution $e^{-ix} j_l(x)$. Multiplying the quantity inside the brackets in (60) by $\exp(ix + ix_s \ln((x - x_s)/x_s))$ we obtain again relation (39).

A formula which will be needed later is the full expansion of $\mathscr{F}_2$ obtained by combining (53), (55), and (56)

$$\mathscr{F}_2[y, x_s] = [\sum_n \Phi_n(y) x_s^{l+n}] \ln x_s + \sum_n \Psi_n(y) x_s^{l+n} . \tag{61}$$

The above expression gives explicitly the behavior of $\mathscr{F}_2[y, x_s]$ near $x_s = 0$ for constant y.

To go from the R-description to the F-description the transformation (12) can be used as it is *or* with the imaginary unit i replaced by $-i$. Then i should be replaced by $-i$ in the relations (13), (15), the formulas of this sections and in Theorems B3 and B4. The new quantities will be denoted by the old symbols with an asterisk. Thus i.e., instead of (40) and (45) we will have $\mathscr{R}_1 = \exp[-i(x + x_s \ln(x - x_s) - x_s \ln x_s)] \mathscr{F}_1^*$ and $\mathscr{F}_1^* = \sum_n \Phi_n^*(y) x_s^n$ respectively. Both cases (with i and $-i$) are needed for the evaluation of K_{ij} in Section 6. If x and x_s are real, then the addition of an asterisk to a symbol reduces to taking its complex conjugate.

4. The Solutions about $x = x_s$

The radial functions $\mathscr{R}_3$ and $\mathscr{R}_4$ are defined [21] as those solutions of (11) which for $0 < |x - x_s| < |x_s|$ are given by the relations

$$\mathscr{R}_3 \equiv \mathscr{R}_3(x, x_s) = e^{ix_s \ln(x - x_s)} \sum_n c_n'(x - x_s)^n \tag{62}$$

$$\mathscr{R}_4 \equiv \mathscr{R}_4(x, x_s) = \mathscr{R}_3^*(x, x_s) \tag{63}$$

with $c_0' = 1$ and

$$(n + 2ix_s) n x_s c_n' + [(n + l)(n - l - 1) + 2x_s^2 + (2n - 1) i x_s] c_{n-1}' + 3x_s c_{n-2}' + c_{n-3}' = 0 . \tag{64}$$

Since $\mathscr{R}_3$ and $\mathscr{R}_4$ are related by (63) we study below only $\mathscr{R}_3$. $\mathscr{R}_4$ satisfies every equation obtained for $\mathscr{R}_3$ if we replace i by $-i$ (see explanation at the end of the previous section).

Because of the factor $e^{ix_s \ln(x-x_s)}$ both $\mathscr{R}_3$ and $\mathscr{R}_4$ behave badly near $x=x_s$ and we cannot speak about their values or the values of their derivatives at $x=x_s$. Thus an R-description would require specification of $\mathscr{R}_3$ and $\mathscr{R}_4$ and of their derivatives at a different point. This procedure introduces difficulties and complicates the R-description.

In the F-description we remove the factor $e^{ix_s \ln(x-x_s)}$ setting

$$\mathscr{R}_3 \equiv \mathscr{R}_3(x, x_s) = e^{i[x-x_s+x_s \ln(x-x_s)]} \mathscr{F}_3(x, x_s) \tag{65}$$

and then we find

$$\mathscr{F}_3 \equiv \mathscr{F}_3(x, x_s) = \sum_n c_n (x-x_s)^n \tag{66}$$

with $c_0 = 1$ and

$$n(n+2ix_s)x_s c_n - [(l+n)(l-n+1)-(2n-1)2ix_s]c_{n-1} + (n-1)2ic_{n-2} = 0\,. \tag{67}$$

If we set $y = x/x_s$ the expansion (66) becomes

$$\mathscr{F}_3 \equiv \mathscr{F}_3[y, x_s] = \sum_n \gamma_n (y-1)^n \tag{68}$$

with $\gamma_0 = 1$, $\gamma_n = c_n x_s^n$ and

$$n(n+2ix_s)\gamma_n - [(l+n)(l-n+1)-(2n-1)2ix_s]\gamma_{n-1} + (n-1)2ix_s\gamma_{n-2} = 0\,. \tag{69}$$

As defined by (68) *$\mathscr{F}_3[y, x_s]$ is an analytic function of x_s all over the complex x_s-plane except at the points $x_s = ni/2$ $(n=1, 2, \ldots)$ for every finite $y \neq 0$. The points $x_s = ni/2$ are first order poles of $\mathscr{F}_3[y, x_s]$.* This property is a consequence of Theorem A1. The coefficients γ_n satisfy the recurrence relation (69) which is similar to (A2) with the quantities

$$((l+n)(l-n+1)-(2n-1)2ix_s)/n(n+2ix_s)\,, \qquad -(n-1)2ix_s/n(n+2ix_s) \tag{70}$$

corresponding to $\varrho_{nm}(x_s)$. In order that these quantities remain bounded we consider as D_{x_s} (which corresponds to D_s of Theorem A1) a finite region of the complex x_s-plane not containing a neighborhood of the points $x_s = ni/2$ $(n=1, 2, \ldots)$. Then according to Theorem A1 the series (68) converges uniformly on D_{x_s} for some y near $y=1$. Since each term of the series is analytic in x_s, the series converges to an analytic function and because of the theorem of Section 2.2 this analyticity property holds for any y different from the singularities of (11). For $y=1$ the property is obvious. To prove that the point $x_s = n_0 i/2$ is a first order pole of $\mathscr{F}_3[y, x_s]$ we repeat the same arguments for the series $(n_0+2ix_s)\sum_n \gamma_n (y-1)^n$. For this series we do not exclude a neighborhood of $x_s = n_0 i/2$ from D_{x_s} in order to have the quantities (70) bounded. Thus $(n_0+2ix_s)\mathscr{F}_3[y, x_s]$ is analytic at $x_s = n_0 i/2$ and, consequently, $\mathscr{F}_3[y, x_s]$ has a first order pole there. Since from (65)

$$\mathscr{R}_3 \equiv \mathscr{R}_3[y, x_s] = e^{ix_s[y-1+\ln(y-1)+\ln x_s]} \mathscr{F}_3[y, x_s] \tag{71}$$

we can easily derive properties of $\mathscr{R}_3[y, x_s]$ from the properties of $\mathscr{F}_3[y, x_s]$. For $\mathscr{R}_4[y, x_s]$ we can make similar statements with the points $x_s = -ni/2$ as poles.

We can set now

$$\mathscr{F}_3[y, x_s] = \sum_n X_n(y) x_s^n\,. \tag{72}$$

This expression converges for every finite $y \neq 0$ and $|x_s| < \frac{1}{2}$, because at $x_s = i/2$ the function $\mathscr{F}_3[y, x_s]$ has the closest to $x_s = 0$ pole. We can have an expansion similar to (72) converging for larger $|x_s|$, if we expand $(1+2ix_s)(2+2ix_s)\dots(n_0+2ix_s)\mathscr{F}_3[y, x_s]$ in powers of x_s. Such an expansion will converge for $|x_s| < (n_0+1)/2$. In any case the expansion (72) or similar expansions give an analytic continuation of $\mathscr{F}_3[y, x_s]$ outside the circle of convergence $|y-1| < 1$ of the original expression (68).

Substituting the series (72) into the differential equation (15) we find that the sequence $X_n(y)$ $(n = 0, 1, \dots)$ satisfies (B19). From (68) we have

$$\mathscr{F}_3[1, x_s] = 1\,, \qquad d\mathscr{F}_3[y, x_s]/dy|_{y=1} = (l(l+1) - 2ix_s)/(1+2ix_s)\,. \tag{73}$$

If we expand the right hand sides of (73) in powers of x_s and combine (72) and (73) we find that the sequence $X_n(y)$ satisfies the conditions (B46) and consequently has the properties proved in Theorem B5. Thus we can write (72) as

$$x_s^l \mathscr{F}_3(x, x_s) = \sum_n X_n(x/x_s) \cdot (x/x_s)^{-l-n} \cdot x^{l+n} \tag{74}$$

and obtain using (B54)

$$\lim_{x_s \to 0} [x_s^l \mathscr{F}_3(x, x_s)] = \sum_n \chi_n x^{l+n}\,. \tag{75}$$

Multiplying by $2^l l! [\chi_0 (2l+1)!]^{-1}$ we find using (B56)

$$\lim_{x_s \to 0} [(-1)^l A_l x_s^l \mathscr{F}_3(x, x_s)] = e^{-ix} j_l(x)\,. \tag{76}$$

Combining this result with (65) we conclude that

$$\lim_{x_s \to 0} [(-1)^l A_l x_s^l \mathscr{R}_3(x, x_s)] = j_l(x)\,. \tag{77}$$

This equation and a similar one with $\mathscr{R}_4$ instead of $\mathscr{R}_3$ relate $\mathscr{R}_3(x, x_s)$ and $\mathscr{R}_4(x, x_s)$ to the solution $j_l(x)$ of the wave equation in flat space-time.

5. The Radial Function $R_0(x, x_s)$

Since all $\mathscr{R}_i$ $(i = 1, 2, 3, 4)$ reduce to $j_l(x)$ the following question is raised naturally: Is there any solution of the differential equation (11) that reduces to $n_l(x)$? In this section we will answer this question in the affirmative and find the radial function which has this property.

In Sections 3 and 4 we started with a solution $\mathscr{R}_i$ and we wrote it as a power series of x_s with coefficients depending on y. Then we used the asymptotic properties of those coefficients to find the flat-space solution $j_l(x)$ in the limit $x_s \to 0$. Here we have to follow the reverse procedure. Let us assume that

$$\mathscr{R}_0(x, x_s) = e^{i[x + x_s \ln(x - x_s) - x_s \ln x_s]} \mathscr{F}_0(x, x_s) \tag{78}$$

is the solution that reduces to $n_l(x)$ with

$$\mathscr{F}_0 \equiv \mathscr{F}_0[y, x_s] = \sum_n U_n(y) x_s^n\,. \tag{79}$$

The sequence $U_n(y)$ must have the appropriate asymptotic properties necessary to find $n_l(x)$ in the limit $x_s \to 0$. This means that (79) written in the form

$$x_s^{-l-1} \mathscr{F}_0(x, x_s) = \sum_n U_n(x/x_s) \cdot (x/x_s)^{l-n+1} \cdot x^{n-l-1} \tag{80}$$

would give

$$\lim_{x_s\to 0}\,[x_s^{-l-1}\mathscr{F}_0(x,x_s)]=\sum_n u_n x^{n-l-1} \tag{81}$$

with the right hand side nothing more than a miltiple of $e^{-ix}n_l(x)$ as given by (7).

To find such a sequence of functions $U_n(y)$ we use Theorem B6. With $U_n(y)$ defined by (B60) and (B61) we have the asymptotic property (d). We can easily prove from (B67) or (B70) that $u_n x/u_{n-1}$ is equal to the ratio of two consecutive terms in the sum (7). Thus multiplying (81) by $-(2l)![2^l l! u_o]^{-1}$ we find

$$\lim_{x_s\to 0}\,[B_l x_s^{-l-1}\mathscr{F}_0(x,x_s)]=e^{-ix}n_l(x)\,, \tag{82}$$

where

$$B_l=(-1)^l(2l)!(2l+1)!/2^{l-1}(l!)^3\,. \tag{83}$$

Finally from (78) and (82) we obtain

$$\lim_{x_s\to 0}\,[B_l x_s^{-l-1}\mathscr{R}_0(x,x_s)]=n_l(x)\,. \tag{84}$$

Thus we have established the existence of a radial function $\mathscr{R}_0$ that reduces to $n_l(x)$ and we have given an expression for $\mathscr{R}_0$, namely the relations (78) and (79) with $U_n(y)$ defined by (B60), (B62) and (B61).

To express $\mathscr{F}_0$ as a linear combination of $\mathscr{F}_1$ and $\mathscr{F}_2$ we observe from Theorem B6 that as $y\to 0$ only U_0 and dU_0/dy diverge while U_n and dU_n/dy go to constants. Separating the $n=0$ term from (79) and using (B18) we find

$$[\mathscr{F}_0[y,x_s]+\tfrac{1}{2}\ln y]_{y=0}=-\sigma(l)+\sum_{n=1}^{\infty} v_n x_s^n \tag{85}$$

and

$$\begin{aligned}[d\mathscr{F}_0[y,x_s]/dy+\tfrac{1}{2}y^{-1}-\tfrac{1}{2}l(l+1)\ln y]_{y=0}&=-\tfrac{1}{2}-\tfrac{1}{2}l(l+1)+l(l+1)\sigma(l)\\&\quad-l(l+1)\sum_{n=1}^{\infty} v_n x_s^n\,.\end{aligned} \tag{86}$$

If now we assume that

$$\mathscr{F}_0[y,x_s]=C_1(x_s)\mathscr{F}_1[y,x_s]+C_2(x_s)\mathscr{F}_2[y,x_s] \tag{87}$$

we can substitute in (85) and (86) and find using (46) and (57)

$$C_1(x_s)=\tfrac{1}{2}\ln x_s+\tfrac{1}{2}-\sigma(l)+\sum_{n=1}^{\infty} v_n x_s^n\,, \tag{88}$$

$$C_2(x_s)=-\tfrac{1}{2}\,. \tag{89}$$

Thus $\mathscr{F}_0$ has been expressed as a linear combination of $\mathscr{F}_1$ and $\mathscr{F}_2$. Because of (78) a similar relation holds for $\mathscr{R}_0$. Using (87), (43), and (53) we can obtain an expansion of $\mathscr{F}_0$ or $\mathscr{R}_0$ in powers of y or x.

6. The Quantities $K_{ij}(x_s)$

Any three of the four solutions $\mathscr{R}_1$, $\mathscr{R}_2$, $\mathscr{R}_3$, and $\mathscr{R}_4$ of the differential equation (11) are related by (18) with the indices i and j ranging from 1 to 4. Since $K_{ij}=-K_{ji}$

there are six only K_{ij} related to the regular singular points $x=0$ and $x=x_s$, namely $K_{12}, K_{13}, K_{14}, K_{23}, K_{24}, K_{34}$. It has been found [21] that

$$K_{12}=-x_s \quad \text{and} \quad K_{34}=-2ix_s^2\,. \tag{90}$$

To evaluate the remaining four K_{ij} we set $y=x/x_s$ in (17) and find that in general

$$K_{ij}(x_s)=x_s y(y-1)\{\mathscr{R}_i[y,x_s]d\mathscr{R}_j[y,x_s]/dy-\mathscr{R}_j[y,x_s]d\mathscr{R}_i[y,x_s]/dy\}\,. \tag{91}$$

For K_{13} we observe that the expressions (22) and (23) do not contain anywhere the imaginary unit i. Thus using (40) we have

$$\mathscr{R}_1[y,x_s]=\mathscr{R}_1^*[y,x_s]=e^{-ix_s[y+\ln(y-1)]}\mathscr{F}_1^*[y,x_s] \tag{92}$$

with

$$\mathscr{F}_1^*[y,x_s]=\sum_n \Phi_n^*(y)x_s^n \tag{93}$$

instead of (45). The use of (92) instead of (40) in the following eliminates the exponential factor and simplifies the calculations. Substituting expressions (92) and (65) into (91) we find that

$$K_{13}(x_s)=x_s e^{ix_s(\ln x_s-1)}y(y-1)(\mathscr{F}_1^* d\mathscr{F}_3/dy-\mathscr{F}_3 d\mathscr{F}_1^*/dy+(2ix_s y/(y-1))\mathscr{F}_1^*\mathscr{F}_3)\,. \tag{94}$$

Expression (94) shows that $K_{13}(x_s)$ *is analytic in* x_s *for all* x_s *except perhaps at the points* $x_s=ni/2$ $(n=1,2,\ldots)$, *where* K_{13} *has at most first order poles, and the branch cut introduced by the factor* $e^{ix_s\ln x_s}$.

For $|x_s|<\frac{1}{2}$ we obtain from (94) using (72) and (93)

$$K_{13}(x)=x_s e^{ix_s(\ln x_s-1)}\sum_n \kappa_n x_s^n \tag{95}$$

where

$$\kappa_n=\sum_m [y(y-1)(\Phi_m^* dX_{n-m}/dy-X_{n-m}d\Phi_m^*/dy)+2iy^2\Phi_m^* X_{n-m-1}]\,. \tag{96}$$

Expansions of K_{13} in power series of x_s converging for $|x_s|$ larger than 1/2 can be obtained using expansions of $\mathscr{F}_3$ valid for $|x_s|>\frac{1}{2}$ according to the method of Section 4.

Although $\mathscr{F}_1^*$, $\mathscr{F}_3$, Φ_n^*, and X_n depend on y, K_{13} and κ_n are independent of y. This is a direct consequence of the Wronskian (17). We can also prove that $d\kappa_n/dy=0$ by direct differentiation of (96) and use of (B19). Since κ_n is a constant it can be evaluated for any y. We choose to evaluate it for $y=0$ using the order relations (B24), (B25), (B50), and (B51). Since these relations hold for $n>0$ we have to consider separately the $m=0$ and $m=n$ terms in the sum (96). After some calculations we find using (B26) and (B52) that for $n\geqq 0$

$$\kappa_n=-\lambda_n=2i\int_0^1 yP_l(d(yX_{n-1})/dy)dy\,. \tag{97}$$

Explicit evaluation with X_0 and X_1 from (B47) and (B48) gives

$$\kappa_0=0\,,\qquad \kappa_1=i(-1)^l\,, \tag{98}$$

$$\kappa_2=(-1)^l[2\sigma(l)-(16l^3+9l^2-19l+5)/2(2l-1)(2l+1)(2l+3)]\,. \tag{99}$$

The evaluation of κ_2 has been made possible after factorial integration of (97) for $n=2$ and use of the formulas

$$\int yQ_l(d(yP_l)/dy)dy = \tfrac{1}{2}y^2P_lQ_l - \tfrac{1}{4}y - \tfrac{1}{4}\ln(y-1)\,, \tag{100}$$

$$\int_0^1 dy(P_l^2-1)/(y-1) = 2\sigma(l)\,. \tag{101}$$

The last relation can be derived from definite integrals [34] containing P_l and Q_l.

It can be shown that for $n=2$ the explicit calculation of $X_2(y)$ from (B49) requires the calculation of the integral

$$\int_1^y dy \ln y/(y-1) \tag{102}$$

in terms of elementary functions, which is not possible. Thus $X_n(y)$ for $n\geqq 2$ cannot be given explicitly as $X_0(y)$ and $X_1(y)$ by (B47) and (B48). Hence for $n>2$ κ_n can be calculated numerically only from (97).

To evaluate $K_{14}(x_s)$ we use $\mathscr{R}_1$ as given by (40) and $\mathscr{R}_4$ as given by (63) and (65). We find

$$K_{14}(x_s) = K_{13}^*(x_s) \tag{103}$$

with K_{13} given by (95) and consequently for $|x_s|<\frac{1}{2}$

$$K_{14}(x_s) = x_s e^{-ix_s(\ln x_s - 1)} \sum_n \kappa_n^* x_s^n\,. \tag{104}$$

Since κ_n is a constant, κ_n^* is the complex conjugate of κ_n. *$K_{14}(x_s)$ is analytic in x_s for all x_s except perhaps for $x_s = -ni/2$ $(n=1, 2, \ldots)$, where it has first order poles at the most.*

The solution $\mathscr{R}_2$ as defined by (32), (33), and (34) does not contain the imaginary unit i, hence

$$\mathscr{R}_2[y, x_s] = \mathscr{R}_2^*[y, x_s] = \mathscr{R}_1^*[y, x_s] \ln x_s + \mathscr{R}_2'^*[y, x_s]\,. \tag{105}$$

Substituting into (91) we find

$$K_{23}(x_s) = K_{13}(x_s) \ln x_s + K_{2'3}(x_s) \tag{106}$$

where

$$K_{2'3}(x_s) = x_s y(y-1)[\mathscr{R}_2'^* d\mathscr{R}_3/dy - \mathscr{R}_3 d\mathscr{R}_2'^*/dy]\,. \tag{107}$$

As given by (106) and (107) *$K_{23}(x_s)$ has a branch cut introduced by* $\ln x_s$ *and perhaps first order poles at $x_s = ni/2$ $(n=1, 2, \ldots)$. It is analytic in x_s in the rest x_s-plane.*

We use now the F-description to calculate $K_{2'3}$. With

$$\mathscr{R}_2'^*[y, x_s] = e^{-ix_s[y+\ln(y-1)]}\mathscr{F}_2'^*[y, x_s] \tag{108}$$

and $\mathscr{R}_3[y, x_s]$ given by (65) expression (107) becomes

$$K_{2'3}(x_s) = x_s e^{ix_s(\ln x_s - 1)} y(y-1)(\mathscr{F}_2'^* d\mathscr{F}_3/dy - \mathscr{F}_3 d\mathscr{F}_2'^*/dy + (2ix_s y/(y-1))\mathscr{F}_2'^*\mathscr{F}_3)\,. \tag{109}$$

Using (56) and (72) we obtain for $|x_s|<\frac{1}{2}$

$$K_{2'3}(x_s) = x_s e^{ix_s(\ln x_s - 1)} \sum_n \kappa_n' x_s^n\,, \tag{110}$$

where

$$\kappa'_n = \sum_m [y(y-1)(\Psi^*_m dX_{n-m}/dy - X_{n-m} d\Psi^*_m/dy) + 2iy^2 \Psi^*_m X_{n-m-1}] . \tag{111}$$

For $n=0$ we have directly from (111) using (B37) and (B47)

$$\kappa'_0 = (-1)^l . \tag{112}$$

For $n>0$ since κ'_n (and $K_{2'3}$) is independent of y we evaluate the right side of (111) for $y=0$. Using the order relations (B39), (B40), (B50), and (B51) we find

$$\kappa'_n = 2\mu_n - [1-2\sigma(l)]\lambda_n \tag{113}$$

with λ_n and μ_n given by (B52) and (B53). For $n=1$ the explicit evaluation of the integrals (B52) and (B53) gives

$$\kappa'_1 = 2i(-1)^l[1-2\sigma(l)] . \tag{114}$$

The calculation of κ'_2 is complicated and will not be given here.

Combining expressions (93), (106), and (110) we obtain for $|x_s|<\frac{1}{2}$

$$K_{23}(x_s) = x_s e^{ix_s(\ln x_s - 1)} \sum_n (\kappa_n \ln x_s + \kappa'_n) x_s^n . \tag{115}$$

Working along the same lines we find

$$K_{24}(x_s) = K^*_{23}(x_s) = x_s e^{-ix_s(\ln x_s - 1)} \sum_n (\kappa^*_n \ln x_s + \kappa'^*_n) x_s^n \tag{116}$$

with κ^*_n and κ'^*_n the complex conjugates of κ_n and κ'_n. Statements similar to those made for $K_{23}(x_s)$ can be made for $K_{24}(x_s)$.

From the expressions found for K_{ij} $(i,j=1,2,3,4)$ we find easily that as $x_s \to 0$

$$\lim K_{ij}(x_s) = 0 , \qquad \lim K_{23}(x_s)/K_{24}(x_s) = -\lim K_{13}(x_s)/K_{14}(x_s) = 1 , \tag{117}$$

which were expected since all $\mathscr{R}_i$ reduce to $j_l(x)$ as $x_s \to 0$ and they become linearly dependent.

7. General Remarks

The main results of this paper can be considered as answers to the three questions raised in the introduction. The Equations (24), (35), (45), (56), (72), and (79) give analytic continuations of radial functions outside their original circle of convergence. The relations (29), (39), (49), (60), (76), (77), and (84) relate the radial functions to the flat-space solutions $j_l(x)$ and $n_l(x)$. Finally the expressions (90), (95), (104), (115), and (116) give the coefficients $K_{ij}(x_s)$ of a linear relation connecting any three $\mathscr{R}_i$. All these results provide a more rigorous mathematical basis for studying weak fields in blackhole space-times.

Beyond the above mentioned concrete results the procedure used in this paper suggests a new approach for studying the radial functions and at the same time shows that there are certain limits in the analytical study of such functions. The new approach consists in studying the properties of sequences of functions satisfying the differential equation (B1) or (B19). Other properties of these sequences beyond those presented in Appendix B can shed more light in the behavior of radial functions. Many relations presented in this paper, i.e. those relating $\mathscr{R}_l$ to $j_l(x)$, can be generalized to order relations. On the other hand the proven fact that members of the sequences Φ_n, Ψ_n, X_n, U_n cannot be determined explicitly for $n>1$ shows that numerical calculations are a necessity beyond this point. Thus i.e. κ_n in (95) has to be found numerically for $n>2$.

A more complete description of the radial functions and of their relationships to the special cases ($k=0$ or $r_s=0$) will be presented in a subsequent paper where similar results will be established for radial functions related to the irregular singular point $x=\infty$.

Appendix

A. Two Theorems on Uniform Convergence

We present here two theorems used repeatedly in this paper to prove the analyticity of a power series with respect to a parameter which appears in the coefficients of the series.

Theorem A1. *We consider the series*

$$\sum_{n=0}^{\infty} f_n(s)\alpha^n \quad \textit{and} \quad \sum_{n=0}^{\infty} n f_n(s)\alpha^n\,, \tag{A1}$$

where α is a positive number. The coefficients $f_n(s)$ are defined by the recurrence relations

$$f_0(s)=1\,, \qquad f_n(s)=\sum_{m=1}^{M} \varrho_{nm}(s) f_{n-m}(s) \quad \textit{for} \quad n>0\,, \qquad M<+\infty\,, \tag{A2}$$

with $\varrho_{nm}(s)$ given functions of s defined on a region D_s of the complex s-plane. If

$$\lim_{n\to+\infty} \varrho_{nm}(s)=\varrho_m(s) \qquad (m=1,2,\ldots,M) \tag{A3}$$

uniformly on D_s and the functions $\varrho_{nm}(s)$, $\varrho_m(s)$ are bounded on D_s then there exists a number α_0 such that for $\alpha<\alpha_0$ the series (A1) *converge uniformly on D_s.*

Proof. Since the limits (A3) exist uniformly on D_s, for every $\varepsilon>0$ there is an integer N (depending on ε only) such that $n\geqq N$ implies

$$|\varrho_{nm}(s)-\varrho_m(s)|<\varepsilon\,. \tag{A4}$$

If λ is an upper bound for all $\varrho_m(s)$ ($m=1,2,\ldots,M$, $s\in D_s$), then we conclude from (A4) that

$$|\varrho_{nm}(s)|<\lambda+\varepsilon\,. \tag{A5}$$

Without any restriction we can assume that $\lambda>1$.

By induction we prove easily that for fixed n $f_n(s)$ is bounded on D_s. Hence there is a number A such that

$$|f_n(s)|<A \quad \text{for all} \quad n\leqq N\,, \qquad s\in D_s\,. \tag{A6}$$

We claim now that for $n=0,1,\ldots,+\infty$ we have

$$|f_n(s)|<AM^n(\lambda+\varepsilon)^n\,. \tag{A7}$$

For $n \leqq N$ this is obvious from inequality (A6). For $n > N$ we prove the relation (A7) by induction. In fact if we assume that (A7) is true for $f_0(s), \ldots, f_{n-1}(s)$, then using (A5) we obtain from (A2)

$$f_n(s) < \left(\sum_{m=1}^{M} |f_{n-m}(s)| \right) (\lambda + \varepsilon) < \left(\sum_{m=1}^{M} AM^{n-m} (\lambda + \varepsilon)^{n-m} \right) (\lambda + \varepsilon) < AM^n (\lambda + \varepsilon)^n , \tag{A8}$$

namely inequality (A7).

If we choose now $\alpha_0 = M^{-1}(\lambda + \varepsilon)^{-1}$, then for $\alpha < \alpha_0$ the series

$$\sum_{n=0}^{\infty} AM^n (\lambda + \varepsilon)^n \alpha^n \quad \text{and} \quad \sum_{n=0}^{\infty} nAM^n (\lambda + \varepsilon)^n \alpha^n \tag{A9}$$

converge. But

$$|f_n(s) \alpha^n| < AM^n (\lambda + \varepsilon)^n \alpha^n , \tag{A10}$$

hence according to Weierstrass's test the series (A1) converge uniformly on D_s for $\alpha < \alpha_0$.

Theorem A2. *We consider the series*

$$\sum_{n=0}^{\infty} f'_n(s) \alpha^n \quad \textit{and} \quad \sum_{n=0}^{\infty} n f'_n(s) \alpha^n , \tag{A11}$$

where α is a positive number. The coefficients $f'_n(s)$ are defined by the recurrence relations

$$f'_0(s) = 1 , \qquad f'_n(s) = \sum_{m=1}^{M'} \sigma_{nm}(s) f'_{n-m}(s) + \sum_{m=1}^{M''} \tau_{nm}(s) f_{n-m}(s) \tag{A12}$$

with $f_n(s)$ satisfying (A2) *and $\varrho_{nm}(s)$, $\sigma_{nm}(s)$, $\tau_{nm}(s)$ given functions of s defined on a region D_s of the complex s-plane. If $\varrho_{nm}(s)$, $\sigma_{nm}(s)$ and $\tau_{nm}(s)$ satisfy uniformly on D_s relations similar to* (A3) *and the functions $\varrho_{nm}(s)$, $\sigma_{nm}(s)$, $\tau_{nm}(s)$, $\varrho_m(s)$, $\sigma_m(s)$, $\tau_m(s)$ are bounded on D_s, then there exist a number α_0 such that for $\alpha < \alpha_0$ the series* (A1) *converge uniformly on D_s.*

Proof. As in Theorem A1 for every $\varepsilon > 0$ there is an integer N such that for $n \leqq N$ $|\varrho_{nm}(s)|$, $|\sigma_{nm}(s)|$ and $|\tau_{nm}(s)|$ are smaller than $\lambda' + \varepsilon$ and $|f'_n(s)|$ smaller than A' [$\lambda' > 1$ is an upper bound for $\varrho_m(s)$, $\sigma_m(s)$, $\tau_m(s)$]. Thus for $n \leqq N$ we have

$$|f'_n(s)| < A'(M' + M'')^n (\lambda' + \varepsilon)^n \tag{A13}$$

and by induction for $n > N$

$$|f'_n(s)| < \left(\sum_{m=1}^{M'} |f'_{n-m}(s)| + \sum_{m=1}^{M''} |f_{n-m}(s)| \right) (\lambda' + \varepsilon) < A'(M' + M'')^n (\lambda' + \varepsilon)^n . \tag{A14}$$

Thus λ', A' and $M' + M''$ replace λ, A and M in (A10) and the series (A11) converge uniformly on D_s for $\alpha < \alpha_0 = (M' + M'')^{-1}(\lambda' + \varepsilon)^{-1}$.

B. The Sequences Φ'_n, Ψ'_n, Φ_n, Ψ_n, X_n, U_n

The main contributions of this paper concerning questions of analytic continuation, the relation of the radial functions to the Bessel functions and the evaluation of $K_{ij}(x_s)$ have been made possible through the study of some sequences of functions Φ'_n, Ψ'_n, Φ_n, Ψ_n, X_n and U_n. The most important properties of these functions are presented here as theorems.

Theorem B1. *If $\Phi'_n(y)$ $(n=0, 1, \ldots, +\infty)$ is a sequence of functions satisfying the differential equations*

$$y(y-1)d^2\Phi'_n/dy^2+(2y-1)d\Phi'_n/dy-l(l+1)\Phi'_n=-(y^3/(y-1))\Phi'_{n-1} \tag{B1}$$

and the conditions

$$\left.\begin{array}{ll}\Phi'_0(0)=1\,, & d\Phi'_0(y)/dy|_{y=0}=-l(l+1)\,,\\ \Phi'_n(0)=0\,, & d\Phi'_n(y)/dy|_{y=0}=0 \quad for \quad n>0\,,\end{array}\right\} \tag{B2}$$

then

(a) $\Phi'_0(y)=P_l(1-2y)\,,$ (B3)

(b) *for $n>0$*

$$\Phi'_n(y)=2P_l\textstyle\int_0^y (y^3/(y-1))Q_l\Phi'_{n-1}dy-2Q_l\int_0^y (y^3/(y-1))P_l\Phi'_{n-1}dy\,, \tag{B4}$$

(c) *for $n\geqq 0$ as $y\to+\infty$*

$$\Phi'_n(y)=\varphi'_n y^{l+2n}+o(y^{l+2n}) \tag{B5}$$

with

$$\varphi'_n=(-1)^{l+n}(2l)!(2l+1)!(l+n)!/(l!)^3n!(2l+2n+1)! \tag{B6}$$

Proof. For $n=0$ (B1) reduces to the Legendre differential equation and its solution $P_l(1-2y)$ satisfies the conditions (B2). For $n>0$ we can easily verify that $\Phi'_n(y)$ as given by (B4) satisfies the differential equation (B1) and the conditions (B2).

To prove properly (c) and some other properties later in the appendix we need the formulas

$$P_l(1-2y)=\sum_{n=0}^{l}(-1)^n(l+n)!((l-n)!(n!)^2)^{-1} \tag{B7}$$

and

$$Q_l(1-2y)=\sum_n(-1)^{l+1}[(l+n)!]^2[2n!(2l+n+1)!]^{-1}y^{-n-l-1} \qquad (y>1)\,. \tag{B8}$$

As $y\to+\infty$ we obtain

$$P_l=p_ly^l+o(y^l)\,, \qquad p_l=(-1)^l(2l)!/(l!)^2 \tag{B9}$$

$$Q_l=q_ly^{-l-1}+o(y^{-l-1})\,, \qquad q_l=(-1)^{l+1}(l!)^2/2(2l+1)!\,. \tag{B10}$$

Thus for $n=0$ property (c) is true with $\varphi'_0=p_l$. If we assume that it is also true for $\Phi'_0, \Phi'_1, \ldots, \Phi'_{n-1}$, then from (B4) we have as $y\to+\infty$

$$\begin{aligned}\Phi'_n(y)&=2[p_ly^l+o(y^l)]\textstyle\int_{y_0}^y[q_l\varphi'_{n-1}y^{2n-1}+o(y^{2n-1})]dy\\ &\quad-2[q_ly^{-l-1}+o(y^{-l-1})]\textstyle\int_{y_0}^y[p_l\varphi'_{n-1}y^{2l+2n}+o(y^{2l+2n})]dy+o(y^{l+1})\\ &=-(2n(2l+2n+1))^{-1}\varphi'_{n-1}y^{l+2n}+o(y^{l+2n})\,,\end{aligned} \tag{B11}$$

where y_0 is an arbitrary number large enough for expansion (B8) to hold (some intermediate calculations have been ommitted). Thus (B5) is true with

$$\varphi'_n = -(2n(2l+2n+1))^{-1}\varphi'_{n-1}\,. \tag{B12}$$

From this recurrence relation we prove easily the general expression (B6).

Theorem B2. *If* $\Psi'_n(y)$ $(n=0,1,\dots,\infty)$ *is a sequence of functions satisfying the differential equations* (B1) *and the conditions*

$$\left.\begin{aligned}&[\Psi'_0-\ln y]_{y=0}=1\,, \quad [d\Psi'_0/dy-y^{-1}+l(l+1)\ln y]_{y=0}=1\,,\\ &\Psi'_n(0)=0\,, \qquad d\Psi'_n/dy|_{y=0}=0 \quad \textit{for} \quad n>0\,,\end{aligned}\right\} \tag{B13}$$

then

(a) $$\Psi'_0(y)=[1-2\sigma(l)]P_l(1-2y)-2Q_l(1-2y) \tag{B14}$$

with [32]

$$\sigma(l)=\sum_{m=1}^{l} m^{-1}\,, \tag{B15}$$

(b) *for* $n>0$ $\Psi'_n(y)$ *satisfies the same integral recurrence relation* (B4) *as* Φ'_n,
(c) *for* $n\geqq 0$ *as* $y\to+\infty$

$$\Psi'_n(y)=\psi'_n y^{l+2n}+o(y^{l+2n}) \tag{B16}$$

with

$$\Psi'_n=[1-2\sigma(l)]\varphi'_n\,. \tag{B17}$$

Proof. We verify easily that $\Psi'_0(y)$ and $\Psi'_n(y)$ satisfy the differential equation (B1). Using the relation [33]

$$Q_l(1-2y)=\tfrac{1}{2}P_l\ln((y-1)/y)-\sum_{m=1}^{l} m^{-1}P_{m-1}P_{l-m} \tag{B18}$$

we find that $\Psi'_0(y)$ satisfies the conditions (B13). From expression (B4) written for $\Psi'_n(y)$ we verify the conditions (B13) for $\Psi'_n(y)$ with $n>0$. Finally the order of $\Psi'_n(y)$ and expression (B17) for ψ'_n are derived as in the case of $\Phi'_n(y)$ in Theorem B1.

Theorem B3. *If* $\Phi_n(y)$ $(n=0,1,\dots,+\infty)$ *is a sequence of functions satisfying the differential equation*

$$y(y-1)d^2\Phi_n/dy^2+(2y-1)d\Phi_n/dy-l(l+1)\Phi_n=-2iyd(y\Phi_{n-1})/dy \tag{B19}$$

and the conditions

$$\left.\begin{aligned}&\Phi_0(0)=1\,, \quad d\Phi_0/dy|_{y=0}=-l(l+1)\,,\\ &\Phi_n(0)=0\,, \quad d\Phi_n/dy|_{y=0}=0 \quad \textit{for} \quad n>0\,,\end{aligned}\right\} \tag{B20}$$

then

(a) $\Phi_0(y) = P_l(1-2y)\,,$ (B21)

(b) $\Phi_1(y) = -i[y + \ln(y-1)]\,P_l\,,$ (B22)

(c) *for* $n > 0$

$$\Phi_n(y) = 4iP_l \int_0^y yQ_l(d/dy)(y\Phi_{n-1})dy - 4iQ_l \int_0^y yP_l(d/dy)(y\Phi_{n-1})dy\,, \tag{B23}$$

(d) *for* $n > 0$ *as* $y \to 0$

$$\Phi_n(y) = \varrho_n y^{2n} + o(y^{2n+1}\ln^n y)\,, \tag{B24}$$

$$d\Phi_n(y)/dy = 2n\varrho_n y^{2n-1} + o(y^{2n}\ln^n y) \tag{B25}$$

with

$$\varrho_n = i^n(2n)!/2^{2n}(n!)^3 \tag{B26}$$

(e) *for* $n \geqq 0$ *as* $y \to +\infty$

$$\Phi_n(y) = \varphi_n y^{l+n} + o(y^{l+n})\,, \tag{B27}$$

$$d\Phi_n(y)/dy = (l+n)\varphi_n y^{l+n-1} + o(y^{l+n-1}) \tag{B28}$$

with

$$\varphi_n = (-1)^{l+n}(2i)^n(2l)!(2l+1)!(l+n)!/(l!)^3 n!(2l+n+1)!\,. \tag{B29}$$

Proof. Properties (a), (b), and (c) can be proved as in Theorems B1 and B2 by direct substitution of (B21), (B22), and (B23) into (B19) and (B20). For $n=1$ we can derive property (d) from expression (B22). Furthermore, as $y \to 0$ we have from (B7) and (B18)

$$P_l = 1 + o(y\ln y)\,, \qquad dP_l/dy = o(\ln y) \tag{B30}$$

and

$$Q_l = -\tfrac{1}{2}\ln y - \sigma(l) + o(y\ln^2 y)\,, \quad dQ_l/dy = -\tfrac{1}{2}y^{-1} + o(\ln^2 y)\,. \tag{B31}$$

If we accept property (d) for $\Phi_1, \Phi_2, \ldots, \Phi_{n-1}$ we obtain from (23) for $n \geqq 2$

$$\begin{aligned}\Phi_n(y) = {} & 4i[1 + o(y\ln y)] \int_0^y [-(2n-1)2^{-1}\varrho_{n-1}y^{2n-1}\ln y - (2n-1)\varrho_{n-1}\sigma(l)y^{2n-1} \\ & + o(y^{2n}\ln^n y)]dy - 4i[-\tfrac{1}{2}\ln y - \sigma(l) + o(y\ln^2 y)] \\ & \cdot \int_0^y [(2n-1)\varrho_{n-1}y^{2n-1} + o(y^{2n}\ln^{n-1} y)]dy \\ = {} & (2n-1)(2n^2)^{-1}i\varrho_{n-1}y^{2n} + o(y^{2n+1}\ln^n y)\,,\end{aligned} \tag{B32}$$

where some intermediate calculations have been ommitted. Thus the relation (B24) has been proved with

$$\varrho_n = (2n-1)(2n^2)^{-1}i\varrho_{n-1}\,. \tag{B33}$$

From this recurrence relation we prove (B26). Working along the same lines we prove (B25). Note that it is not permissible to derive (B25) from (B24) by differentiation.

Property (e) can be easily verified for Φ_0 and Φ_1 from expressions (B21) and (B22). Assuming (e) for $\Phi_0, \Phi_1, \ldots, \Phi_{n-1}$ we have from (B23) using (B7), (B8), (B9) and (B10)

$$\begin{aligned}\Phi_n(y) &= 4i[p_l y^l + o(y^l)] \int_{y_0}^{y} [(l+n) q_l \varphi_{n-1} y^{n-1} + o(y^{n-1})]\,dy \\ &\quad - 4i[q_l y^{-l-1} + o(y^{-l-1})] \int_{y_0}^{y} [(l+n) p_l \varphi_{n-1} y^{2l+n} + o(y^{2l+n})]\,dy + o(y^{l+1}) \\ &= -2i(l+n)n^{-1}(2l+n+1)^{-1}\varphi_{n-1} y^{l+n} + o(y^{l+n}), \qquad \text{(B34)}\end{aligned}$$

namely the relation (B27) with

$$\varphi_n = -2i(l+n)n^{-1}(2l+n+1)^{-1}\varphi_{n-1}. \qquad \text{(B35)}$$

Working along the same lines we prove (B28) and from the recurrence relation (B35) we derive the expression (B29) for φ_n.

Theorem B4. *If $\Psi_n(y)$ $(n=0, 1, \ldots, +\infty)$ is a sequence of functions satisfying the differential equation* (B19) *and the conditions*

$$\left.\begin{aligned}&[\Psi_0 - \ln y]_{y=0} = 1, \quad [d\Psi_0/dy - y^{-1} + l(l+1)\ln y]_{y=0} = 1, \\ &\Psi_n(0) = 0, \quad d\Psi_n/dy|_{y=0} = 0 \quad \text{for} \quad n>0,\end{aligned}\right\} \qquad \text{(B36)}$$

then

(a) $$\Psi_0(y) = [1 - 2\sigma(l)]P_l(1-2y) - 2Q_l(1-2y), \qquad \text{(B37)}$$

(b) $$\Psi_1(y) = -i[y + \ln(y-1)]\Psi_0(y), \qquad \text{(B38)}$$

(c) *for $n>0$ $\Psi_n(y)$ satisfies an integral recurrence relation similar to* (B23),

(d) *for $n>0$ as $y \to 0$*

$$\Psi_n(y) = \sigma_n y^{2n} \ln y + \tau_n y^{2n} + o(y^{2n+1} \ln^{n+1} y), \qquad \text{(B39)}$$

$$d\Psi_n(y)/dy = 2n\sigma_n y^{2n-1} \ln y + (\sigma_n + 2n\tau_n) y^{2n-1} + o(y^{2n} \ln^{n+1} y) \qquad \text{(B40)}$$

with

$$\sigma_n = i^n (2n)!/2^{2n}(n!)^3, \qquad \text{(B41)}$$

(e) *for $n \geqq 0$ as $y \to +\infty$*

$$\Psi_n(y) = \psi_n y^{l+n} + o(y^{l+n}), \qquad \text{(B42)}$$

$$d\Psi_n(y)/dy = (l+n)\psi_n y^{l+n-1} + o(y^{l+n-1}) \qquad \text{(B43)}$$

with

$$\psi_n = [1 - 2\sigma(l)]\varphi_n. \qquad \text{(B44)}$$

Proof. Properties (a), (b), and (c) can be proved by direct substitution into the differential equation (B19) and the conditions (B36). For $n=1$ we derive property (d) from (B38). For $n>1$ property (d) can be derived from the integral recurrence relation satisfied by Ψ_n [property (c)]. The detailed calculations show that we have to keep *two* terms in (B39) (and not one as was the case with Φ_n in Theorem B3). We find

$$\sigma_n = (2n-1)(2n^2)^{-1} i\sigma_{n-1} \quad \text{and} \quad \tau_n = -(n-1)(2n^3)^{-1} i\sigma_{n-1} + (2n-1)(2n^2)^{-1} i\tau_{n-1} \qquad \text{(B45)}$$

from which we obtain (B41) and a more complicated formula for τ_n. Finally we prove property (e) working along the same lines as for property (e) in Theorem B3.

Theorem B5. *If* $X_n(y)$ $(n=0, 1, \ldots, +\infty)$ *is a sequence of functions satisfying the differential equation* (B19) *and the conditions*

$$\left.\begin{array}{ll} X_0(1)=1\,, & dX_0/dy|_{y=1}=l(l+1)\,, \\ X_n(1)=0\,, & dX_n/dy|_{y=1}=[l(l+1)+1](-2i)^n \quad \text{for} \quad n>0\,, \end{array}\right\} \tag{B46}$$

then

(a) $$X_0(y)=(-1)^l P_l(1-2y)\,, \tag{B47}$$

(b) $$X_1(y)=[1-2\sigma(l)]i(-1)^l P_l+2i(-1)^l Q_l-i(-1)^l[y+\ln(y-1)]P_l\,, \tag{B48}$$

(c) *for* $n>0$

$$X_n(y)=4iP_l\int_1^y yQ_l(d/dy)(yX_{n-1})dy-4iQ_l\int_1^y yP_l(d/dy)(yX_{n-1})dy\,, \tag{B49}$$

(d) *for* $n>0$ *as* $y\to 0$

$$X_n(y)=\lambda_n\ln y+(2\mu_n+2\sigma(l)\lambda_n)+o(1)\,, \tag{B50}$$

$$dX_n(y)/dy=\lambda_n y^{-1}-l(l+1)\lambda_n\ln y+o(\ln y) \tag{B51}$$

with

$$\lambda_n=-2i\int_0^1 yP_l(d/dy)(yX_{n-1})dy\,, \tag{B52}$$

$$\mu_n=-2i\int_0^1 yQ_l(d/dy)(yX_{n-1})dy\,, \tag{B53}$$

(e) *for* $n\geqq 0$ *as* $y\to+\infty$

$$X_n(y)=\chi_n y^{l+n}+o(y^{l+n})\,, \tag{B54}$$

$$dX_n(y)/dy=(l+n)\chi_n y^{l+n-1}+o(y^{l+n-1}) \tag{B55}$$

with

$$\chi_n=(-2i)^n(2l)!(2l+1)!(l+n)!/(l!)^3 n!(2l+n+1)!\,. \tag{B56}$$

Proof. We prove easily that (B47), (B48), and (B49) satisfy (B19) and that X_0 and X_1 fulfill the conditions (B46). If $X_{n-1}(y)$ satisfies (B46) then as $y\to 1$

$$X_{n-1}(y)=o(1) \quad \text{and} \quad dX_{n-1}(y)/dy=[l(l+1)+1](-2i)^{n-1}+o(1)\,. \tag{B57}$$

Thus (B49) gives as $y\to 1$

$$\begin{aligned} X_n(y)&=4iP_l\int_1^y[\tfrac{1}{2}(-1)^l\ln(y-1)+o(\ln(y-1))][(l(l+1)+1)(-2i)^{n-1}+o(1)]dy \\ &\quad-4iQ_l\int_1^y[(-1)^l+o(1)][(l(l+1)+1)(-2i)^{n-1}+o(1)]dy \\ &=o(1) \end{aligned} \tag{B58}$$

while a similar calculation gives

$$dX_n(y)/dy=[l(l+1)+1](-2i)^n+o(1)\,. \tag{B59}$$

The relations (B58) and (B59) imply the conditions (B46) for X_n.

Property (d) can be verified for $n=1$ from (B48) and then proved by induction from (B49). Property (e) is proved as in Theorems B3 and B4. In the proof the behavior of P_l, Q_l and their derivatives near $y=0$, $y=1$, and $y=+\infty$ is obtained from (B7), (B8), (B9), (B10), (B18), (B30), and (B31).

Theorem B6. *If $U_n(y)$ $(n=0, 1, \ldots, +\infty)$ is a sequence of functions defined by the relations*

$$U_0(y)=Q_l(1-2y) \tag{B60}$$

and for $n>0$

$$U_n(y)=4iP_l \int_{y_n}^{y} yQ_l(d/dy)(yU_{n-1})dy-4iQ_l \int_0^y yP_l(d/dy)(yU_{n-1})dy \tag{B61}$$

with $y_n=+\infty$ for $n\leqq 2l+1$ and $y_n=0$ for $n>2l+1$, then

(a) *the sequence $U_n(y)$ satisfies the differential equation* (B19),

(b) $$U_1(y)=-i[y+\ln(y-1)]Q_l\,, \tag{B62}$$

(c) *for $n>0$ as $y\to 0$*

$$U_n(y)=v_n+o(1)\,,\qquad dU_n(y)/dy=-l(l+1)v_n+o(1) \tag{B63}$$

with

$$v_n=\begin{cases}-4i\int_0^\infty yQ_l(d(yU_{n-1})/dy)dy & \text{for} \quad n\leqq 2l+1\,,\\ 0 & \text{for} \quad n>2l+1\,,\end{cases} \tag{B64}$$

(d) *for $n\geqq 0$ as $y\to+\infty$*

$$U_n(y)=u_ny^{n-l-1}+o(y^{n-l-1})\,, \tag{B65}$$

$$dU_n(y)/dy=(n-l-1)u_ny^{n-l-2}+o(y^{n-l-2}) \tag{B66}$$

with

$$u_n=\begin{cases}(-1)^{l+1}(l!)^3(-2i)^n(2l-n)!/2(2l)!(2l+1)!(l-n)! & \text{for} \quad n\leqq l\,,\\ 0 & \text{for} \quad n>l\,.\end{cases} \tag{B67}$$

Proof. We prove easily properties (a) and (b). For $0\leqq n\leqq l$ we prove property (d) by induction. We verify (B65), (B66), and (B67) for $n=0$ and $n=1$. Assuming property (d) for $0, 1, \ldots, n-1$ we have from (B61)

$$\begin{aligned}U_n(y)&=4iP_l \int_{+\infty}^{y} [(n-l-1)q_lu_{n-1}y^{n-2l-2}+o(y^{n-2l-2})]dy\\ &\quad-4iQ_l \int_0^y [(n-l-1)p_lu_{n-1}y^{n-1}+o(y^{n-1})]dy \qquad \text{(B68)}\\ &=u_ny^{n-l-1}+o(y^{n-l-1})\,, \qquad \text{(B69)}\end{aligned}$$

where

$$u_n=-2i(l+1-n)n^{-1}(2l+1-n)^{-1}u_{n-1}\,. \tag{B70}$$

When $n=l+1$ the first term in the integrals of (B68) vanishes. Thus we obtain (B67). Furthermore, since the term $o(y^{n-2l-2})$ in the first integral of (B68) is equal to const. $\times\, y^{n-2l-3}$, the integral diverges when $n>2l+1$ and we have to change

y_n from $+\infty$ to 0. Property (c) can be proved easily by induction from (B61). It should be stressed that whenever an integrand becomes infinite at $y=1$ we obtain the Cauchy principal value of the integral.

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