Commun. math. Phys. 47, 117—129 (1976)

Communications in Mathematical Physics

Some Remarks on Fröhlich's Condition in $P(\phi)_2$ Euclidean Field Theory

A. Kishimoto

Department of Physics, Kyoto University, Kyoto, Japan

Abstract. We derive Fröhlich's condition as the KMS condition on a suitable algebra and time translation. Next we consider Fröhlich's condition and its variance and prove their equivalence in a general setting. Finally we mention some results which follow from the latter condition.

§ 1. Introduction

In the $P(\phi)_2$ Euclidean field theory there are some "equilibrium equations" of probability measures on (Q, Σ) where Q may be realized as $\mathscr{D}' = \mathscr{D}'_{\text{real}}(R^2)$ or $\mathscr{S}' = \mathscr{S}'_{\text{real}}(R^2)$ and Σ as the σ-algebra generated by cylinder sets. They are expected to completely characterize infinite volume theories with given interaction and bare mass. One of them is given by Guerra, Rosen and Simon [5], which is a version of the DLR equations in classical statistical mechanics. Recently Fröhlich [3] observed that such measures are quasi-invariant and obtained another characterization which is expressed by Radon-Nikodym derivatives: Let ν be a probability measure and satisfy

$$\frac{d\nu(\phi+g)}{d\nu(\phi)} = \exp\{-\phi(\mu^2 g) - \tfrac{1}{2}\|\mu g\|^2_{L^2} - [\mathscr{U}(\phi+g) - \mathscr{U}(\phi)]\} \tag{1.1}$$

for any g in $\mathscr{D}(R^2)$ where μ^2 denotes $-\Delta + m_0^2$ and

$$[\mathscr{U}(\phi+g) - \mathscr{U}(\phi)] = \int \{:P(\phi(x)+g(x)): - :P(\phi(x)):\} d^2x \tag{1.2}$$

for an interaction polynomial P bounded below and a bare mass m_0. In this case ν should be called an "equilibrium measure" if it further satisfies the physical conditions (e.g. Osterwalder-Schrader-Nakano positivity). Fröhlich showed that the two characterizations are equivalent.

In the present note our first purpose is to show the equivalence of the condition (1.1) with the KMS condition relative to suitable time translation automorphisms as Brascamp [2] did in the case of classical lattice gas (see Theorem 3). In this way the similarity of the $P(\phi)_2$ Euclidean theory with classical statistical mechanics becomes more complete.

Fröhlich [3] also gave a "differential form" of his original condition in terms of Euclidean field operators and proved their equivalence. Our second purpose is to prove the equivalence in a general measure theoretic setting. Finally we mention some results which follow from the differential form of Fröhlich's condition.

In concluding this section we remark that the following three sections are rather independent of each other.

§ 2. Algebras and the KMS Condition

Let ν_0 be the Gaussian measure on $\mathscr{D}'$ with mean zero and covariance

$$\mu^{-2}=(-\Delta+m_0^2)^{-1},$$

where m_0 is a positive number and is fixed throughout this note. Let $\phi(f)=\langle\phi,f\rangle$ for $\phi\in\mathscr{D}'$ and for $f\in\mathscr{D}$. $U(f)=e^{i\phi(f)}$ is a multiplication operator in $L^2(\mathscr{D}',\nu_0)$ and $V(g)=e^{i\pi(g)}$ is a translation operator defined by

$$(V(g)F)(\phi)=\sqrt{\frac{d\nu_0(\phi+g)}{d\nu_0(\phi)}}\,F(\phi+g),\qquad F\in L^2(\mathscr{D}',\nu_0).\tag{2.1}$$

$U(f)$ and $V(g)$ are unitaries and satisfy the canonical commutation relation (CCR)

$$V(g)U(f)=e^{i(f,g)}U(f)V(g)\tag{2.2}$$

where (f,g) is the inner product of the real Hilbert space $L^2(R^2)$. If $\Lambda\subset R^2$, let Σ_Λ be the σ-algebra generated by the set of $\phi(f)$ with f in $\mathscr{D}$ and $\operatorname{supp} f\subset\Lambda$.

Definition 1. For $\Lambda\subset R^2$ compact, we define "local algebras" $\mathfrak{M}(\Lambda)$ and $\mathfrak{A}(\Lambda)$:

$$\mathfrak{M}(\Lambda)=\{U(f);f\in C_0^\infty(\Lambda)\}''\tag{2.3}$$

$$\mathfrak{A}(\Lambda)=\bar{\mathfrak{L}}\{AV(g);A\in\mathfrak{M}(\Lambda),g\in C_0^\infty(\Lambda)\}\tag{2.4}$$

both on $L^2(\mathscr{D}',\nu_0\restriction\Sigma_\Lambda)$ where $\bar{\mathfrak{L}}\{\cdots\}$ means the norm closure of linear span of $\{\cdots\}$. Let $\mathfrak{M}$ and $\mathfrak{A}$ be "quasi-local algebras" defined by

$$\mathfrak{M}=\cup\{\mathfrak{M}(\Lambda);\Lambda\text{ compact}\}^-\tag{2.5}$$

$$\mathfrak{A}=\cup\{\mathfrak{A}(\Lambda);\Lambda\text{ compact}\}^-.\tag{2.6}$$

We remark that $\mathfrak{M}(\Lambda)$'s are von Neuman algebras and that the others are C^*-algebras. The "classical algebras" $\mathfrak{M}(\Lambda)$ (respectively $\mathfrak{M}$) are abelian and are embedded into the "quantum algebras" $\mathfrak{A}(\Lambda)$ (respectively $\mathfrak{A}$). We now define one parameter groups of automorphisms which act on $\mathfrak{M}$ trivially.

Definition 2. Let $\{\alpha_t^0;t\in R\}$ and $\{\alpha_t;t\in R\}$ be one parameter groups of *-automorphisms on $\mathfrak{A}$ such that for all f and g in $\mathscr{D}$,

$$\alpha_t^0(U(f))=U(f)\tag{2.7}$$

$$\alpha_t^0(V(g))=V(g)\cdot\exp(-it\{\phi(\mu^2g)-\tfrac{1}{2}\|\mu g\|^2\})\tag{2.8}$$

$$\alpha_t(U(f))=U(f)\tag{2.9}$$

$$\alpha_t(V(g))=V(g)\exp itG_g(\phi-g)=\exp itG_g(\phi)\cdot V(g)\tag{2.10}$$

where $G_g(\phi)=-\phi(\mu^2g)-\frac{1}{2}\|\mu g\|^2-[\mathscr{U}(\phi+g)-\mathscr{U}(\phi)]$.

α_t^0 may be interpreted as free time automorphisms because it is formally implemented by the "free Hamiltonian"

$$H_0=\tfrac{1}{2}\int :(\nabla\phi)^2(x)+m_0^2\phi^2(x):d^2x=\tfrac{1}{2}(\mu\phi,\mu\phi) \tag{2.11}$$

where we use "commutation relations"

$$\begin{aligned}&[H_0,\phi(f)]=0\\ &[H_0,\pi(g)]=i\phi(\mu^2 g)\,.\end{aligned} \tag{2.12}$$

We note here that the "Bogoliubov transformations"

$$\begin{aligned}&\phi(f)\to e^{itH_0}\phi(f)e^{-itH_0}=\phi(f)\\ &\pi(g)\to e^{itH_0}\pi(g)e^{-itH_0}=\pi(g)-\phi(\mu^2 g)t\end{aligned} \tag{2.13}$$

are not unitarily implementable. On the other hand α_t is formally induced by

$$H=H_0+\int :P(\phi(x)):d^2x \tag{2.14}$$

where : : denotes the Wick product with respect to ν_0. In fact α_t may be defined by

$$\alpha_t(Q)=\lim_{\Lambda\to R^2} e^{i\mathcal{U}_\Lambda t}\alpha_t^0(Q)e^{-i\mathcal{U}_\Lambda t},\qquad Q\in\mathfrak{A} \tag{2.15}$$

where $\mathcal{U}_\Lambda=\int_\Lambda :P(\phi(x)):d^2x$ for compact Λ. For $Q\in\mathfrak{A}(\Lambda)$ we need not take the limit since

$$e^{i\mathcal{U}_\Lambda t}\alpha_t^0(Q)e^{-i\mathcal{U}_\Lambda t}=e^{i\mathcal{U}_{\Lambda'} t}\alpha_t^0(Q)e^{-i\mathcal{U}_{\Lambda'} t}$$

for $\Lambda'\supset\Lambda$. Finally we note that $\alpha_t(\mathfrak{A}(\Lambda))=\mathfrak{A}(\Lambda)$ and that α_t is continuous in the weak operator topology but not in the strong topology, i.e. $\lim_{t\to 0}\|\alpha_t(Q)-Q\|=0$ does not necessarily hold.

Let us consider states of $\mathfrak{A}$. A state is a positive linear functional of norm 1 on the algebra. A locally Fock state on $\mathfrak{A}$ is a state on $\mathfrak{A}$ such that the restriction to any local algebra is Fock, i.e. it can be extended to a normal state on $\mathfrak{A}(\Lambda)''$ in $L^2(\mathscr{D}',\nu_0\restriction\Sigma_\Lambda)$. A locally normal state on $\mathfrak{M}$ is a state on $\mathfrak{M}$ such that the restriction to $\mathfrak{M}(\Lambda)$ is normal, i.e. a locally normal state on $\mathfrak{M}$ is given by a probability measure on $(\mathscr{D}',\Sigma)$ which is locally absolutely continuous with respect to ν_0.

Proposition 1. *Let ω be an α_t-KMS state on $\mathfrak{A}$ such that $\omega\restriction\mathfrak{M}$ is locally normal. Let $\nu=\omega\restriction\mathfrak{M}$ be a measure on $\mathscr{D}'$. Then ν is quasi-invariant and*

$$\frac{d\nu(\phi+g)}{d\nu(\phi)}=\exp G_g(\phi)\,.\quad\square \tag{2.16}$$

Proof. By the KMS condition (see, for example [1]), for any Q_1 and Q_2 in $\mathfrak{A}$, there exists a function $F(z)$ of a complex number z, which is holomorphic for $\operatorname{Im} z\in(0,1)$, continuous for $\operatorname{Im} z\in[0,1]$ and satisfies

$$F(t)=\omega(Q_1\alpha_t(Q_2)),\ F(t+i)=\omega(\alpha_t(Q_2)Q_1),\ -\infty<t<+\infty\,. \tag{2.17}$$

Let $Q_1 = V(g)A$ and $Q_2 = V(-g)$ for $g \in \mathscr{D}$, $A \in \mathfrak{M}$. Then

$$\begin{aligned} F(t) &= \omega(V(g)AV(-g)e^{itG_{-g}(\cdot+g)}) \\ &= \int (V(g)AV(-g))(\phi)e^{-itG_g(\phi)}d\nu(\phi) \\ &= \int A(\phi+g)e^{-itG_g(\phi)}d\nu(\phi), \end{aligned} \tag{2.18}$$

$$\begin{aligned} F(t+i) &= \omega(e^{itG_{-g}(\cdot)}A) \\ &= \int A(\Phi+g)e^{-itG_g(\phi)}d\nu(\phi+g) \end{aligned} \tag{2.19}$$

where we have used

$$G_g(\phi) = -G_{-g}(\phi+g). \tag{2.20}$$

Now we have: for any $A \in L^\infty(\nu)$, there exists a continous function $F(z)$ for $\operatorname{Im} z \in [0, 1]$ which is holomorphic for $\operatorname{Im} z \in (0, 1)$ and satisfies $(G(\phi) \equiv G_g(\phi))$

$$F(t) = \int A(\phi)e^{-itG(\phi)}d\nu(\phi) \tag{2.21}$$

$$F(t+i) = \int A(\phi)e^{-itG(\phi)}d\nu(\phi+g). \tag{2.22}$$

If $\nu(\Delta)=0$ for $\Delta \in \Sigma$, let $A = \chi_\Delta$. Then we have $F(t) \equiv 0$ and so $\nu(\Delta+g)=0$. Hence ν is quasi-invariant. Let $S_N = \{\phi : G(\phi) \leqq N\}$ and let $A = \chi_{S_N} \cdot B$, $B \in L^\infty(\nu)$. Then $F(t+i)$ is

$$\int_{S_N} B(\phi)e^{-itG(\phi)}e^{G(\phi)}d\nu(\phi) \qquad \text{[by (2.21)]}$$

$$= \int_{S_N} B(\phi)e^{-itG(\phi)}\frac{d\nu(\phi+g)}{d\nu(\phi)}d\nu(\phi) \qquad \text{[by (2.22)]}. \tag{2.23}$$

From the case $B \equiv 1$ and $t=0$, we have $\int_{S_N} e^{G(\phi)}d\nu(\phi) \leqq 1$ and by the monotone convergence theorem $\int e^{G(\phi)}d\nu(\phi) \leqq 1$. We obtain the desired result (2.16) by letting $N \to \infty$ in the equality (2.23).

A state on $\mathfrak{A}$ is called a classical state if it vanishes on $AV(g)$ for any A in $\mathfrak{M}$ and g in $\mathscr{D}$ with $g \neq 0$. A classical state is not locally Fock because $s \in R \to \omega(V(sg))$ is not continuous at $s=0$ for any non zero $g \in \mathscr{D}$.

Hence the GNS representation space $\mathscr{H}_\omega$ associated with a classical state ω is not $L^2(\mathscr{D}', \nu)$ even if $\nu = \omega \upharpoonright \mathfrak{M}$ exists as a measure on $\mathscr{D}'$. In fact $\mathscr{H}_\omega$ is non-separable as easily shown: The uncountable set $\{\pi_\omega(V(g))\Omega_\omega; g \in \mathscr{D}\}$ is an orthonormal family of vectors of $\mathscr{H}_\omega$. [Here $(\mathscr{H}_\omega, \pi_\omega, \Omega_\omega)$ is the GNS triple associated with ω.]

Proposition 2. *If ω is an α_t-KMS state on $\mathfrak{A}$, then ω is classical.*

Proof. Let $Q = AV(g)$ for $A \in \mathfrak{M}$ and $g \in \mathscr{D}$. We can calculate $\omega(U(h)QU(-h))$ by using the commutation relations and by using the KMS condition, i.e.

$$\begin{aligned} \omega(U(h)QU(-h)) &= e^{-i(h,g)}\omega(Q) \qquad \text{(by CCR)} \\ &= \omega(Q) \qquad \text{(by KMS)}. \end{aligned}$$

If $g \neq 0$, there is h in $\mathscr{D}$ such that $(h, g) \notin 2\pi Z$, and hence $\omega(Q)=0$.

Let us consider the converse of Proposition 1. When we have a measure satisfying (2.16), we want to construct a state on $\mathfrak{A}$ satisfying the KMS condition.

Let $\mathfrak{A}_0$ be the *-algebra algebraically generated by

$$\{AV(g): A\in\mathfrak{M}, g\in\mathscr{D}\}.$$

If v is a locally normal state on $\mathfrak{M}$, we can define a state on $\mathfrak{A}_0$ by

$$\omega(Q)=v(A_0) \quad \text{for} \quad Q=A_0+\textstyle\sum_{i=1}^{n} V(g_i)A_i \tag{2.24}$$

where $A_i\in\mathfrak{M}$ and g_i's are mutually different non-zero elements in $\mathscr{D}$. In fact ω is a state: for Q above

$$\begin{aligned}\omega(Q^*Q)&=\textstyle\sum_{i=0}^{n} v(A_i^*A_i)\geqq 0\\ \omega(1)&=v(1)=1\,.\end{aligned} \tag{2.25}$$

If v is a quasi-invariant measure, there is a natural representation π_v of $\mathfrak{A}_0$ similar to (2.1). If we define a vector state ω_Ψ for $\Psi\in L^2(\mathscr{D}', v)$ with $\|\Psi\|=1$ by $\omega_\Psi(Q)=(\Psi, \pi_v(Q)\Psi)$, $Q\in\mathfrak{A}_0$, then for Q in (2.24),

$$\omega(Q)=\lim_{T\to\infty} 1/2T \textstyle\int_{-T}^{T} dt\omega_{U(tf)\Omega}(Q)$$

with $\Omega\equiv 1$ and with f in $\mathscr{D}$ such that $(f, g_i)\neq 0$ for $i=1,\dots,n$. Hence we have $|\omega(Q)|\leqq\|\pi_v(Q)\|$. Since v is locally absolutely continuous with respect to v_0, we have $\|\pi_v(Q)\|\leqq\|\pi_{v_0}(Q)\|\equiv\|Q\|$ for $Q\in\mathfrak{A}_0$, and so we can extend ω to a state on $\mathfrak{A}\equiv\bar{\mathfrak{A}}_0$ by the continuity. And by the construction it is clear that such extension to classical states on $\mathfrak{A}$ is unique.

We have made a classical state ω on $\mathfrak{A}$ for a given measure v satisfying (2.16). Now we want to show that ω satisfies the KMS condition. It is easy to show that for any $g, h\in\mathscr{D}$ and $A, B\in\mathfrak{M}$, there exists a bounded continuous function $F(z)$ for $\operatorname{Im} z\in[0,1]$ which is holomorphic for $\operatorname{Im} z\in(0,1)$ and satisfies (2.17) with $Q_1=V(g)A$ and $Q_2=V(h)B$ [cf. (2.18) and (2.19)]. Note that if $g+h\neq 0$, $F\equiv 0$. Hence for any Q_1 and Q_2 in $\mathfrak{A}_0$ we have $F(z)$ satisfying (2.17). Since $\mathfrak{A}$ is the norm closure of $\mathfrak{A}_0$, we obtain, for any Q_1 and Q_2 in $\mathfrak{A}$, $F(z)$ satisfying (2.17) by a limiting procedure.

Now we summarize:

Theorem 3. *There is a one-one correspondence between*

(i) *the set of KMS states ω on $\mathfrak{A}$ whose restriction to $\mathfrak{M}$ are locally normal and*

(ii) *the set of probability measures v on $\mathscr{D}'$ satisfying* (2.16),

the correspondence being given by $\omega\restriction\mathfrak{M}=v$. □

§ 3. Differential Form of Fröhlich's Equation

In this section we are concerned with the following differential form of Fröhlich's equation

$$\int A(\phi)\{-\mu^2\phi(x)-\mathscr{U}'(\phi(x))\}dv(\phi)=-\int\frac{\delta A(\phi)}{\delta\phi(x)}dv(\phi) \tag{3.1}$$

where $\mathscr{U}'(\phi(x))=:P'(\phi(x)):$ and $\delta/\delta\phi(x)$ is a functional derivative. More precisely, for any integer $n\geqq 0$, for any bounded C^1 function A of n variables with bounded derivatives and for $n+1$ elements $f_1, \dots, f_n, g$ in $\mathscr{D}$,

$$\int A(\phi(f_1), \dots, \phi(f_n)) \{-\phi(\mu^2 g) - \mathscr{U}'(\phi, g)\} d\nu(\phi)$$
$$= -\sum_{i=1}^{n} (f_i, g) \int \partial_i A(\phi(f_1), \dots, \phi(f_n)) d\nu(\phi) \tag{3.1'}$$

where $\partial_i A$ is a derivative of A by the i'th variable and $\mathscr{U}'(\phi, g) = \int :P'(\phi(x)): g(x) dx$.

In the following, we will show the equivalence of (3.1) with (1.1) under some assumptions. Since the original proof of Fröhlich [3] uses some detailed estimates of the $P(\phi)_2$ theory, we think it is worthwhile to give a sufficient condition under which the equivalence holds in a general setting.

Before doing it, we give below the formal derivation of (3.1) from (1.1) and a simple example where (3.1) can be "solved".

The formal derivation is as follows: Starting from the identity

$$\int A(\phi) \frac{1}{s} \{d\nu(\phi + sg) - d\nu(\phi)\} = -\int \frac{1}{s} \{A(\phi) - A(\phi - sg)\} d\nu(\phi) \tag{3.2}$$

we let s tend to zero. If (1.1) holds, then

$$\frac{1}{s}\left\{\frac{d\nu(\phi + sg)}{d\nu(\phi)} - 1\right\} \to -\phi(\mu^2 g) - \mathscr{U}'(\phi, g) \quad \text{as} \quad s \to 0 \tag{3.3}$$

at least pointwise and we obtain (3.1).

As an example we consider the free case, i.e. $P \equiv 0$. In this case we can immediately find the unique measure satisfying (3.1). Equation (3.1) with $A = e^{is\phi(f)}$ yields

$$\int e^{is\phi(f)} \phi(f) d\nu = is \|\mu^{-1} f\|^2 \int e^{is\phi(f)} d\nu$$

after being smeared out by $\mu^{-2} f$. Therefore the characteristic function $J(sf) = \int e^{is\phi(f)} d\nu$ satisfies the following differential equation

$$\frac{d}{ds} J(sf) = -s \|\mu^{-1} f\|^2 J(sf).$$

We have $J(sf) = \exp\left\{-\frac{s^2}{2} \|\mu^{-1} f\|^2\right\}$. The measure is the Gaussian measure ν_0 which satisfies (3.1) with $\mathscr{U}' = 0$ as is easily seen. The case $P(\phi) = a\phi^2 + b\phi$ can be treated similarly.

Now we turn to the main problem. First let us explain some definitions. We have two random fields over $\mathscr{D}$ whose common underlying probability space is $(\mathscr{D}', \Sigma, \nu)$. One field is the usual ϕ and the other is denoted by $K(x) = K(\phi(x))$ which might not be defined everywhere on $\mathscr{D}'$ but ν almost everywhere. If ν is quasi-invariant, let $\{\beta_g : g \in \mathscr{D}\}$ be *-automorphisms on $L^\infty(\mathscr{D}', \nu)$ defined by $(\beta_g Q)(\phi) = Q(\phi + g)$ for Q in L^∞. β_g is also defined for any measurable function on $(\mathscr{D}', \nu)$.

Theorem 4. *Let ϕ, K, and ν as above. Suppose that*

(a) *ν is quasi-invariant and for some $p > 1$*

$$g \in \mathscr{D} \to \frac{d\nu(\phi + g)}{d\nu(\phi)} \in L^p$$

is continuous,

(b) $f \in \mathscr{D} \to K(f) \equiv K(\phi, f) \in L^q$ *is linear for some* $q > 0; p^{-1} + q^{-1} \leqq 1$
(c) $g \in \mathscr{D} \to \beta_g K(f) \in L^q$ *is continuous for any* f *in* $\mathscr{D}$.
Then the following are equivalent

(i) $\int e^{i\phi(f)} K(\phi, g) dv(\phi) = -i(f, g) \int e^{i\phi(f)} dv(\phi)$ *for any* f *and* g *in* $\mathscr{D}$ (3.4)

(ii) $\dfrac{dv(\phi+g)}{dv(\phi)} = \exp \int_0^1 \beta_{sg} K(g) ds$ *for any* g *in* $\mathscr{D}$. (3.5)

Proof. First we assume (ii). We note that

$$\frac{dv(\phi+tg)}{dv(\phi)} = \exp \int_0^t \beta_{sg} K(g) ds . \tag{3.6}$$

We use (3.2) with $A = e^{i\phi(f)}$ and so we have to compute

$$\frac{1}{t}(\exp \int_0^t \beta_{sg} K(g) ds - 1) = \frac{1}{t} \int_0^t ds \beta_{sg} K(g) \cdot \exp \int_0^s du \beta_{ug} K(g) . \tag{3.7}$$

Since $s \to \beta_{sg} K(g) \cdot \exp \int_0^s du \beta_{ug} K(g) \in L^1$ is continuous by the assumption, (3.7) tends to $K(g)$ in L^1 as $t \to 0$. Hence we have (i).

Next we assume (i). From (3.4) we can get the equation of type (3.1′) with $A \in \mathscr{S}(R^n)$. For let us rewrite (3.4) as follows:

$$\int e^{i\{\sum_1^n s_j \phi(f_j)\}} K(\phi, g) dv(\phi) = -\sum_1^n i s_j (f_j, g) \int e^{i\{\sum_1^n s_j \phi(f_j)\}} dv(\phi) .$$

Then we multiply both sides by $\hat{A}(s_1, \ldots, s_n)$ and integrate them by s_j, $j = 1, \ldots, n$. By Fubini's theorem we can interchange the order of integrations and obtain

$$\int A(\phi(f_1), \ldots, \phi(f_n)) K(\phi, g) dv = -\sum_1^n (f_i, g) \int \partial_i A(\phi(f_1), \ldots, \phi(f_n)) dv . \tag{3.8}$$

For a set $M \subset \mathscr{D}$, let $\Sigma(M)$ be the σ-algebra generated by $\phi(f)$ with f in M. Let $\{f_1, \ldots, f_n\}$ be a linearly independent subset of $\mathscr{D}$ and M be a linear span of them. Then (3.8) can be written as

$$\begin{aligned} &\int A(s_1, \ldots, s_n) \{\textstyle\sum_1^n \lambda_i K_i(s_1, \ldots, s_n)\} dv_M(s_1, \ldots, s_n) \\ &\quad = -\textstyle\sum_1^n \sum_1^n \lambda_j (f_j, f_i) \int \partial_i A(s_1, \ldots, s_n) dv_M(s_1, \ldots, s_n) \quad \text{for} \quad g = \sum_1^n \lambda_i f_i \end{aligned} \tag{3.9}$$

where $K_i(s_1, \ldots, s_n)$ is the conditional expectation $E(K(f_i)|\Sigma(M))$ of $K(f_i)$ with respect to $\Sigma(M)$ and $v_M(s_1, \ldots, s_n)$ is the restriction of v to $\Sigma(M)$. Here M^* is identified with R^n relative to the dual base of the base $(f_1, \ldots, f_n)$ in M and $s \equiv (s_1, \ldots, s_n)$ is the point in R^n so identified with M^*. We assume $(f_i, f_j) = \delta_{ij}$ which can always be attained by a linear transformation. Then we have for $i = 1, \ldots, n$

$$\int A(s) K_i(s) dv_M(s) = -\int \partial_i A(s) dv_M(s) . \tag{3.10}$$

Since dv_M is equivalent to the Lebesgue measure ds on R^n by the assumption of quasi-invariance, there is an a.e. (almost everywhere) positive function $\varrho(s)$ on R^n such that $dv_M = \varrho(s) ds$. Equation (3.10) with $dv_M = \varrho ds$ implies by the formal calculation

$$\frac{\partial}{\partial s_i} \log \varrho(s) = K_i(s) \tag{3.11}$$

which we justify later. In the following let $i=1$. The meaning of (3.11) is as follows: for a.a. (almost all) fixed $(s_2,\dots,s_n)$, $\varrho>0$ for every s_1, $\log\varrho$ is absolutely continuous in s_1 and (3.11) holds for a.a. s_1. If we assume (3.11),

$$\frac{\varrho(s_1+\alpha, s_2,\dots,s_n)}{\varrho(s_1,\dots,s_n)} = \exp \int_{s_1}^{s_1+\alpha} K_1(t, s_2,\dots,s_n)dt \tag{3.12}$$

for a.a. $(s_1,\dots,s_n)$. We can rewrite it as follows:

$$\frac{dv(\phi+\alpha f_1)\upharpoonright\Sigma(M)}{dv(\phi)\upharpoonright\Sigma(M)} = \exp \int_0^\alpha dt\beta_{tf_1}E(K(f_1)|\Sigma(M)) . \tag{3.13}$$

Since $\mathscr{D}$ is separable, there is an increasing sequence $\{M_n\}$ of finite dimensional subspaces such that $\Sigma(M_n)\uparrow\Sigma$. By Doob's theorem

$$\frac{dv(\phi+\alpha f_1)\upharpoonright\Sigma(M_n)}{dv(\phi)\upharpoonright\Sigma(M_n)} = E\left(\frac{dv(\phi+\alpha f_1)}{dv(\phi)}\middle|\Sigma(M_n)\right) \to \frac{dv(\phi+\alpha f_1)}{dv(\phi)}, \tag{3.14}$$
$$E(K(\Phi,f_1)|\Sigma(M_n))\to K(\phi,f_1), \quad \text{as } n\to\infty$$

where the convergence are in L^1 norm.

Hence we know that

$$\int_0^\alpha dt\beta_{tf_1}E(K(f_1)|\Sigma(M_n))\to \int_0^\alpha dt\beta_{tf_1}K(f_1)$$

in L^1 norm. Therefore

$$\frac{dv(\phi+\alpha f_1)}{dv(\phi)} = \exp \int_0^\alpha dt\beta_{tf_1}K(f_1) = \exp \int_0^1 dt\beta_{t\cdot\alpha f_1}K(\alpha f_1) . \tag{3.15}$$

Since αf_1 is arbitrary, we have the theorem.

In order to prove (3.11), we give some lemmas.

Lemma 5. *Under the assumption of the theorem,*

$$g\in\mathscr{D}\to\beta_g E_M K(f)\in L^1 \tag{3.16}$$

is continuous where E_M is the conditional expectation $E(\cdot|\Sigma(M))$.

Proof. First we prove $\beta_g E_M K(f)\in L^1$. By Jensen's inequality for $q\geqq 1$

$$|E_M K|^q \leqq E_M|K|^q \tag{3.17}$$

where $K=K(f)$. We can compute by using Hölder's and Jensen's inequalities,

$$\begin{aligned}\int |(\beta_g E_M K)(\phi)|dv(\phi) &= \int |(E_M k)(\phi+g)|dv(\phi)\\ &= \int |(E_M K)(\phi)| \frac{dv(\phi-g)}{dv(\phi)} dv(\phi)\\ &\leqq \left(\int |E_M K|^{p'}dv\right)^{1/p'}\left(\int\left(\frac{dv(\phi-g)}{dv(\phi)}\right)^p dv(\phi)\right)^{1/p}\\ &\leqq \left(\int |K|^{p'}dv\right)^{1/p'}\left(\int\left(\frac{dv(\phi-g)}{dv(\phi)}\right)^p dv(\phi)\right)^{1/p}\end{aligned} \tag{3.18}$$

where $p^{-1}+p'^{-1}=1$ and p is given in (a). As the right hand side is finite, we have $\beta_g E_M K(f) \in L^1$.

Let $E_M K(f) = K_f(s_i, \ldots, s_n)$ as in the proof of Theorem 4. Then we have $\beta_g E_M K(f) = K_f(s_1+\lambda_1, \ldots, s_n+\lambda_n) \equiv K_f(s+\lambda) \equiv \hat{\beta}_\lambda K_f$ where $(g, f_i) = \lambda_i$. Hence we only have to prove that

$$\lambda \in \mathrm{R}^n \rightarrow \hat{\beta}_\lambda K_f \in \mathrm{L}^1(\mathrm{d}\nu_M) \tag{3.19}$$

is continuous. Now

$$\begin{aligned} &\int |\hat{\beta}_{\lambda'} K_f - \hat{\beta}_\lambda K_f| \varrho ds \\ &\leqq \int |(\hat{\beta}_\lambda K_f \cdot \varrho)(s+\lambda'-\lambda) - (\hat{\beta}_\lambda K_f \cdot \varrho)(s)| ds \\ &\quad + \int |\hat{\beta}_{\lambda'} K_f(s)| \, |\varrho(s) - \varrho(s+\lambda'-\lambda)| ds . \end{aligned} \tag{3.20}$$

As $\lambda' \rightarrow \lambda$, the first term tends to zero because $\hat{\beta}_\lambda K_f \cdot \varrho$ is integrable with respect to the Lebesgue measure. The second term is majorized by

$$\left(\int |K_f(s+\lambda')|^{p'} \varrho(s) ds\right)^{1/p'} \cdot \left(\int \left| \frac{\varrho(s+\lambda'-\lambda)}{\varrho(s)} - 1 \right|^p \varrho(s) ds \right)^{1/p}$$

which tends to zero as $\lambda' \rightarrow \lambda$.

Lemma 6. *$K_f(s) \equiv E_M K(f)$ is integrable on any bounded set with respect to the Lebesgue measure.*

Proof. By Lemma 5, $g \rightarrow |\beta_g E_M K(f)| \in L^1$ is continuous, i.e.

$$\lambda \in R^n \rightarrow \hat{\beta}_\lambda K_f \in L^1(d\nu_M) \tag{3.21}$$

is continuous. Hence the function

$$s \in R^n \rightarrow \int_I |K_f(s+\lambda)| d\lambda \tag{3.22}$$

is measurable for any bounded set $I \subset R^n$, because by Fubini's theorem

$$\int_I d\lambda \int ds \varrho(s) |K_f(s+\lambda)| = \int ds \varrho(s) \int_I d\lambda |K_f(s+\lambda)| .$$

If for some point $s^0 \in R^n$ and bounded set I^0

$$\int_{I^0} d\lambda |K_f(s^0+\lambda)| = +\infty$$

then

$$\int_I d\lambda |K_f(s+\lambda)| = +\infty$$

on $\{s; s^0+I^0 \subset s+I\}$ whose measure is non zero for large enough I which is a contradiction. Hence (3.22) takes finite values everywhere.

Lemma 7. *Let K_1 and ϱ be as in the proof of Theorem 4. In particular they satisfy for any $A \in \mathcal{S}(R^n)$*

$$\begin{aligned} &\int A(s_1, \ldots, s_n) K_1(s_1, \ldots, s_n) \varrho(s_1, \ldots, s_n) ds_1 \ldots ds_n \\ &\quad = -\int \frac{\partial}{\partial s_1} A(s_1, \ldots, s_n) \varrho(s_1, \ldots, s_n) ds_1 \ldots ds_n . \end{aligned} \tag{3.23}$$

Then

$$\varrho(s_1,\dots,s_n)=\int_{-\infty}^{s_1} dtK_1(t,s_2,\dots,s_n)\varrho(t,s_2,\dots,s_n) \text{ a.e. } \square \tag{3.24}$$

Proof. Let ϱ_1 be defined by the right hand side of (3.24). Then

$$\int\left(\frac{\partial}{\partial s_1}A\right)\varrho_1 ds_1=-\int AK_1\varrho ds_1 .$$

In comparison with (3.23) we know that

$$\int\frac{\partial}{\partial s_1}A(s_1,\dots,s_n)(\varrho_1(s_1,\dots,s_n)-\varrho(s_1,\dots,s_n))ds_1=0$$

for any $A\in\mathscr{S}(R^n)$. Hence

$$\varrho_1(s_1,\dots,s_n)-\varrho(s_1,\dots,s_n)=c(s_2,\dots,s_n) \quad \text{a.e.}$$

with some s_1-independent function c. By the condition at $s_1=-\infty$, $c=0$ a.e.

Now we can conclude the proof of Theorem 4. We know that $K_1(s_1,\dots,s_n)$ is integrable on any bounded set in R^n by Lemma 6 and $\varrho(s_1,\dots,s_n)$ is absolutely continuous in s_1 for a.a. $(s_2,\dots,s_n)$ by Lemma 7. There is a null set N in R^{n-1} such that for $(s_2,\dots,s_n)\notin N$

$$\frac{\partial}{\partial s_1}\varrho(s_1,\dots,s_n)=K_1(s_1,\dots,s_n)\varrho(s_1,\dots,s_n) \quad \text{a.a. } s_1$$

$$K_1(s_1,\dots,s_n) \text{ is integrable in } s_1 \text{ on bounded sets} \tag{3.25}$$

and

$$\varrho(s_1,\dots,s_n)>0 \quad \text{a.a. } s_1 .$$

We can assume that $\varrho(s_1,\dots,s_n)$ is continuous in s_1 for $(s_2,\dots,s_n)\notin N$. On the set $\{s_1:\varrho(s_1,\dots,s_n)>0\}$ for fixed $(s_2,\dots,s_n)\notin N$, $\log\varrho$ is absolutely continuous in s_1 and

$$\frac{\partial\log\varrho(s_1,\dots,s_n)}{\partial s_1}=K_1(s_1,\dots,s_n) \quad \text{a.a. } s_1 .$$

If there is an interval $[a,b)$ such that $\varrho>0$ on $[a,b)$ and $\varrho=0$ at b as a function in s_1 then we have a contradiction:

$$\lim_{x\uparrow b}\int_a^x K_1(s_1,\dots,s_n)ds_1=-\infty .$$

Hence $\varrho(s_1,\dots,s_n)>0$ for any s_1 and $\log\varrho$ is absolutely continuous in s_1 and satisfies

$$\frac{\partial\log\varrho(s_1,\dots,s_n)}{\partial s_1}=K_1(s_1,\dots,s_n)$$

which is (3.11) with $i=1$.

Let us return to the case $K(\phi(x)) = -\mu^2\phi(x) - \mathscr{U}'(\phi(x))$. For g in $\mathscr{D}$, $K(\phi, g)$ is defined by $K(\phi, g) = -\phi(\mu^2 g) - \int :P'(\phi(x)):g(x)d^2x$. We compute

$$\begin{aligned}\int_0^1 \beta_{sg}K(g)ds &= \int_0^1 K(\phi+sg, g)ds\\ &= -\int_0^1 \{\phi(\mu^2 g) + s\|\mu g\|^2 + \int :P'(\phi(x)+sg(x)):g(x)dx\}ds\\ &= -\phi(\mu^2 g) - \tfrac{1}{2}\|\mu g\|^2 - \int \{:P(\phi(x)+g(x)): - :P(\phi(x)):\}dx\,.\end{aligned}$$

Hence we have the equivalence of (1.1) with (3.1) under the assumptions in Theorem 4.

Remark 1. From Theorem 4, $K(x)$ satisfies

$$\begin{aligned}&\exp \int_0^1 \beta_{sf} E(K(f)|\Sigma(M))ds\\ &\quad = E(\exp \int_0^1 \beta_{sf} K(f)ds|\Sigma(M))\end{aligned}$$

for any f in $\mathscr{D}$ and any finite dimensional space $M \ni f$. Here we cannot interchange the order of $E(\cdot|\Sigma(M))$ and β_{sf} in the left hand side.

Remark 2. If we consider the equation

$$\int A(\phi)\{-\mu^2\phi(x) - g(x)\mathscr{U}'(\phi(x))\}dv(\phi) = -\int \frac{\delta A(\phi)}{\delta\phi(x)} dv(\phi)$$

where the space cutoff g is involved, we can construct the measure v as usual,

$$dv = e^{-\mathscr{U}_g}dv_0 / \int e^{-\mathscr{U}_g}dv_0$$

with

$$\mathscr{U}_g = \int :P(\phi(x)):g(x)dx\,.$$

Remark 3. A connection between (3.1) and the relativistic field equation is seen from the following phenomena: Let $f_1, \ldots, f_n$ in $\mathscr{D}$ have mutually disjoint supports and $F_i = \mu^2\phi(f_i)$ or $-\int :P'(\phi(x)):f_i(x)dx$. Then $\int F_1 \ldots F_n dv$ does not depend whether $F_i = \mu^2\phi(f_i)$ or $F_i = -\int :P'(\phi(x)):f_i(x)dx$. In other words, we cannot distinguish the two fields $\mu^2\phi(x)$ and $-:P'(\phi(x)):$ at non-coincident points.

§ 4. Additional Remarks

We now consider the case $P(\phi) = a\phi^4 + b\phi^2 - c\phi$ $(a > 0)$. Suppose that we have a measure v satisfying (3.1) with some additional properties which enable us to construct the corresponding Wightman functions. The existence of such measures is proved for half-Dirichlet theories with $P(\phi) = \sum_{m=1}^{n} a_m\phi^{2m} - c\phi$ $(a_n > 0)$ [3]. We also assume the uniqueness of vacuum. This is proved for $c \neq 0$ or we can decompose the Euclidean measure into ergodic measures which correspond to quantum field theories with unique vacuum [3].

We shall not deal with a measure v but only Schwinger type functions which are expectation values of products of monomials and Wick powers of ϕ's and which satisfy

$$\begin{aligned}&4a\langle\phi(x_1)\dots\phi(x_{n-1}):\phi^3:(x_n)\rangle+2b\langle\phi(x_1)\dots\phi(x_n)\rangle\\&\quad+\langle\phi(x_1)\dots\phi(x_{n-1})\mu^2\phi(x_n)\rangle-c\langle\phi(x_1)\dots\phi(x_{n-1})\rangle\\&=\sum_1^{n-1}\delta(x_n-x_i)\langle\phi(x_1)\dots\phi(x_{i-1})\phi(x_{i+1})\dots\phi(x_{n-1})\rangle\end{aligned}\tag{4.1}$$

where $\langle\dots\rangle=\int\dots dv$. This is obtained by substitution of $A=\phi(x_1)\dots\phi(x_{n-1})$ into (3.1). We can justify it by a suitable limiting procedure, where we use the bounded convergence theorem, if all the Schwinger type functions involved exist as distributions. This is true, of course, for the cases we mentioned above.

We remark here that equations of type (4.1) have a rather old history in Euclidean field theory (see e.g. [4]).

Let us recall the Källen-Lehman representation about $S_2^T(x-y)\equiv\langle\phi(x)\phi(y)\rangle-\langle\phi(z)\rangle\langle\phi(y)\rangle$. There is a positive measure ϱ on $(0,\infty)$ with

$$S_2^T(x-y)=\int d\varrho(m^2)S_m(x-y)=\int d\varrho(m^2)\frac{1}{(2\pi)^2}\int d^2p\,\frac{e^{ip(x-y)}}{p^2+m^2}.\tag{4.2}$$

We assume $\int d\varrho(m^2)=1$ which holds in the cases above where the Schwinger functions are the limits of the cutoff Schwinger functions [8]. We note that $\langle:\phi^3:(x):\phi^3:(y)\rangle^T$ is also represented in terms of a spectral measure in the same manner as $S_2^T(x-y)$.

Now we represent some quantities in terms of ϱ in (4.2):

(i) The "magnetization" $M=\langle\phi(x)\rangle$ is given by

$$M^2=\int d\varrho(m^2)\left(\frac{1}{4\pi}\log(m^2/m_0^2)+m^2/12a\right)-(m_0^2+2b)/12a$$

(ii) $\langle:\phi^2:\rangle=\dfrac{1}{12a}\int d\varrho(m^2)m^2-(m_0^2+2b)/12a$

(iii) $\langle:\phi^3:(x):\phi^3:(y)\rangle^T=\dfrac{1}{(4a)^2}\int d\varrho(m^2)(m^2-m_0^2-2b)^2S_m(x-y)$

(iv) $\langle:\phi^4:\rangle=\dfrac{M}{4a}\{c-M(m_0^2+2b)\}$.

Remark. From (i) if there is a mass gap uniformly as $b\to-\infty$ for fixed a and c we have $M^2\geqq O(|b|)$.

The proof of the above result is easv. We use (4.1) with $n\leqq 4$.

Acknowledgements. I would like to thank Professor H. Araki for critical reading of the manuscript and for many helpful suggestions. I thank Professor H. Totoki for useful discussions and the referee for valuable comments.

References

1. Araki, H.: In: Varenna lectures on C^*-algebras and physical applications. Italian Physical Society (1973)
2. Brascamp, H. J.: Commun. math. Phys. **18**, 82 (1970)
3. Fröhlich, J.: Helv. Phys. Acta **47**, 265 (1974); Schwinger functions and their generating functionals II, (to appear)

4. Glimm, J., Jaffe, A.: Two and three body equations in quantum field models. Commun. math. Phys. **44**, 293—320 (1975)
5. Guerra, F., Rosen, L., Simon, B.: The $P(\phi)_2$ Euclidean quantum field theory as classical statistical mechanics, (to appear in Ann. Math.)
6. Guerra, F., Rosen, L., Simon, B.: Commun. math. Phys. **41**, 19 (1975)
7. Simon, B.: Correlation inequalities and the mass gap in $P(\phi)_2$ II, (to appear)
8. Simon, B., Griffith, R. B.: Commun. math. Phys. **33**, 145 (1973)

Communicated by A. S. Wightman

Received April 8, 1974; in revised form October 31, 1975