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Dissipations and Derivations

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Abstract. We show a usefulness of the notion of "dissipative operators" in the study of derivations of C^*-algebras and prove that the closure of a normal $*$-derivation of UHF algebra satisfying a special condition is a generator of a one-parameter group of $*$-automorphisms.

§ 1. Introduction

Recently various authors have studied unbounded derivations of C^*-algebras [2–4, 6, 7, 10, 11, 13]. In particular Powers and Sakai [10] have studied unbounded derivations of UHF algebra.

The purpose of the present note is to show a usefulness of the notion of "dissipative operators" [9, 17] in the study of derivations of C^*-algebras.

Our first result is that an everywhere defined "dissipation" is bounded, which implies the well-known theorem concerning derivations [5, 12].

Our second result is about a normal $*$-derivation of UHF algebra satisfying a special condition discussed in [1, 10, 14, 15]. For such a $*$-derivation, we prove that its closure is a generator of a one-parameter group of $*$-automorphisms. As its application we consider one-dimensional lattice system.

§ 2. Bounded Derivation

Let $\mathfrak{A}$ be a Banach space. For each $x\in\mathfrak{A}$ there is at least one non-zero element f of the dual Banach space $\mathfrak{A}^*$ such that $\langle x,f\rangle=\|x\|\cdot\|f\|$ by the Hahn-Banach theorem. An f_x denotes one of them throughout this note.

Definition 1. [9] A linear map γ with domain $\mathscr{D}(\gamma)$ in a Banach space is called dissipative if there is an f_x such that

$$\mathrm{Re}\langle\gamma x, f_x\rangle\leqq 0$$

for each $x\in\mathscr{D}(\gamma)$.

Definition 2. A linear map δ with domain $\mathscr{D}(\delta)$ in a Banach space is called derivative if there is an f_x such that

$$\mathrm{Re}\langle\delta x, f_x\rangle=0$$

for each $x\in\mathscr{D}(\delta)$.

Let $\mathfrak{A}$ be a C^*-algebra. A linear map δ of $\mathfrak{A}$ is called a derivation if it satisfies

$$\delta(xy)=\delta(x)y+x\delta(y)$$

for $x, y \in \mathscr{D}(\delta)$, where $\mathscr{D}(\delta)$, the domain of δ, is a *-subalgebra in $\mathfrak{A}$. A derivation δ is a *-derivation if $\delta(x)^* = \delta(x^*)$ for $x \in \mathscr{D}(\delta)$. In the following we will be concerned with only *-derivation and so omit *.

A linear map δ of $\mathfrak{A}$ is a derivation if δ and $-\delta$ are dissipations whose definition is:

Definition 3. [8] A linear map γ of a C^*-algebra $\mathfrak{A}$ is called a dissipation if it satisfies

$$\gamma(x)^* = \gamma(x^*)$$

$$\gamma(x^*x) \geqq \gamma(x^*)x + x^*\gamma(x)$$

for each $x \in \mathscr{D}(\gamma)$, where $\mathscr{D}(\gamma)$, the domain of γ, is a *-subalgebra.

Remark 1. Call γ an "n-dissipation" if $\gamma \otimes \iota;\ \mathfrak{A} \otimes F_n \to \mathfrak{A} \otimes F_n$ is a dissipation where F_n is an algebra of all $n \times n$ matrices and ι is an identity map. If γ is a $2n$-dissipation of a C^*-algebra with identity and $\mathscr{D}(\gamma) \ni 1$, then γ' defined by $\gamma'(x) = \gamma(x) - \frac{1}{2}\{\gamma(1)x + x\gamma(1)\}$ is an n-dissipation. Note $\gamma(1) \leqq 0$ and $\gamma'(1) = 0$. (See [8] for the arguments of bounded complete dissipations; a complete dissipation is defined to be an n-dissipation for all n.)

Lemma 1. *Let γ be a dissipation with domain $\mathscr{D}(\gamma)$. Suppose that for any positive $x \in \mathscr{D}(\gamma)$ there is an f_x such that* $\mathrm{Re}\langle \gamma x, f_x \rangle \leqq 0$. *Then γ is dissipative.*

Proof. Note that f_x is positive for a positive $x \in \mathfrak{A}$ [12]. If we define fx^* and xf in $\mathfrak{A}^*$ for $x \in \mathfrak{A}$ and $f \in \mathfrak{A}^*$ by $\langle a, fx^* \rangle = \langle x^*a, f \rangle$ and $\langle a, xf \rangle = \langle ax, f \rangle$ $(a \in \mathfrak{A})$, then $xf_{x^*x} = f_{x^*}$ and $f_{x^*x}x^* = f_x$. For any $x \in \mathscr{D}(\gamma)$, there is an $f = f_{x^*x}$ such that $\langle \gamma(x^*x), f \rangle \leqq 0$. Then we have

$$\begin{aligned} 0 &\geqq \langle \gamma(x^*x), f \rangle \\ &\geqq \langle \gamma x^*, xf \rangle + \langle \gamma x, fx^* \rangle \\ &= 2\,\mathrm{Re}\langle \gamma x, fx^* \rangle\,. \end{aligned}$$

Lemma 2. (Lemmas 3.3 and 3.4 in [9]). *A dissipative operator with dense domain in a Banach space is closable and its closure is also dissipative.*

Sketch of the proof. Let γ be the dissipative operator. Let $x_n \in \mathscr{D}(\gamma)$ with $x_n \to 0$ and $\gamma x_n \to y$. For any $a \in \mathscr{D}(\gamma)$ and $\lambda \in \mathbb{R}$, let $f_{n,\lambda} = f_{a+\lambda x_n}$ with $\|f_{n,\lambda}\| = 1$ and $\mathrm{Re}\langle \gamma(a + \lambda x_n), f_{n,\lambda} \rangle \leqq 0$. We may suppose $f_{n,\lambda} \to f_\lambda (n \to \infty)$ and $f_\lambda \to f'(\lambda \to \infty)$. Then we have $f' = f_a$ and $\mathrm{Re}\langle y, f' \rangle \leqq 0$. We may suppose $f' \to f(a \to y)$. Then $f = f_y$ and $\|y\| = \mathrm{Re}\langle y, f \rangle \leqq 0$, i.e. $y = 0$. The rest of the proof is easy.

In the rest of this section we will treat only everywhere defined operators.

Theorem 1. *A dissipation γ of a C^*-algebra $\mathfrak{A}(=\mathscr{D}(\gamma))$ is dissipative and bounded.*

Proof. We suppose $\mathfrak{A} \ni 1$. If $\mathfrak{A} \not\ni 1$, we can consider a dissipation γ_1 of $\mathfrak{A}_1 = \mathfrak{A} + \mathbb{C} \cdot 1$ defined by $\gamma_1(x + \lambda 1) = \gamma(x)(x \in \mathfrak{A}, \lambda \in \mathbb{C})$.

Let $x \in \mathfrak{A}$ be positive. Setting $h \equiv (\|x\| \cdot 1 - x)^{1/2}$, we have for $f = f_x$,

$$\begin{aligned} \langle \gamma x, f \rangle &\leqq \langle \gamma(x - \|x\| \cdot 1), f \rangle \\ &= -\langle \gamma h^2, f \rangle \\ &\leqq -\langle (\gamma h)h, f \rangle - \langle h\gamma h, f \rangle \\ &= 0 \end{aligned}$$

where we have used the Schwartz inequality and the fact $\langle h^2, f\rangle = 0$ and $f \geqq 0$. Hence γ is dissipative by Lemma 1 and closed by Lemma 2. An everywhere defined closed operator is bounded by the closed graph theorem.

Corollary. *A derivation of a C^*-algebra is derivative and bounded.*

Proof. The proof is quite similar to the above. Or it follows from the above theorem by the following remark.

Remark 2. From the proof of Theorem 1 we can conclude that if γ is a dissipation, for any f_x, $\mathrm{Re}\langle \gamma x, f_x\rangle \leqq 0$. It is immediate for $x \geqq 0$. For a general $x \in \mathfrak{A}$, any f_x is equal to $f x^*$ where $f = f_{x^*x} = \|x\|^{-1}|f_x|$. (Let $x = u|x|$ be the polar decomposition of x in the enveloping von Neumann algebra of $\mathfrak{A}$. Then $|f_x| = f_x u$, from which we can deduce $|f_x| = f_{|x|} = f_{x^*x}$.) The same situation prevails for derivations. (See Remark 2 in [9].)

Remark 3. [6] A dissipation γ generates a uniformly continuous one-parameter semi-group of positive contractions $\Phi_t = e^{t\gamma}$. Lindblad showed the equivalence of (i) and (ii);

(i) Φ_t is uniformly continuous, $\Phi_t(1) = 1$ and

$$\Phi_t(x^*)\Phi_t(x) \leqq \Phi_t(x^*x)\,.$$

(ii) γ is a dissipation with $\gamma(1) = 0$.

Finally we remark the following property of a derivation δ. Let x be self-adjoint and $C(x)$ be the commutative C^*-subalgebra generated by x and 1. Let φ be a character of $C(x)$ and $\bar{\varphi}$ be any norm-preserving extension of φ ($\bar{\varphi}$ is a state). Then $\langle \delta x, \bar{\varphi}\rangle = 0$ which is considered as generalization of derivativeness (see [5]).

This is easily seen; if a polynomial $P(x)$ of x satisfies $\langle P'(x), \varphi\rangle = P'(\langle x, \varphi\rangle) = 0$, then $\langle \delta P(x), \bar{\varphi}\rangle = 0$. The set of such $P(x)$ is dense in $C(x)$ and so $\langle \delta x, \bar{\varphi}\rangle = 0$ by the continuity of δ.

§ 3. Unbounded Derivations

In the following the domain of a derivation or dissipation of a C^*-algebra is a dense *-subalgebra.

Theorem 2. *Let γ be a dissipation of a C^*-algebra $\mathfrak{A}$. If $\mathscr{D}(\gamma)$ is closed under the square root operation of positive elements, then γ is dissipative and hence closable.*

Proof [4, 10]. The proof that γ is dissipative is quite similar to that of Theorem 1. By Lemma 2 it is closable.

Let $\mathfrak{A}$ be a uniformly hyperfinite C^*-algebra (UHF algebra). A derivation δ in $\mathfrak{A}$ is said to be normal [10] if $\mathscr{D}(\delta)$ is the union of an increasing sequence of finite type I subfactors $\{\mathfrak{A}_n | n = 1, 2, \ldots\}$ in $\mathfrak{A}$.

Corollary. *A normal derivation of a UHF algebra is derivative and hence closable. Its closure is also a derivative derivation.*

Let τ be a unique tracial state on a UHF algebra $\mathfrak{A}$. A derivation δ in $\mathfrak{A}$ is said to be regular [10] if $\langle \delta(a), \tau\rangle = 0$ for $a \in \mathscr{D}(\delta)$.

Let δ be a normal derivation. Since $\langle ab, \tau\circ\delta\rangle = \langle ba, \tau\circ\delta\rangle$ for $a, b \in \mathscr{D}(\delta) \equiv \cup\mathfrak{A}_n$ and $\langle 1, \tau\circ\delta\rangle = 0$, $\tau\circ\delta|\mathfrak{A}_n = 0$ for any n. Hence δ is regular [10].

Theorem 3. *If a derivation δ in a UHF algebra is regular, then δ is derivative.*

Proof. Let $L^2(\mathfrak{A}, \tau)$ be a Hilbert space completion of a UHF algebra $\mathfrak{A}$ with inner product $\langle x, y\rangle_\tau = \langle y^*x, \tau\rangle$. Let x be a positive element of $\mathscr{D}(\delta)$ and $L^2(C(x), \tau)$ be the closed subspace spanned by $C(x)$. Let E_x be the orthogonal projection onto $L^2(C(x), \tau)$. If δ is regular,

$$\begin{aligned} 0 &= \langle x^n, \tau \circ \delta\rangle \\ &= n\langle x^{n-1}\delta(x), \tau\rangle \\ &= n\langle \delta(x), x^{n-1}\rangle_\tau . \end{aligned}$$

Hence $E_x\delta(x)=0$. Let φ be a character of $C(x)$ and $\hat{\varphi}$ be any norm-preserving extension of φ into $L^\infty(C(x), \tau)^*$. Since $E_x : \mathfrak{A}(\subset L^\infty(\mathfrak{A}, \tau)) \to L^\infty(C(x), \tau)$ is a contraction, $\bar{\varphi} = \hat{\varphi} \circ E_x$ is an element of $\mathfrak{A}^*$. Let φ be a character such that $\langle x, \varphi\rangle = \|x\| \, \|\varphi\| = \|x\|$ and let $\bar{\varphi} = \hat{\varphi} \circ E_x$. Then $\bar{\varphi} = f_x$ and $\langle \delta x, \bar{\varphi}\rangle = 0$. Now the proof is completed by Lemma 1.

Let δ be a normal derivation in $\mathfrak{A}$. Let $\tilde{\delta}$ be the greatest linear extension of δ in all linear extensions γ satisfying

$$\gamma(axb) = \delta(a)xb + a\gamma(x)b + ax\delta(b)$$

$$\langle x, \tau \circ \gamma\rangle = 0, \qquad a, b \in \mathscr{D}(\delta), \ x \in \mathscr{D}(\gamma) .$$

$\tilde{\delta}$ is called the greatest regular extension of a normal derivation δ [10].

Theorem 4. *Let δ be a normal derivation. Suppose that $\tilde{\delta}$ is a derivation (or $\tilde{\delta}$ is derivative) and that there is an infinitesimal generator δ_1 of a strongly continuous group of *-automorphisms such that $\delta_1 \supseteq \delta$. Then $\delta_1 = \tilde{\delta}$.*

Proof. Since δ_1 is regular [10], $\delta_1 \subseteq \tilde{\delta}$. As $(1 \pm \tilde{\delta})\mathscr{D}(\tilde{\delta}) \supseteq (1 \pm \delta_1)\mathscr{D}(\delta_1) = \mathfrak{A}$ and $\tilde{\delta}$ is derivative by Theorem 3, $\tilde{\delta}$ is an infinitesimal generator by the following theorem and remark. Hence $\delta_1 = \tilde{\delta}$.

Theorem 5. *Let δ be a derivation of a C^*-algebra $\mathfrak{A}$. If δ is derivative and closed and $(1 \pm \delta)\mathscr{D}(\delta)$ is dense in $\mathfrak{A}$, then δ is an infinitesimal generator of a strongly continuous group of *-automorphisms.*

Proof. If f_x satisfies $\operatorname{Re}\langle \delta x, f_x\rangle = 0$ and $\|f_x\| = 1$,

$$\begin{aligned} \|(\delta + \lambda)x\| &\geqq \pm \operatorname{Re}\langle(\delta + \lambda)x, f_x\rangle \\ &= \pm \operatorname{Re}\lambda \|x\| \end{aligned}$$

i.e. $\|(\delta + \lambda)x\| \geqq |\operatorname{Re}\lambda| \cdot \|x\|$.

The rest of the proof is standard [2–4].

Remark 4. The assumption that δ is a derivation in Theorem 5 can be replaced as follows: Let δ be a linear operator with dense domain $\mathscr{D}(\delta)$ such that $\mathscr{D}(\delta) \ni 1$ and $\delta(1) = 0$. It is shown as follows: By a result in the Hill-Yosida semi-group theory [17] δ generates a strongly continuous group of contractions ϱ_t on $\mathfrak{A}$. Since $\varrho_t(1) = 1$ (by the assumption $\delta(1) = 0$) and $\|\varrho_t\| = 1$ they are positive contractions. As they form a group, they are order-isomorphisms. Thus ϱ_t is a strongly continuous one-parameter group of Jordan automorphisms [cf. 16]. Then it is known [18, Theorem 3.4] that ϱ_t is a group of *-automorphisms.

Remark 5. $\tilde{\delta}$ is in general not a derivation (see Problem 1 of [10]). For if δ is a normal derivation which has more than two different extensions to infinitesimal generators, then $\tilde{\delta}$ is not a derivation, as easily shown by using Theorem 4. (We can construct such δ. See Remark 3 of [10].)

Let P_n be the canonical conditional expectation of $\mathfrak{A}$ onto $\mathfrak{A}_n$. Let h_n be a self-adjoint element of $\mathfrak{A}$ such that $\delta(a)=[ih_n, a]\equiv\delta_{ih_n}(a)$ for all $a\in\mathfrak{A}_n$. Then $P_n\tilde{\delta}(x)=P_n\delta_{ih_n}(x)$ for $x\in\mathscr{D}(\tilde{\delta})$ [10]. For if $a\in\mathfrak{A}_n$,

$$\begin{aligned}\langle aP_n\tilde{\delta}(x), \tau\rangle &= \langle a\tilde{\delta}(x), \tau\rangle\\ &= \langle ax, \tau\circ\tilde{\delta}\rangle - \langle(\delta a)x, \tau\rangle\\ &= -\langle(\delta_{ih_n}a)x, \tau\rangle\\ &= \langle a\delta_{ih_n}x, \tau\rangle\\ &= \langle aP_n\delta_{ih_n}x, \tau\rangle .\end{aligned}$$

In [10] $W\subset\mathscr{D}(\tilde{\delta})$ is defined by

$$W\equiv\{x\in\mathscr{D}(\tilde{\delta});\ \lim P_n\tilde{\delta}(1-P_n)x=0\} .$$

If we set $P_n(h_n)=k_n$,

$$W=\{x\in\mathscr{D}(\tilde{\delta});\ \lim\delta_{ik_n}P_nx=\tilde{\delta}(x)\} .$$

In [6] an operator ex-lim δ_{ik_n} (the extended limit of the $\delta_{ik_n}|\mathfrak{A}_n$) is defined, whose graph is the set of $(x, y)\in\mathfrak{A}\times\mathfrak{A}$ such that there is a sequence $x_n\in\mathfrak{A}_n$, with $\|x_n-x\|\to 0$ and $\|\delta_{ik_n}(x_n)-y\|\to 0$.

In [7] an operator $\hat{\delta}$ (the graph limit of the δ_{ik_n}) is defined, whose graph is the set of $(x, y)\in\mathfrak{A}\times\mathfrak{A}$ such that there is a sequence $x_n\in\mathfrak{A}$, with $\|x_n-x\|\to 0$ and $\|\delta_{ik_n}(x_n)-y\|\to 0$.

Then

$$\delta\subset\tilde{\delta}|W\subset\text{ex-lim}\,\delta_{ik_n}\subset\hat{\delta}\subset\tilde{\delta} .$$

Theorem 6. *$\hat{\delta}$ is derivative.*

Proof. Let $x\in\mathscr{D}(\hat{\delta})$ and $\{x_n\}$ be a sequence such that $x_n\to x$ and $\delta_{ik_n}(x_n)\to\hat{\delta}(x)$. Let $f_n=f_{x_n}$ be of norm 1. We may suppose $f_n\to f$. Then $f=f_x$ and

$$\begin{aligned}\mathrm{Re}\langle\hat{\delta}x, f\rangle &= \lim\mathrm{Re}\langle\delta_{ik_n}x_n, f_n\rangle\\ &= 0\end{aligned}$$

where we have used Remark 2.

Remark 6. [6, 7] $\hat{\delta}$ and ex-lim δ_{ik_n} are closed derivations.

Lemma 3. *If $\{\|h_n-k_n\|\}$ is uniformly bounded, $\tilde{\delta}$ is derivative.*

Proof. Let $x\in\mathscr{D}(\tilde{\delta})$ and $f_n=f_{P_nx}$ with $\|f_n\|=1$. We may suppose $f_n\to f$. Then $f=f_x$ and

$$\begin{aligned}\mathrm{Re}\langle\tilde{\delta}x, f\rangle &= \lim\mathrm{Re}\langle P_n\tilde{\delta}x, f_n\rangle\\ &= \lim\mathrm{Re}\langle P_n\tilde{\delta}(1-P_n)x, f_n\rangle\\ &= \lim\mathrm{Re}\langle P_n\delta_{ih_n-ik_n}(1-P_n)x, f_n\rangle\end{aligned}$$

where we have used $\operatorname{Re}\langle P_n\tilde{\delta}P_n x, f_n\rangle = 0$, $P_n\delta_{ik_n}(1-P_n)=0$ and $\delta_{ih_n - ik_n} = \delta_{ih_n} - \delta_{ik_n}$. The last term is dominated by

$$2\|h_n - k_n\| \cdot \|(1-P_n)x\|$$

which tends to zero as $n \to \infty$.

Theorem 7. *Let δ be a normal derivation. If $\{\|h_n - k_n\|\}$ is uniformly bounded, $\bar{\delta}$, the closure of δ, is an infinitesimal generator of a strongly continuous group of *-automorphisms and $\bar{\delta} = \tilde{\delta}$.*

Proof. Suppose that $(1+\delta)\mathscr{D}(\delta)$ is not dense in $\mathfrak{A}$. Then there is an element f in $\mathfrak{A}^*$ such that $\|f\|=1$ and $\langle x+\delta x, f\rangle = 0$ for all $x \in \mathscr{D}(\delta)$. There are $x_n \in \mathfrak{A}_n \subset \mathscr{D}(\delta) \equiv \cup \mathfrak{A}_n$ such that $\langle x_n, f\rangle = \|x_n\| \, \|f|\mathfrak{A}_n\| = \|f|\mathfrak{A}_n\|$. Then

$$\begin{aligned} 0 &= \lim \operatorname{Re}\{\langle x_n, f\rangle + \langle \delta x_n, f\rangle\} \\ &= \lim \operatorname{Re}\{\|f|\mathfrak{A}_n\| + \langle \delta_{ih_n} x_n, f\rangle\} \\ &= \|f\| + \lim \operatorname{Re}\langle \delta_{ih_n - ik_n} x_n, f\rangle \\ &\geqq 1 - \overline{\lim}\, 2 \cdot \|h_n - k_n\| \end{aligned}$$

where we have used $\operatorname{Re}\langle \delta_{ik_n} x_n, f\rangle = 0$. Suppose $\|h_n - k_n\| < 1/2 - \varepsilon (\varepsilon > 0)$. Then it is a contradiction and hence $(1+\delta)\mathscr{D}(\delta)$ is dense in $\mathfrak{A}$. Quite similarly we can conclude that $(1-\delta)\mathscr{D}(\delta)$ is dense in $\mathfrak{A}$. Since $\bar{\delta}$ is derivative by Corollary of Theorem 3, $\bar{\delta}$ is an infinitesimal generator by Theorem 5. If $\|h_n - k_n\| < C$ for any n, we may consider $\delta/3C$ instead of δ. $\bar{\delta} = \tilde{\delta}$ follows from Theorem 4 and Lemma 3.

Remark 7. Under the assumption of Theorem 7 the one-parameter group ϱ_t generated by $\bar{\delta}$ satisfies

$$\varrho_t(x) = \lim e^{t\delta_{ik_n}}(x), \qquad x \in \mathfrak{A}$$

where the convergence is uniform in t on every compact subset of $(-\infty, \infty)$. This follows from Theorem 7 combined with Theorems 6 and 8 in [10] (cf. the proof of Theorem 8 below).

As an application of Theorem 7, we consider one-dimensional lattice system. Let $\{\mathfrak{A}_j : j \in Z\}$ be a family of type I finite factors and let $\mathfrak{A} = \bigotimes_{j\in Z} \mathfrak{A}_j$ be the infinite tensor product of them. Let Φ be a map from the family $P_f(Z)$ of finite subsets of Z into $\mathfrak{A}$ such that $\Phi(\emptyset)=0$ and $\Phi(\Lambda)$ is a self-adjoint element of $\mathfrak{A}(\Lambda) = \bigotimes_{j\in\Lambda} \mathfrak{A}_j$. Put

$$\|\Phi\|_\alpha = \sup_j \sum_{\Lambda \ni j} e^{\alpha N(\Lambda)} \|\Phi(\Lambda)\|$$

where $N(\Lambda)$ denotes the number of points in Λ and $\alpha \geqq 0$.

It is known (cf. [1]) that if $\|\Phi\|_\alpha < \infty$ for $\alpha > 0$, there exists a one-parameter group of *-automorphisms such that

$$\varrho_t(Q) = \lim_\Lambda e^{itU(\Lambda)} Q e^{-itU(\Lambda)} = \lim e^{t\delta_{iU(\Lambda)}} Q, \qquad Q \in \mathfrak{A}$$

$$U(\Lambda) = \sum_{J \subset \Lambda} \Phi(J).$$

Now we give another sufficient condition for the existence of the above automorphism group:

Theorem 8. *Suppose that* (i) $\|\Phi\|_0 < \infty$ *and* (ii) *there is an increasing sequence* $\{\Lambda_n\} \subset P_f(Z)$ *such that* $\cup \Lambda_n = Z$ *and the following element* $W(\Lambda_n)$ *of* $\mathfrak{A}$ *is bounded in norm uniformly in* n*:*

$$W(\Lambda_n) = \sum_J \{\Phi(J); J \in P_f(Z), J \cap \Lambda \neq \emptyset, J \cap \Lambda^c \neq \emptyset\}$$

where Λ^c *denotes the complement of* Λ *in* Z. *Then there exists a strongly continuous one-parameter group of *-automorphisms such that*

$$\varrho_t(Q) = \lim_n e^{t\delta_n}(Q) \qquad (*)$$

where $\delta_n = \delta_{iU(\Lambda_n)}$ *and the convergence is uniformly in* t *on every compact interval of* t.

Proof. By (i), $W(\Lambda_n)$ is well-defined. Let $\mathfrak{A}_n = \mathfrak{A}(\Lambda_n)$ and let $h_n = U(\Lambda_n) + W(\Lambda_n)$. Let δ be the normal derivation such that

$$\delta|\mathfrak{A}_n = \delta_{ih_n}, \qquad \mathscr{D}(\delta) = \cup \mathfrak{A}_n .$$

Then [1]

$$\begin{aligned} \|h_n - k_n\| &\leqq \|h_n - U(\Lambda_n)\| + \|U(\Lambda_n) - k_n\| \\ &\leqq 2\|W(\Lambda_n)\| \end{aligned}$$

where $k_n = P_n(h_n)$. Hence $\bar{\delta}$ is an infinitesimal generator by Theorem 7. Now the proof of the convergence in (*) follows as in [10]: It is shown by (i) that $\lim \delta_n = \delta$ on $\mathscr{D}(\delta)$. Then for $x \in \mathscr{D}(\delta)$

$$\begin{aligned} &\|\{(1 \pm \delta_n)^{-1} - (1 \pm \bar{\delta})^{-1}\}(1 \pm \bar{\delta})x\| \\ &\quad = \|(1 \pm \delta_n)^{-1}\{(1 \pm \bar{\delta})x - (1 \pm \delta_n)x\}\| \\ &\quad \leqq \|(1 \pm \bar{\delta})x - (1 \pm \delta_n)x\| \\ &\quad \leqq \|\bar{\delta}x - \delta_n x\| \\ &\qquad \to 0 \quad \text{as} \quad n \to \infty . \end{aligned}$$

where we have used $\|(1 \pm \delta_n)^{-1}\| \leqq 1$.

Hence $\lim (1 \pm \delta_n)^{-1} = (1 \pm \bar{\delta})^{-1}$ since $(1 \pm \delta)\mathscr{D}(\delta)$ is dense in $\mathfrak{A}$. By the Trotter-Kato theorem [cf. 17] we get (*).

Finally we remark that the assumption (i) can be weakened by (i′) $\sum_{\Lambda \ni j} \|\Phi(\Lambda)\| < \infty$ for any $j \in Z$.

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