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Bounds in the Yukawa$_2$ Quantum Field Theory: Upper Bound on the Pressure, Hamiltonian Bound and Linear Lower Bound*

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Abstract. We prove bounds of the form $Z_\Lambda \leqq e^{a|\Lambda|}$ and $(\mathrm{SZ})_\Lambda \leqq e^{a|\Lambda|}$ in the Y_2 Euclidean field theory and from this obtain Glimm's Hamiltonian bound and Schrader's linear lower bound.

I. Introduction

One of the most basic infinite volume bounds in constructive field theory is a linear lower bound on the energy per unit volume. Such bounds were first proven in $P(\phi)_2$ theories by Glimm and Jaffe [6], Y_2 theories by Schrader [18] and ϕ_3^4 theories by Glimm and Jaffe [7]. The Euclidean translation of these bounds fits into the view of Euclidean field theories as statistical mechanical systems [10, 8] for the "essentially equivalent" Euclidean bound is an upper bound on the pressure (see [10], § VI).

Our goal in this paper is to provide a new proof of and, we feel, new insight into Schrader's bound. Along the way we will establish Glimm's basic result [4, 5] that the (renormalized) Yukawa$_2$ Hamiltonian spatially cutoff is bounded from below. (This result is a basic input in Schrader's proof.) We also prove a volume independent bound on the Euclidean pressure – in fact this is our main input in proving the results of Glimm and Schrader. Conversely, we should note that given the connection between the Hamiltonian and Euclidean theories (see [15], § III for this connection, which uses the Euclidean Fermi fields of Osterwalder-Schrader), Schrader's result implies a bound on the pressure.

A "semi-Euclidean" proof of Schrader's result has been obtained by Brydges [1]. When our own work on this subject was completed in a preliminary draft, we received a preprint from McBryan [15] with similar results. McBryan also works

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in the Matthews-Salam formalism but his methods of estimation are quite different from ours.

We deal throughout in the formalism of Matthews and Salam [12, 13] (see also Schwinger [22]) in which the fermions have been integrated out. For the Euclidean Y_2 theory with a space-time cutoff, the renormalization cancellations have been controlled in this formalism by Seiler [19]. We follow the notation of that paper with one change: namely, for $A \in \mathscr{C}_n (\mathscr{C}_n = \{A \mid \|A\|_n^n \equiv \mathrm{Tr}((A^*A)^{n/2}) < \infty\})$ we define $\det_n$ by:

$$\det_n(1+A) = \det[(1+A)\exp[\textstyle\sum_{k=1}^{n-1}(-A)^k/k]] \tag{1.1}$$

(this is called $\det_{(n-1)}$ in [13]; both conventions are used in the literature).

The Matthews-Salam formalism is critical for our method of proof. In the first place, we do not appear to be in possession of an analytically powerful Euclidean Fermi field theory; while the fields of Osterwalder-Schrader [17] have provided a useful bridge between the Euclidean and Hamiltonian worlds, they have not yet proven to be useful in proving estimates. More critically we use certain L^p properties which are formally false in a theory before the fermions are integrated out. For let $d\mu_0$ be the free Bose field and let $d\nu$ denote a formal symbol for a putative fermion integration. If $g\mathrm{U}(\Lambda)$ is the basic Yukawa interaction in region Λ (with g a coupling constant and the Fermi fields Wick-ordered), the counter-terms are quadratic in g, say of the form $g^2\mathrm{C}(\Lambda)$. Then, formally, we expect that $[\int d\mu_0 \int d\nu \exp(-g\mathrm{U}(\Lambda) - g^2\mathrm{C}(\Lambda))] < \infty$. But then for $p > 1$ $\{\int d\mu_0 [\exp(-g\mathrm{U} - g^2\mathrm{C}(\Lambda)]^p\} = \infty$ since the p^{th} power takes $g\mathrm{U}$ to $pg\mathrm{U}$ and $g^2\mathrm{C}$ to $pg^2\mathrm{C}$ rather than $p^2g^2\mathrm{C}$. Thus methods such as ours formally fail in a theory where the fermions have not been eliminated by a preliminary "integration". The moral of [19] is that $\int d\mu_0 [\int d\nu \exp(-g\mathrm{U} - g^2\mathrm{C})]^p$ is finite, so integrating out the fermions helps.

The basic decoupling of distant regions we will exploit is a consequence of the hypercontractivity and Markov property of the free Bose field, essentially in a form discovered by Nelson [16]. We need the following "checker-board" estimate [10] which generalizes Nelson's basic idea:

Theorem 1.1 ([10]). *Let $d\mu_0$ be the free Bose field of mass m_0 in $\mathbb{R}^2$. Let $\{\Lambda_\alpha\}_{\alpha \in \mathbb{Z}^2}$ be the partition of $\mathbb{R}^2$ into squares of side l with centers at points $l\alpha$. Let f_α be a function of the fields in Λ_α. Then for any p:*

$$\left[\int \left(\textstyle\prod_\alpha f_\alpha\right)^p d\mu_0\right]^{1/p} \leqq \textstyle\prod_\alpha \|f_\alpha\|_{\beta p}, \tag{1.2a}$$

where

$$\beta = 4/(1-e^{-m_0 l})^2 . \tag{1.2b}$$

The point of (1.2) is, of course, that the L^q norm, $\|\cdot\|_{\beta p}$ on the right is independent of the number of factors on the left, a vast improvement of Hölder's inequality. For the $P(\phi)_2$ case where $\exp(-\mathrm{U}(\bigcup_{\alpha \in X} \Lambda_\alpha)) = \prod_\alpha f_\alpha$ [with $f_\alpha = \exp(-\mathrm{U}(\Lambda_\alpha)]$, (1.2) immediately provides the bound $\int \exp[-\mathrm{U}(\bigcup_{\alpha \in X} \Lambda_\alpha)] d\mu_0 \leqq C^{|X|}$ (this is essentially Nelson's proof of the linear lower bound).

Our goal in this paper is to prove a bound of the form

$$\int d\mu_0 \det_{\mathrm{ren}}[1 + K(\textstyle\bigcup_{\alpha \in X} \Lambda_\alpha)] \leqq C^{|X|} . \tag{1.3}$$

In (1.3), $\det_{\mathrm{ren}}$ is a "renormalized" determinant defined in [19] which is formally just $\det(1+K)\exp(\text{counterterms})$ where the counterterms involve linear Wick ordering ($\mathrm{Tr}(K)$), the local quadratic mass counterterm and the second order vacuum energy renormalization. To describe $K(\Lambda)$, we introduce some simple notation. We work on the Hilbert space $L^2(\mathbb{R}^2, dx; \mathbb{C}^2)$ of square integrable $\mathbb{C}^2$-value functions. iP denotes the gradient operator. Then

$$K(\Lambda)=[(P^2+m^2)^{-1/4}W(P)]\phi(x)\chi_\Lambda(x)[(P^2+m^2)^{-1/4}], \tag{1.4a}$$

where

$$W(P)=(P+m)(P^2+m^2)^{-1/2}\Gamma \tag{1.4b}$$

with Γ either 1 (scalar) or $i\gamma_5$ (pseudoscalar). In all our estimates we will ignore the unitary $W(P)$, and as a result the spin degrees of freedom. It is straightforward to make the necessary modifications to include the factors of $W(P)$. We also take coupling constant $g=1$ for simplicity. We also remark that the K of (1.4) differs from that in [19, 14] where $(P^2+m^2)^{-1/2}W(P)\chi_\Lambda(x)\phi(x)$ is considered on $L^2(\mathbb{R}^2, \sqrt{P^2+m^2}\,d^2p, \mathbb{C}^2)$. But clearly, under the natural unitary equivalence of $L^2(\mathbb{R}^2, d^2x)$ and $L^2(\mathbb{R}^2, \sqrt{P^2+m^2}\,d^2p)$ the two are equivalent.

Since $\det(1+K(\bigcup_{\alpha\in X}\Lambda_\alpha)) \neq \prod_{\alpha\in X}\det(1+K(\Lambda_\alpha))$, we cannot directly use (1.2) to prove (1.3). Our philosophy in this paper is to use determinant inequalities to bound $\det_{\mathrm{ren}}(1+K(\bigcup_{\alpha\in X}\Lambda_\alpha))$ by a product of the form $\prod_\alpha u_\alpha$ and then use checkerboard estimates. Were it not for renormalization, this would be easy. Explicitly, if $K(\Lambda)$ were trace class, we could use

$$\begin{aligned}|\det(1+K(\textstyle\bigcup_\alpha\Lambda_\alpha))| &\leqq \exp(\|K(\textstyle\bigcup_\alpha\Lambda_\alpha)\|_1)\\ &= \textstyle\prod_\alpha \exp(\|K(\Lambda_\alpha)\|_1)\,.\end{aligned} \tag{1.5}$$

Notice that (1.5) implies a bound on the pressure in a theory where $S_F(x-y)$ is bounded even if there is no exponential falloff in the Fermi degrees of freedom!

Our basic determinant inequality appears in § 4. It will require us to control "error" terms of a form close to $\exp(\sum_{\alpha,\beta\in X}\|K_\alpha K_\beta\|_1)$. Bounds on trace ideal norms of such integral operators appear in § 2. Those bounds allow us to bound $\|K_\alpha K_\beta\|_1$ by $e^{-D|\alpha-\beta|}F(\phi\chi_\alpha)^{1/2}F(\phi\chi_\beta)^{1/2}$ where F is "quadratic" in ϕ. Integrability of $\exp(F(\phi\chi_\alpha))$ is discussed in § 3. Then, by checkerboard estimates, we have

$$\begin{aligned}\int \exp(\textstyle\sum_{\alpha,\beta\in X}\|K_\alpha K_\beta\|_1)d\mu_0 &\leqq \int\exp(\textstyle\sum_{\alpha,\beta} e^{-D|\alpha-\beta|}F(\phi\chi_\alpha)^{1/2}F(\phi\chi_\beta)^{1/2})d\mu_0\\ &\leqq \int\exp(c\textstyle\sum_\alpha F(\phi\chi_\alpha))d\mu_0 \leqq C^{|X|}\,.\end{aligned}$$

We put everything together in § 5 to prove (1.3). More general bounds on $(\mathrm{SZ})_\Lambda$ and a sketch of the proof of the Glimm and Schrader bounds appear in § 6.

II. Trace Ideal Properties of Some Integral Operators

Lemma 2.1. *Let $2\leqq p\leqq\infty$. Let $f, g\in L^p(\mathbb{R}^2, d^2x)$. Then $f(P)g(X)$ is a bounded operator on $L^2(\mathbb{R}^2, d^2x)$ in the trace ideal $\mathscr{C}_p$ and*

$$\|f(P)g(X)\|_p \leqq \|f\|_p\|g\|_p\,. \tag{2.1}$$

In particular, if $h \in L^{p'}$, the dual space of L^p, and A is the operator with integral kernel $h(x-y)g(y)$, then $A \in \mathscr{C}_p$.

Remark. T. Kato (private communication) has obtained similar estimates by similar methods.

Proof. The final statement follows from (2.1) and the remark that convolution with h is the operator $(2\pi)\hat{h}(p)$ and $\hat{h} \in L^p$ by the Hausdorff-Young inequality. (2.1) is obvious when $p=\infty$. When $p=2$, $f(p)g(x)$ has integral kernel $(2\pi)^{-1}\check{f}(x-y)g(y)$ where $\check{f}$ is the inverse Fourier transform. Again (2.1) holds. Now consider general p. Since $f(P)g(X)=\mathrm{U}[|f|(P)][|g|(X)]\mathrm{V}$ with U, V unitary, we can suppose f, g a.e. non-negative. In that case, let

$$F(z)=f^{\alpha(z)}(P)g^{\alpha(z)}(X),$$

where $\alpha(z)=pz/2$. Then $F(z) \in \mathscr{C}_2$ if $\mathrm{Re}\,z=1$ and is bounded if $\mathrm{Re}\,z=0$. The theorem now follows by interpolation (see e.g. [9]; more detailed references and history on interpolation can be found in [20]). □

Theorem 2.2. *Fix l. Let χ_α, $\alpha \in \mathbb{Z}^2$, be the characteristic function of the square with center at $l\alpha$ and side l. Fix $q \geqq 1$, $k \geqq 0$. Suppose operators A, B are given with $(P^2+m^2)^{-k}\chi_\alpha A$ in $\mathscr{C}_q$ and $B\chi_\beta(P^2+m^2)^{-k} \in \mathscr{C}_{q'}$. Then, for all α, β with $|\alpha-\beta|>\sqrt{2}$, (i.e. non-touching squares), $B\chi_\beta(P^2+m^2)^{-1/4}\chi_\alpha A \in \mathscr{C}_1$ and*

$$\|B\chi_\beta(P^2+m^2)^{-1/4}\chi_\alpha A\|_1 \leqq Ce^{-D|\alpha-\beta|}\|B\chi_\beta(P^2+m^2)^{-k}\|_{q'}\|(P^2+m^2)^{-k}\chi_\alpha A\|_q \quad (2.2)$$

Proof. Given α, β let $\eta_{\alpha,\beta}$ be a smooth function on $\mathbb{R}_+$ with support in $(|\alpha-\beta|-\sqrt{2}-\varepsilon, |\alpha-\beta|+\sqrt{2}+\varepsilon)$, identically one in $(|\alpha-\beta|-\sqrt{2}, |\alpha-\beta|+\sqrt{2})$ where ε is chosen with $\sqrt{2}+\varepsilon<2$. By translating a fixed function, we can bound $\|D^k\eta_{\alpha,\beta}\|_\infty$ uniformly in α, β. Let $(P^2+m^2)^{-1/4}$ be convolution with some function G. Then $\chi_\alpha(x)G(x-y)\chi_\beta(y)=\chi_\alpha(x)\eta_{\alpha,\beta}(l^{-1}|x-y|)G(x-y)\chi_\beta(y)$. Now $(-\Delta+m^2)^{2k}\cdot[\eta_{\alpha,\beta}(l^{-1}|x|)G(x)] \equiv f_{\alpha,\beta}(x)$ is a L^1 function with $\|f_{\alpha\beta}\|_1 \leqq C_1 e^{-D|\alpha-\beta|}$ because of the exponential falloff of G and its derivatives. Thus, letting $F_{\alpha\beta}$ be convolution with $f_{\alpha\beta}$:

$$B\chi_\beta(P^2+m^2)^{-1/4}\chi_\alpha A=[B\chi_\beta(P^2+m^2)^{-k}]F_{\alpha\beta}[(P^2+m^2)^{-k}\chi_\alpha A]$$

so the result (2.2) follows from Hölder's inequality for operators and the fact that Young's inequality implies that $F_{\alpha\beta}$ is a bounded operator with norm $\|f_{\alpha\beta}\|_1$. □

Lemma 2.3. *Let χ_α be the characteristic function of the square of side l and center $l\alpha$. Then:*

(a) *If $0 \leqq \nu < 1/8$ and $\lambda-2\nu>1/8$, then*

$$[(P^2+1)^{-\lambda}, \chi_\alpha](P^2+1)^\nu \in \mathscr{C}_4\,. \quad (2.3)$$

(b) *Under the hypothesis of* (a)

$$[(P^2+1)^\nu, \chi_\alpha](P^2+1)^{-\lambda} \in \mathscr{C}_4\,. \quad (2.4)$$

(c) *If $0 \leqq \nu < 1/16$, $\lambda-2\nu>1/16$, then*

$$[(P^2+1)^\nu, \chi_\alpha](P^2+1)^{-\lambda} \in \mathscr{C}_8\,. \quad (2.5)$$

Proof. (a) The operator in question has a momentum space integral kernel

$$A(p,q)=[(p^2+1)^{-\lambda}-(q^2+1)^{-\lambda}]\hat{\chi}_\alpha(p-q)(q^2+1)^\nu .$$

We first note the bound

$$|\hat{\chi}_\alpha(p)|\leqq C(1+|p_1|)^{-1}(1+|p_2|)^{-1}, \tag{2.6}$$

where p_1 and p_2 are the components of p. Next we note that to prove A is a $\mathscr{C}_4$ kernel, it suffices to find a $\mathscr{C}_4$ kernel L with $|A(p,q)|\leqq L(p,q)$ pointwise. For the operator with integral kernel A is in $\mathscr{C}_4$ if and only if $\int \overline{A(p,q)}A(r,q)dq\in L^2$. It thus suffices to find $f,\tilde{f}\in L^4$ and $g,\tilde{g}\in L^{4/3}$ with

$$|A(p,q)|\leqq f(q)g(p-q); \quad |p|\geqq|q|, \tag{2.7a}$$

$$|A(p,q)|\leqq \tilde{f}(p)\tilde{g}(p-q); \quad |p|\leqq|q|, \tag{2.7b}$$

and appeal to Lemma 2.1.

Suppose first, that $|p|\geqq|q|$. From the bounds

$$|(p^2+1)^{-\lambda}-(q^2+1)^{-\lambda}|\leqq 2(q^2+1)^{-\lambda}$$

$$|(p^2+1)^{-\lambda}-(q^2+1)^{-\lambda}|\leqq C_0|p-q|(q^2+1)^{-\lambda-1/2}$$

we conclude that for any $\beta\in[0,1]$

$$|(p^2+1)^{-\lambda}-(q^2+1)^{-\lambda}|\leqq C(p-q)^\beta(q^2+1)^{-\lambda-1/2\beta}. \tag{2.8}$$

Thus (2.7a) holds with $f(q)=(q^2+1)^{-\lambda-1/2\beta+\nu}$ and $g(x)=C|x|^\beta(1+|x_1|)^{-1}\cdot(1+|x_2|)^{-1}$. For $f\in L^4$ we need

$$\lambda+1/2\beta-\nu>1/4$$

and for $g\in L^{4/3}$

$$\beta<1/4.$$

Such a β exists since $\lambda-\nu>1/8$.

Now suppose $|p|\leqq|q|$. Then (2.8) is replaced by:

$$|(p^2+1)^{-\lambda}-(q^2+1)^{-\lambda}|\leqq C(p-q)^\beta(p^2+1)^{-\lambda-1/2\beta}.$$

We also have

$$(q^2+1)\leqq[1+(|q-p|+|p|)^2]^\nu\leqq C(1+|p-q|^{2\nu})(|p|^2+1)^\nu$$

so (2.7b) holds with $\tilde{f}(p)=(p^2+1)^{-\lambda-1/2+\nu}$ and $\tilde{g}(x)=C[|x|^\beta+|x|^{\beta+2\nu}](1+|x_1|)^{-1}\cdot(1+|x_2|)^{-1}$. For $f\in L^4$, we need

$$\lambda+1/2\beta-\nu>1/4 \tag{2.9a}$$

and for $g\in L^{4/3}$

$$\beta+2\nu<1/4. \tag{2.9b}$$

Taking $\beta=1/4-2\nu-\varepsilon$, (2.9b) holds and (2.9a) becomes $\lambda+1/8-2\nu-1/2\varepsilon>1/4$ which can be arranged by choosing a suitable ε.

(b) Since $[A, B]C = [AC, B] - A[C, B]$, we have:

$$[(P^2+1)^\nu, \chi_\alpha](P^2+1)^{-\lambda} = [(P^2+1)^{-\lambda+\nu}, \chi_\alpha] - (P^2+1)^\nu[(P^2+1)^{-\lambda}, \chi_\alpha]$$

so the result follows from (a).

(c) The proof is similar to (a), (b). $\square$

Theorem 2.4. *Let $\alpha \neq \beta$. Then*
(a) $\chi_\alpha(P^2+m^2)^{-\lambda}\chi_\beta \in \mathscr{C}_4$ *if* $\lambda > 1/8$.
(b) $(P^2+m^2)^\nu\chi_\alpha(P^2+m^2)^{-\lambda}\chi_\beta(P^2+m^2)^\nu \in \mathscr{C}_4$ *if* $0 \leqq \nu < 1/16$, $\lambda - 4\nu > 1/8$.

Proof. (a) Since $\chi_\alpha\chi_\beta = 0$, we have:

$$\chi_\alpha(P^2+m^2)^{-\lambda}\chi_\beta = [\chi_\alpha, (P^2+m^2)^{-\lambda}]\chi_\beta$$

is in $\mathscr{C}_4$ by Lemma 2.3(a).

(b) Since $ABCDE = [A, B]C[D, E] + B(ACE)D + [A, B]CED + BAC[D, E]$, we have that

$$(P^2+m^2)^\nu\chi_\alpha(P^2+m^2)^{-\lambda}\chi_\beta(P^2+m^2)^\nu = A_1 + A_2 + A_3 + A_4 .$$

$A_1 = [(P^2+1)^\nu, \chi_\alpha](P^2+m^2)^{-\lambda/2}(P^2+m^2)^{-\lambda/2}[\chi_\beta, (P^2+m^2)^\nu]$ is in $\mathscr{C}_4$ by Lemma 2.3(c). A_2 is in $\mathscr{C}_4$ by part (a) above. $A_3 = [(P^2+1)^\nu, \chi_\alpha](P^2+1)^{-\lambda+\nu}\chi_\beta$ is in $\mathscr{C}_4$ by Lemma 2.3(b). A_4 is similarly in $\mathscr{C}_4$. $\square$

III. Integrability of Some Functions of the Free Bose Field

In our discussion in the Introduction of the kind of cross terms that occur in our determinant inequalities we were led to consideration of a condition

$$\int d\mu_0 \exp[F(\phi\chi_\alpha)] < \infty$$

for suitable functions F "quadratic" in ϕ. In this section, we wish to prove such integrability estimates. We first prove some general integrability results for general Gaussian random processes. We follow the notation of [21] for these general processes.

Theorem 3.1. *Let $d\mu_0$ be the measure associated with the Gaussian random process indexed by $\mathscr{H}$. Let $u(\phi)$ be a function in $\bigoplus_{m=0}^k \Gamma_m(\mathscr{H})$ (the "polynomials" of degree less than $k=1$). Suppose that $\|u\|_2 < (k/2e)^{k/2}$. Then*

$$\int d\mu_0 \exp(|u|^{2/k}) < \infty .$$

Proof. We must show that $\sum_{n=0}^\infty (n!)^{-1}\|u^{2n/k}\|_1 < \infty$. For $2n/k \leqq 2$, we have $\|u^{2n/k}\|_1 \leqq \|1+u^2\|_1 < \infty$. For $p = 2n/k \geqq 2$, hypercontractivity implies that

$$\|u\|_p \leqq (p-1)^{k/2}\|u\|_2 \leqq p^{k/2}\|u\|_2$$

so

$$\begin{aligned}\|u^{2n/k}\|_1 &= \|u\|_{2n/k}^{2n/k} \leqq ((2n/k)^{k/2}\|u\|_2)^{2n/k}\\ &= n^n[2/k\|u\|_2^{2/k}]^n \leqq C_\delta n![(2e^{(1+\delta)}/k)\|u\|_2^{2/k}]^n\end{aligned}$$

for any $\delta > 0$. Thus the sum in question converges. $\square$

Theorem 3.2. *Let A be a strictly positive Hilbert-Schmidt operator on $\mathscr{H}$ (i.e. one with zero null space). Let u be a homogeneous polynomial of degree k in the Gaussian process indexed by $\mathscr{H}$. Then for all α sufficiently large:*

$$\int d\mu_0 \exp(|u|^{2/k}) \exp(-\alpha{:}(\phi, A\phi){:}) < \infty .$$

Remark. By a homogeneous polynomial of degree k, we mean $u_F = \langle F, \phi(\cdot_1)\dots\phi(\cdot_k)\rangle$ where F is in the n-fold symmetric tensor product $\bigotimes_{j=1}^{k}{}_s\mathscr{H}$, with F such that $u_F \in \Gamma(\mathscr{H})$ (certain "partial traces" finite).

Proof. $dv \equiv d\mu_0 \exp(-\alpha{:}(\phi, A\phi){:})$/Normalization is also a Gaussian process but with covariance

$$(\phi(f), \phi(g))_v = 1/2(f, (1+\alpha A)^{-1} g) .$$

[Note: Following the convention in [10], $d\mu_0$ has covariance $1/2(f, g)$.] Let $\{e_n\}_{n=1}^{\infty}$ be a basis of $\mathscr{H}$ consisting of eigenvectors of A; $Ae_n = \lambda_n e_n$. Then u_F can be written

$$u_F = \sum_{n_1, \dots, n_k} F_{n_1 \dots n_k} \phi(e_{n_1}) \dots \phi(e_{n_k})$$

and we obtain

$$\int |u_F|^2 dv = \sum_{m=0}^{[k/2]} T_m(\alpha)$$

with

$$\begin{aligned} T_m(\alpha) = \binom{k}{2m}^2 \begin{Bmatrix} 2m \\ m \end{Bmatrix}^2 & \sum \overline{F_{n_1 n_1 \dots n_m n_m l_1 \dots l_{k-2m}}} \\ & \cdot F_{r_1 r_1 \dots r_m r_m l_1 \dots l_{k-2m}} \prod_{i=1}^{m} (1+\alpha\lambda_{n_i})^{-1} (1+\alpha\lambda_{r_i})^{-1} \\ & \cdot \prod_{j=1}^{k-2m} (1+\alpha\lambda_{l_j})^{-1} , \end{aligned}$$

where the sum runs over all n_i, r_i, l_j; $\binom{k}{2m}$ is a binomial coefficient and $\begin{Bmatrix} 2m \\ m \end{Bmatrix}$ is the number of ways of pairing $2m$ elements. The point is: since all $\lambda_i > 0$ and each $T_m(\alpha)$ is finite for $\alpha = 0$, $T_m(\alpha) \to 0$ as $\alpha \to \infty$. Thus for α sufficiently large $\int |u_F|^2 dv < (k/2e)^{k/2}$, so

$$\int d\mu_0 \exp(|u|^{2/k}) \exp(-\alpha{:}(\phi, A\phi){:}) = \text{const} \int dv \exp(|u|^{2/k}) < \infty$$

by Theorem 3.1. $\square$

Corollary 3.3. *Let μ_0 be the measure of the free Euclidean Bose field of mass m_0 in two dimensions; let $A(\phi)$ be a linear map from (random) distributions to operators, so that for a.e. $(\mu_0)\phi$ $A(\phi) \in \mathscr{C}_{2k}$ for an integer k and*

$$\int d\mu_0 \|A(\phi)\|_{2k}^{4k} < \infty .$$

Suppose moreover that $A(\phi)$ is only dependent on $\chi_\alpha \phi$. Then for all sufficiently large M:

$$\int d\mu_0 \exp(\|A(\phi)\|_{2k}^2) \exp\left(-M \int {:}\phi^2(x){:} \chi_\alpha(x) d^2x\right) < \infty .$$

Proof. $\|A(\phi)\|_{2k}^{2k} = \mathrm{Tr}((A(\phi)^* A(\phi))^k)$ is a homogeneous polynomial of degree $2k$, so the corollary follows from Theorem 3.2 if we can prove the following: Let $\mathscr{H}$ be the Hilbert space obtained by completing $\chi_\alpha L^2$ in the norm $\|f\|^2 = 2\int |\phi(f)|^2 d\mu_0$

(i.e., $\mathscr{H}$ is N_{Λ_α} in the notation of [10]). We must prove that the operator A on $\mathscr{H}$ given by $:(\phi, A\phi)_{\mathscr{H}}: = \int \chi_\alpha(x):\phi^2(x):d^2x$ is a strictly positive Hilbert-Schmidt operator. Now A is Hilbert-Schmidt since $:(\phi, A\phi):\in \Gamma(\mathscr{H})$. To see it is positive let us compute (f, Af) by using

$$\int :(\phi, A\phi)_{\mathscr{H}}:\phi(f)\phi(g)d\mu_0 = 1/4(f, Ag)_{\mathscr{H}}\,.$$

Thus

$$\begin{aligned}(f, Af)_{\mathscr{H}} &= 4\int \chi_\alpha(x) f(y) f(z) G_0(x-y) G_0(x-z) dx dy dz \\ &= 4\int \chi_\alpha(x)|(f * G_0)(x)|d^2x\,.\end{aligned}$$

As a result, $(f, Af)=0$ implies that $f * G_0 = 0$ on Λ_α and this implies that $f = (-\Delta + \mu^2)(f * G_0) = 0$ on Λ_α. Since $f \in \mathscr{H}$, $f = 0$. □

Theorem 3.4. *Let $q \geqq 2$. Let $v, \lambda > 0$ with $(v+\lambda)q > 1$. Let $A(\phi) = (P^2+m^2)^{-v}\cdot \phi\chi_\beta(P^2+m^2)^{-\lambda}$. Then $A(\phi)$ is a.e. in $\mathscr{C}_q$ and for any c, there is an M_0 so that for $M > M_0$:*

$$\int \exp(c\|A(\phi)\|_q^2)\exp(-M\int \chi_\beta(x):\phi^2(x):d^2x)d\mu_0 < \infty\,.$$

Proof. Consider first the case $q = 2k$ with k an integer. By Corollary 3.3, we need only prove that $\int d\mu_0 \|A(\phi)\|_{2k}^{4k} = \int d\mu_0 \operatorname{Tr}((A^*A)^k)^2 < \infty$. Consider the case $q=4$. Also without loss we can suppose $v, \lambda < 1$ since we can always decrease them. Then:

$$\|A(\phi)\|_4^4 = \int d^2x d^2y |\int d^2z d^2w F(x-z)\phi(z)\chi_\beta(z) H(z-w)\phi(w)\chi_\beta(w) F(w-y)|^2\,,$$

where F is (up to a constant) the Fourier transform of $(P^2+m^2)^{-v}$ and H is the Fourier transform of $(P^2+m^2)^{-2\lambda}$. Letting $J = F * F$, we see that

$$\|A(\phi)\|_4^4 = \int \textstyle\prod_{i=1}^4 d^2x_i \phi(x_i)\chi_\beta(x_i) J(x_1-x_2) H(x_2-x_3) J(x_3-x_4) H(x_4-x_1)\,.$$

We conclude that

$$\int \|A(\phi)\|_4^8 d\mu_0 = \int [\textstyle\prod_{i=1}^8 d^2x_i\chi_\beta(x_1)] A(x_1,\dots,x_4) A(x_5,\dots,x_8) B(x_1,\dots,x_8)\,,$$

where

$$B(x_1,\dots,x_8) = \textstyle\sum_{\text{pairings}} G_0(x_{i_1}-x_{j_1})\dots G_0(x_{i_4}-x_{j_4})$$

and

$$A(y_1,\dots,y_4) = J(y_1-y_2) H(y_2-y_3) J(y_3-y_4) H(y_4-y_1)\,.$$

Now $B \in L^p(\Lambda_\beta^8)$ so we need only prove that $A \in L^s(\Lambda_\beta^4)$ for some $s > 1$. Now for x small $|J(x)| \leqq C|x|^{-2+2v}$ and $|H(x)| \leqq C|x|^{-2+2\lambda}$. Let $\tilde{J}$ (resp. $\tilde{H}$) be the function which equals J (resp. H) if $|x| \leqq \sqrt{2}$ and equals 0 if $|x| \geqq \sqrt{2}$. Then $\tilde{J} \in L^r$ and $\tilde{H} \in L^t$ so long as $r^{-1} > 1 - v$, $t^{-1} > 1 - \lambda$. By Young's inequality and the bound $v + \lambda > 1/4$, we easily find that $A \in L^s$ for some $s > 1$.

When q is not an even integer, we can interpolate between two even integers (as in [20], Lemma 2.7) to obtain:

$$\|A(\phi)\|_q \leqq \|A_1(\phi)\|_{q_1}^{\theta} \cdot \|A_2(\phi)\|_{q_2}^{1-\theta}\,,$$

where q_1 and q_2 are even integers and A_1 and A_2 are of the form just discussed but with suitably changed v, λ. Thus

$$\|A(\phi)\|_q^2 \leqq \|A_1(\phi)\|_{q_1}^2 + \|A_2(\phi)\|_{q_2}^2$$

so the result follows from the case of even integers and Hölder's inequality. $\square$

For later purposes we note the following which is similar to Lemma 2.5 of [20]:

Theorem 3.5. *Let θ be an arbitrary function of compact support. Then, a.e. (w.r.t. μ_0)*

$$C(\phi) = \chi_\alpha (P^2 + m^2)^{-1/4} \theta(P)(\phi\chi_\alpha)(P^2 + m^2)^{-1/4} \chi_\alpha$$

is trace class and for any λ:

$$\int \exp(\lambda \|C(\phi)\|_1) d\mu_0 < \infty .$$

Proof. Write $C(\phi) = BDA(\phi)\chi_\alpha$ where $A(\phi) = (P^2 + m^2)^{-1/2} \phi\chi_\alpha (P^2 + m^2)^{-1/4}$, $B = \chi_\alpha (P^2 + m^2)^{-3/4}$ and $D = (P^2 + m^2)\theta(P)$. Then D is bounded, B is Hilbert-Schmidt (by Lemma 2.1) and $A(\phi)$ is a.e. Hilbert-Schmidt by Theorem 3.4. It follows that $C(\phi)$ is a.e. trace class and

$$\|C(\phi)\|_1 \leqq (\mathrm{const}) \|A(\phi)\|_2 .$$

Since, by the method of Theorem 3.4 (using Theorem 3.1 in place of Theorem 3.2), $\exp(\lambda_0 \|A(\phi)\|_2^2) \in L^1(d\mu_0)$ for λ_0 sufficiently small the result follows by using $x \leqq \delta x^2 + 1/4\delta^{-1}$. $\square$

IV. Determinant Inequalities

Theorem 4.1. *Let A, B, C be trace class operators. Then*

$$|\det(1 + A + B + C) e^{-\mathrm{Tr}(A+B+C)}|^2 \leqq e^{a\|AB\|_1} e^{b\|C\|_1} e^{\|B^*B\|_1} J(A) , \tag{4.1}$$

where

$$J(A) = \det(1 + \mathrm{O}_{A+}) e^{-\mathrm{Tr}(\mathrm{O}_{A-}) - 1/2\, \mathrm{Tr}(\mathrm{O}_{A-})} e^{-2\mathrm{Re}(\mathrm{Tr}(A))} \tag{4.2}$$

[$\mathrm{O}_A = A + A^* + A^*A$; O_{A+} is its positive part and O_{A-} its negative part.] $a = 2e^{5/4}$, $b = 2 + 2e^{5/4}$.

Remark 1. In our eventual application of this inequality, B will be the part of $K(\bigcup A_\alpha)$ which "links" distinct squares, C will be the low momentum piece of the "diagonal" part of K and A its high momentum piece.

2. This is an improvement of some bounds in [20].

We prove a series of preliminary lemmas:

Lemma 4.2. *If $0 \leqq R \leqq 1$ is in $\mathscr{C}_n$, then*

$$\|\textstyle\bigwedge^m (1 - R)^{-1} \det_n(1 - R)\| \leqq \exp m(\sum_{k=1}^{n-1} 1/k) . \tag{4.3}$$

Remark. If $n = 1$, $\sum_{k=1}^{0}$ is by convention 0.

Proof. This is Lemma 3.4 of [20] but the proof is so short we repeat it. Let $\alpha_1 \geqq \alpha_2 \geqq \ldots \geqq 0$ be the eigenvalues of R. Then the left-side of (4.3) is:

$$\prod_{i \geqq 1} (1 - \alpha_i) e^{\sum_{k=1}^{n-1} \alpha_i^k / k} \prod_{i \leqq m} (1 - \alpha_i)^{-1} \leqq \prod_{i \leqq m} \exp(\textstyle\sum_{k=1}^{n-1} \alpha_i^k / k) \leqq e^{m(\sum_{k=1}^{n-1} 1/k)}$$

where we have used $(1 - x)\exp(\sum_{k=1}^{n-1} x^k / k) \leqq 1$ for $0 < x < 1$. $\square$

Lemma 4.3. *If $R \geqq -1$ is trace class, then for any n*

$$\|\bigwedge^m (R+1)^{-1} \det(1+R)\| \leqq \det(1+R_+) e^{m\sum_1^n 1/k} \exp[-\sum_{k=1}^n (1/k)\operatorname{Tr}(R_-^k)] .$$

Proof. If $\min(m, \operatorname{Rank}(R_-)) = k$, then

$$\begin{aligned}&\|\bigwedge^m (1+R)^{-1} \det(1+R)\| \\ &= \|\bigwedge^k (1-R_-)^{-1} \det(1-R_-)\| \, \|\bigwedge^{m-k}(1+R_+)^{-1} \det(1+R_+)\| .\end{aligned} \tag{4.4}$$

We now use Lemma 4.2 on the second factor on the right of (4.3) and the trivial bound $\|\bigwedge^k (1+R_+)^{-1}\| \leqq 1$ on the first factor. □

Lemma 4.4. *For any operator B which is Hilbert-Schmidt*

$$\|\det_2(1+B) \bigwedge^m (1+B)^{-1}\| \leqq e^{1/2m} e^{1/2\|B^*B\|_1} . \tag{4.5}$$

Proof. Using standard limiting arguments (see the appendix of [20]) we can suppose B is trace class. The bound $\det(1+R_+) \leqq e^{\operatorname{Tr}(R_+)}$ and Lemma 4.3 with $n=1$ imply that for $R \geqq -1$

$$\|\bigwedge^m (1+R)^{-1} \det(1+R)\| \leqq e^m e^{\operatorname{Tr}(R)} .$$

Thus

$$\begin{aligned}\|\det_2(1+B) \bigwedge^m (1+B)^{-1}\|^2 &= e^{-2\operatorname{Re}(\operatorname{Tr}(B))} \|\det(1+\mathrm{O}_B) \bigwedge^m (1+\mathrm{O}_B)^{-1}\| \\ &\leqq e^m e^{\operatorname{Tr}(\mathrm{O}_B)} e^{-\operatorname{Tr}(B+B^*)} = e^m e^{\|B^*B\|_1} . \quad \Box\end{aligned}$$

Lemma 4.5. $\|\det_2(1+A) \bigwedge^m (1+A)^{-1}\|^2 \leqq e^{3m/2} J(A)$ *with $J(A)$ given by* (4.2).

Proof. This is just Lemma 4.3 with $n=2$ applied to $\|\det(1+A) \bigwedge^m (1+A)^{-1}\|^2 = \|\det(1+\mathrm{O}_A) \bigwedge^m (1+\mathrm{O}_A)^{-1}\|$. □

Proof of Theorem 4.1.

$$\det_2(1+A+B+C) = \det_2(1+A)\det_2(1+B)\det(1+D) e^{-\operatorname{Tr}(C)} , \tag{4.6}$$

where $D = (1+B)^{-1}(1+A)^{-1}[-AB+C]$ since

$$(1+A+B+C) = (1+A)(1+B)(1+D) .$$

Now, by the expansion $\det(1+D) = \sum_{k=0}^\infty \operatorname{Tr}(\bigwedge^k(D))$, and the bound

$$\begin{aligned}|\operatorname{Tr}(\bigwedge^k(D))| &\leqq \|\bigwedge^k (1+A)^{-1}\| \, \|\bigwedge^k (1+B)^{-1}\| \, |\operatorname{Tr}(\bigwedge^k (C-AB))| \\ &\leqq \|\bigwedge^k (1+A)^{-1}\| \, \|\bigwedge^k (1+B)^{-1}\| \, \|C-AB\|_1^k / k!\end{aligned}$$

(4.6) implies that

$$\begin{aligned}&|\det_2(1+A+B+C)| \leqq e^{\|C\|_1} \\ &\cdot \sum_{k=0}^\infty [\|\bigwedge^k (1+A)^{-1} \det_2(1+A)\| \, \|\bigwedge^k (1+B)^{-1} \det_2(1+B)\|] \\ &\cdot (\|C\|_1 + \|AB\|_1)^k / k!\end{aligned}$$

which using Lemmas 4.5 and 4.4 becomes

$$\begin{aligned}&\leqq e^{\|C\|_1} \sum_{k=0}^\infty e^{1/2\|B^*B\|_1} J(A)^{1/2} (e^{5/4})^k (\|C\|_1 + \|AB\|_1)^k / k! \\ &= e^{1/2b\|C\|_1} e^{1/2a\|AB\|_1} e^{1/2\|B^*B\|} F(A)^{1/2} . \quad \Box\end{aligned}$$

V. Upper Bound

In this section, we prove the main technical result of this paper by combining the estimates of §§ 2–4 and the renormalization cancellation mechanism of [19].

Theorem 5.1. *Let* $K_\alpha = (P^2+m^2)^{-1/4} W(P)\phi(x)\chi_\alpha(x)(P^2+m^2)^{-1/4}$ *where* $W(P)$ *is given by* (1.4b) *and* χ_α *is the characteristic function of the square of side* l *and center* $l\alpha$. *Then for any* p, *there is a constant* C *so that for all* $X \subset \mathbb{Z}^2$:

$$\int |\det_{\mathrm{ren}}(1+\textstyle\sum_{\alpha\in X} K_\alpha)|^p d\mu_0 \leqq C^{|X|}\,. \tag{5.1}$$

Remarks. 1. In the preliminary steps below (before Lemma 5.2), we should put in an upper momentum cutoff which we take away after making the renormalization cancellations explicit. Having remarked this we proceed formally without doing this.

2. There is an explicit formula for $\det_{\mathrm{ren}}$ in [19] obtained by making the renormalization cancellations explicit. However, we prefer to think of $\det_{\mathrm{ren}}$ as

$$\det_{\mathrm{ren}}(1+\textstyle\sum_\alpha K_\alpha) = \det_2(1+\sum_\alpha K_\alpha)\exp(\delta E_2(X))\exp(-\delta\mu^2 \sum_\alpha Q_\alpha)\,, \tag{5.2}$$

where $Q_\alpha = \int \chi_\alpha : \phi^2(x) : d^2x$, $\delta\mu^2$ is the (infinite) renormalization counterterm corresponding to subtraction at zero momentum and δE_2 is the 2$^{\text{nd}}$ order energy renormalization. The advantage of (5.2) over the formula for $\det_{\mathrm{ren}}$ which makes sense without ultraviolet cutoff is that the counterterm factors easily into contributions for each basic box. Essentially what we do below is first bound the right side of (5.2) into a product of functions of the fields in each box. We then do the renormalization cancellations in each box separately.

3. While we describe the result with mass counterterm corresponding to subtraction at zero momentum, we can make an arbitrary finite mass renormalization (of either sign) with simple modifications in our proof.

Proof of Theorem 5.1 (Beginning). Let θ be the characteristic function of $\{p \,|\, |p| \leqq \zeta\}$ where ζ is a constant to be determined later in the proof. Let

$$A_\alpha = \chi_\alpha(1-\theta(P))K_\alpha\chi_\alpha$$

$$C_\alpha = \chi_\alpha\theta(P)K_\alpha\chi_\alpha$$

$$B_{\beta\alpha\gamma} = \chi_\beta K_\alpha \chi_\gamma$$

$$B_\alpha = \sum_{\substack{\beta \neq \alpha \text{ or } \gamma \neq \alpha \\ \beta,\gamma\in\mathbb{Z}^2}} B_{\beta\alpha\gamma}\,.$$

Thus

$$K_\alpha = A_\alpha + B_\alpha + C_\alpha \quad \text{and} \quad K = A+B+C\,,$$

where $A = \sum_{\alpha\in X} A_\alpha$, etc. Thus, by Theorem 4.1

$$|\det_2(1+K)| \leqq e^{1/2a\|AB\|_1} e^{1/2b\|C\|_1} e^{1/2\|B^*B\|_1} J(A)^{1/2} \tag{5.3}$$

with $J(A)$ given by (4.2). Now $A^*A = \sum_\alpha A_\alpha^* A_\alpha$ since $A_\beta^* A_\alpha = 0$ for $\beta \neq \alpha$. Since the $A_\alpha^* A_\alpha$ act on orthogonal subspaces $\chi_\alpha L^2$, all parts of $J(A)$ factor, i.e.

$$J(A) = \prod_{\alpha\in X} J(A_\alpha)\,.$$

Thus by (5.2) and (5.3) we have

$$\det_{\mathrm{ren}}(1+K) \leqq e^{1/2a\|AB\|_1} e^{1/2b\|C\|_1} e^{1/2\|B^*B\|_1} \cdot \left[\prod_{\alpha\in X} J(A_\alpha)^{1/2} e^{\delta E_2(\alpha)} e^{-\delta\mu^2 Q_\alpha}\right] e^{\Delta E_2}. \tag{5.4}$$

In (5.4), ΔE_2 is the difference of the "global" second order counterterms

$$1/2 \int d\mu_0 \operatorname{Tr}((\textstyle\sum_{\alpha\in X} K_\alpha)^2)$$

and the local second order counterterms

$$1/2 \textstyle\sum_{\alpha\in X} \int d\mu_0 \operatorname{Tr}(K_\alpha^2).$$

Thus

$$\Delta E_2 = 1/2 \sum_{\substack{\alpha \neq \beta \\ \alpha,\beta\in X}} \int d\mu_0 \operatorname{Tr}(K_\alpha K_\beta).$$

By computing with Feynman diagrams, or by using the methods we develope to control cross terms below [see the proof of (5.13)], one sees that for α fixed

$$\sum_{\substack{\beta\in\mathbb{Z}^2 \\ \beta\neq\alpha}} \left|\int d\mu_0 \operatorname{Tr}(K_\alpha K_\beta)\right| < \infty$$

so that $\Delta E_2 \leqq \mathrm{const}|X|$.

Now we have:

Lemma 5.2. *For any constant s there is a ζ so that*

$$(J(A_\alpha))^{1/2} e^{-\delta\mu^2 Q_\alpha} e^{\delta E_2(\alpha)} e^{sQ_\alpha} \in \bigcap_{p<\infty} L^p(Q, d\mu_0).$$

Proof. When $s=0$ and A_α replaced by K_α this is essentially the main result of [19]. In [20] (see also [14]) we explained how putting in a lower momentum cutoff allows the possibility of $s>0$ (actually, for this part of the argument in taking $s>0$, the basic idea is already in [19]). We will thus be sketchy in describing the modifications in proof needed to accommodate the change from K_α to A_α. By simple manipulations one first rewrites:

$$J(A_\alpha)^{1/2} e^{-\delta\mu^2 Q_\alpha} e^{\delta E_2(\alpha)} = u_{\zeta,\alpha} \exp[1/2 \operatorname{Tr}(:A_\alpha^* A_\alpha:) - \delta\mu^2 Q_\alpha + c], \tag{5.5}$$

$$u_{\zeta,\alpha}^2 = \det_3(1+\mathrm{O}_{A_\alpha^\dagger}) \exp(-2 \operatorname{Re} \operatorname{Tr}(A_\alpha^2 A_\alpha^*) - 1/2 \operatorname{Tr}(A_\alpha^* A_\alpha)^2 - 1/2 \operatorname{Tr}:(A_\alpha A_\alpha^*)^2:), \tag{5.6}$$

and c is the finite constant $1/2 \int d\mu_0 \operatorname{Tr}(K_\alpha^2 - A_\alpha^2)$.

We first claim that for ζ, the lower momentum cutoff, suitably chosen, we have

$$u_{\zeta,\alpha} e^{sQ_\alpha} \in \bigcap_{p<\infty} L^p. \tag{5.7}$$

The proof can be taken over from [19, 20] if one notes that multiplication (on the left or right!) by χ_α is a contraction on the $\mathscr{C}_p$ spaces so that all estimates on K_α dominate estimates on A_α. As for the second factor in (5.5) we note that since (by this last remark)

$$\operatorname{Tr}(A_\alpha^* A_\alpha) \leqq \operatorname{Tr}(K_\alpha^* K_\alpha)$$

we have

$$1/2 \operatorname{Tr}(:A_\alpha^* A_\alpha:) - \delta\mu^2 Q_\alpha \leqq 1/2 \operatorname{Tr}(:K_\alpha^* K_\alpha:) - \delta\mu^2 Q_\alpha + \Delta W, \tag{5.8}$$

where ΔW is the Wick constant

$$\Delta W = \int d\mu_0 (\operatorname{Tr} A_\alpha^* A_\alpha - \operatorname{Tr} K_\alpha^* K_\alpha) < \infty. \tag{5.9}$$

ΔW is finite by an explicit Feynman diagram computation and the other term on the right side of (5.8) has an exponential in $\bigcap_{p<\infty} L^p$ as in [19]. [Alternatively, we can control $1/2\,\mathrm{Tr}(:A_\alpha^* A_\alpha:) - \delta\mu^2 Q_\alpha$ by following the appendix to [19] replacing the Feynman diagrams from there by ones with lower momentum cutoffs and "$\hat{\chi}_\alpha$"-insertions.] $\square$

Proof of Theorem 5.1 (Continuation). On account of (5.4) and Hölder's inequality the theorem is clearly proven if we show:

$$\int e^{r\|AB\|_1} e^{-s_1 \Sigma Q_\alpha} d\mu_0 \leqq C_1^{|X|}, \tag{5.10a}$$

$$\int e^{r\|B^*B\|_1} e^{-s_2 \Sigma Q_\alpha} d\mu_0 \leqq C_2^{|X|}, \tag{5.10b}$$

$$\int e^{r\|C\|_1} d\mu_0 \leqq C_3^{|X|}, \tag{5.10c}$$

$$\int \left[\prod_{\alpha\in X} J(A_\alpha)^{1/2} e^{-\delta\mu^2 Q_\alpha} e^{+sQ_\alpha}\right]^r d\mu_0 \leqq C_4^{|X|}, \tag{5.10d}$$

where (5.10a) and (5.10b) hold in the sense that for any r we can find some s_1, s_2 so that (5.10a) and (5.10b) hold (with s_2 *independent* of ζ) and (5.10d) in the sense that given s, r we can choose ζ so that it holds.

Now (5.10d) holds by Lemma 5.2 and the checkerboard estimates (Theorem 1.1) and (5.10c) follows from $\|C\|_1 \leqq \sum_{\alpha\in X} \|C_\alpha\|_1$, the checkerboard estimate and Theorem 3.5. We thus turn to (5.10a) and (5.10b).

Define

$$F_\alpha(\phi) = \|(P^2+m^2)^{-1/64} \chi_\alpha \phi(x) (P^2+m^2)^{-1/4}\|_4^2. \tag{5.11}$$

We first claim that for all β, α, γ with either β or $\gamma \neq \alpha$ and some fixed $\varepsilon > 0$:

$$\|B_{\beta\alpha\gamma}\|_{2-\varepsilon} \leqq C_0 e^{-D_0(|\alpha-\beta|+|\alpha-\gamma|)} F_\alpha(\phi)^{1/2}. \tag{5.12}$$

By symmetry it clearly suffices to prove that for all $\beta \neq \alpha$, and all γ:

$$\|B_{\beta\alpha\gamma}\|_{2-\varepsilon} \leqq C_1 e^{-2D_0|\alpha-\beta|} F_\alpha(\phi)^{1/2}. \tag{5.13}$$

If β and α are not neighboring squares, by Theorem 2.2

$$\begin{aligned}\|B_{\beta\alpha\gamma}\|_{2-\varepsilon} &\leqq \|B_{\beta\alpha\gamma}\|_1 \leqq C_2 e^{-D|\alpha-\beta|} \|\chi_\beta (P^2+m^2)^{-k}\|_{4/3} \\ &\quad \|(P^2+m^2)^{-k} \chi_\alpha \phi (P^2+m^2)^{-1/4}\|_4.\end{aligned} \tag{5.14}$$

On the other hand, if α and β are neighboring squares we have by Theorem 2.4(b):

$$\|B_{\beta\alpha\gamma}\|_{2-\varepsilon} \leqq \|\chi_\alpha (P^2+m^2)^{-1/64}\|_\delta \, \|T\|_4 F_\alpha(\phi)^{1/2}, \tag{5.15}$$

where $\|T\|_4 = \|(P^2+m^2)^{1/64} \chi_\alpha (P^2+m^2)^{-1/4} \chi_\beta (P^2+m^2)^{1/64}\|_4 < \infty$ and δ is determined by $\delta^{-1} + 1/2 = 1/2 - \varepsilon$. If we now choose ε so that $\delta > 64$, then $\|\chi_\alpha (P^2+m^2)^{-1/64}\|_\delta < \infty$ by Theorem 2.1 so that (5.13) follows from (5.14) and (5.15). Now, since $\|\cdot\|_2 \leqq \|\cdot\|_{2-\varepsilon}$

$$\begin{aligned}\|B^*B\|_1 &\leqq \sum_{\substack{\alpha,\nu\in X\\ \beta,\gamma,\lambda\in\mathbb{Z}^2}} \|B_{\beta\nu\lambda}^* B_{\beta\alpha\gamma}\|_1 \\ &\leqq C_0^2 \sum_{\substack{\alpha,\nu\in X\\ \beta,\gamma,\lambda\in\mathbb{Z}^2}} e^{-D_0(|\alpha-\beta|+|\alpha-\gamma|+|\nu-\lambda|+|\nu-\beta|)} F_\alpha(\phi)^{1/2} F_\nu(\phi)^{1/2} \\ &\leqq C_3 \sum_{\alpha\in X} F_\alpha(\phi).\end{aligned} \tag{5.16}$$

In the last step, we do λ and γ sums trivial and use Young's inequality on $l^p(\mathbb{Z}^2)$

to do the β sum (explicitly, if $f(\gamma)=e^{-D_0|\gamma|}$, then

$$\sum F_\alpha(\phi)^{1/2}F_\nu(\phi)^{1/2}f(\alpha-\beta)f(\beta-\nu)\leqq\sum(f*f)(\alpha-\nu)F_\alpha(\phi)^{1/2}F_\nu(\phi)^{1/2}$$
$$\leqq\|F_\alpha\|_2\|F_\gamma\|_2\|f*f\|_1=\|F_\alpha\|_2\|F_\gamma\|_2\|f\|_1\|f_1\|).$$

By (5.16), Theorem 3.4 and the checkerboard estimate, (5.10b) follows.

Now let

$$G_\alpha(\phi)=\|(P^2+m^2)^{-1/4}\phi\chi_\alpha(P^2+m^2)^{-1/4}\|_{2+\varepsilon'}^2$$

where ε' is chosen with $(2+\varepsilon')^{-1}+(2-\varepsilon)^{-1}=1$ [ε given in (5.12)]. Then clearly

$$\begin{aligned}\|A_\alpha\|_{2+\varepsilon}&=\|\chi_\alpha(1-\theta(P))W(P)(P^2+m^2)^{-1/4}\phi\chi_\alpha(P^2+m^2)^{-1/4}\chi_\alpha\|_{2+\varepsilon}\\&\leqq G_\alpha(\phi)^{1/2}.\end{aligned}\tag{5.17}$$

By (5.12), (5.17), and $\|ST\|_1\leqq\|S\|_{2+\varepsilon'}\|T\|_{2-\varepsilon'}$:

$$\begin{aligned}\|AB\|_1&\leqq\sum_{\substack{\alpha,\nu\in X\\ \beta\in\mathbb{Z}^2}}\|A_\alpha B_{\alpha\nu\beta}\|_1\\&\leqq C_0\sum_{\substack{\alpha,\nu\in X\\ \beta\in\mathbb{Z}^2}}G_\alpha(\phi)^{1/2}F_\nu(\phi)^{1/2}e^{-D_0(|\alpha-\nu|+|\nu-\beta|)}\\&\leqq 2C_4\Big[\sum_{\alpha\in X}G_\alpha(\phi)\Big]^{1/2}\Big[\sum_{\alpha\in X}F_\alpha(\phi)\Big]^{1/2}\\&\leqq C_4\sum_{\alpha\in X}[G_\alpha(\phi)+F_\alpha(\phi)].\end{aligned}\tag{5.18}$$

Again using Theorem 3.4 and checkerboard estimates, (5.18) leads to (5.10a). □

VI. Bounds on Schwinger Functions; Glimm and Schrader Bounds

As in [19], we define

$$\begin{aligned}&(ZS)_X(h_1,\ldots,h_n;f_1,\ldots,f_k;g_1,\ldots,g_k)\\&\quad=\int\prod_{r=1}^n\phi(h_r)D_k(f_i,g_j;\phi)\,\det_{\mathrm{ren}}(1+K)d\mu_0(\phi),\end{aligned}\tag{6.1}$$

where

$$\begin{aligned}D_k(f_i,g_j;\phi)=&((P^2+m^2)^{-1/4}f_1\wedge\ldots\wedge(P^2+m^2)^{-1/4}f_k,[\textstyle\bigwedge^k(1+K)^{-1}\\&\textstyle\bigwedge^k((P^2+m^2)^{1/2}S_F)](P^2+m^2)^{-1/4}g_1\wedge\ldots\wedge(P^2+m^2)^{-1/4}g_k).\end{aligned}\tag{6.2}$$

Theorem 6.1. *Let each h_i be localized in some square Λ_α with exactly n_α localized in square α. Then for suitable constants C_1, C_2:*

$$\begin{aligned}&|(ZS)_X(h_1,\ldots,h_n;f_1,\ldots,f_k;g_1,\ldots,g_k)|\\&\quad\leqq C_1^{|X|}C_2^{n+k}\prod_\alpha(n_\alpha!)^{1/2}\prod_{r=1}^n\|h_r\|_{-1}\prod_{i=1}^k\|f_i\|_{-1/2}\|g_i\|_{-1/2}.\end{aligned}\tag{6.3}$$

Remark. For fixed X and $\prod_\alpha(n_\alpha!)^{1/2}$ replaced by $(n!)^{1/2}$, this result appears in [19]. McBryan [15] has also obtained (6.3).

Proof. We use the method of Fröhlich [2, 3]; namely we obtain bounds on the Schwinger generating function and use Cauchy estimates to bound Schwinger functions. Let $a_i=(P^2+m^2)^{-1/4}f_i$; $b_j=(P^2+m^2)^{1/4}S_Fg_j$. By homogeneity, we can suppose that $\|a_i\|=\|b_i\|=1$ in $L^2(\mathbb{R}^2;\mathbb{C}^2)$. Let C_i be the rank one operator $C_iu=a_i(b_i,u)$. Then

$$\begin{aligned}&(ZS_X)(h;f;g)=(\partial^{n+k}/\partial\mu_1\ldots\partial\mu_n\partial\lambda_1\ldots\partial\lambda_k)\int\exp\Big(\sum_{i=1}^n\mu_i\phi(h_i)\Big)\\&\det_{\mathrm{ren}}\Big(1+K+\sum_{j=1}^k\lambda_jC_j\Big)d\mu_0|_{\mu=\lambda=0},\end{aligned}\tag{6.4}$$

where $\det_{\text{ren}}(1+K(\Lambda)+C)$ means $\det(1+K+C)\exp$ (counterterms) with the same counterterms as before. Call the function whose derivative occurs on the right of (6.4), $G(\mu_1,\ldots,\mu_n;\lambda_1,\ldots,\lambda_k)$. Then it is easy to see that G is an entire function on $\mathbb{C}^{n+k}$. By using our determinant inequalities (including $\sum_{j=1}^{k}\lambda_j C_j$ with the C term), checkerboard estimates and the fact that

$$\int |\exp(\mu\phi(h))|^r d\mu_0 \leqq \exp(c|\mu|^2\|h\|_{-1}^2)$$

we find that if $\|h_i\|_{-1}=1$; $\|a_i\|=\|b_i\|=1$:

$$|G(\mu_i,\lambda_j)|\leqq C_1^{|X|}e^{a(\Sigma|\lambda_i|+1)}\exp\left[a\textstyle\sum_{\alpha\in\mathbb{Z}^2}\left(\sum_{i\in S_\alpha}|\mu_i|\right)^2\right],$$

where S_α is the set of i with $\operatorname{supp} h_i\subset\Lambda_\alpha$. (Thus, e.g. $\#(S_\alpha)=n_\alpha$). Thus, by Cauchy estimates

$$|\partial^{n+k}G/\partial\mu_1\ldots\partial\lambda_k|\leqq C_3^{n+k}C_1^{|X|}\textstyle\prod_{\alpha\in\mathbb{Z}^2}R_\alpha^{-n_\alpha}\exp[a(n_\alpha R_\alpha)^2]$$

Taking $R_\alpha=n_\alpha^{-1/2}$, we get

$$|(ZS_X)(h,f,g)|\leqq C_1^{|X|}C_2^{n+k}\textstyle\prod_\alpha (n_\alpha!)^{1/2}.$$

Since $\|a_i\|=\|f_i\|_{-1/2}$; $\|b_i\|=\|g_i\|_{-1/2}$, homogeneity yields (6.3). $\square$

We want to conclude this paper with a brief sketch of how one can obtain the Glimm and Schrader bounds from Theorem 6.1. We only provide a sketch because we hope to return in detail to the general connection between the Hamiltonian picture and the Matthews-Salam picture:

(1) Given $f_1,\ldots,f_k$, $g_1,\ldots,g_q$, $h_1,\ldots,h_n$ with supports temporally ordered in some way (i.e. if the supports are $\{S_i\}_{i=1}^{k+q+n}$ then for some permutation π on $k+q+n$ letters and some $t_0=l/2<t_1<\ldots<t_{k+q+n}$, $S_{\pi(i)}\subset\{(x,t)|t_{i-1}<t<t_i\}$, there is a natural vector $\eta_+(f,g,h)$ associated in the free field Hilbert space. η_+ is defined via the theory of Osterwalder-Schrader fields [17] (or directly with the analytic continuation of the vector-valued Wightman distributions; see Jost [11] or Simon [21]).

(2) As k, q, n run through all of $\mathbb{Z}_+$ and f, g, h run through all r-tupels ($r=k$, q, n resp.) with support in $\{(x,t)|-l/2<x<l/2;\, l/2<t<3l/2\}$,the $\eta_+(f,g,h)$ run through a dense set. This assertion is a simple extension of the classic Reeh-Schlieder argument.

(3) By the Feynman-Kac formula of Osterwalder-Schrader [17], if $L=2nl$; $t=ml(n,m\in\mathbb{Z}_+)$, then

$$(\eta_+(\underline{f},\underline{g},\underline{h}),e^{-tH_L(\kappa)}\eta_+(\underline{f},\underline{g},\underline{h}))=(ZS)_{X,\kappa}(\underline{h},\tilde{\underline{h}}_t;\underline{f},\tilde{\underline{f}}_t;\underline{g},\tilde{\underline{g}}_t)c_{t,L}(\kappa) \tag{6.5}$$

where $\tilde{\underline{h}}_t,\tilde{\underline{f}}_t,\tilde{\underline{g}}_t$ arise from $\underline{h},\underline{f},\underline{g}$ by reflection in the line $t=0$ and then translation by t units, where X is $\{(a,b)\in\mathbb{Z}^2|-n\leqq a\leqq n; 0\leqq b\leqq m\}$, where κ represents a suitable ultraviolet cutoff and where $c_{t,L}(\kappa)$ is a term representing the difference between the Hamiltonian and Euclidean second order "energy" counterterms.

(4) By explicit formulas for $c_{t,L}(\kappa)$ (see e.g. [15], § 3) $c_{t,L}(\kappa)\leqq 1$.

(5) By (6.5) and Theorem 6.1 $(\eta_+(\underline{f},\underline{g},\underline{h}),e^{-tH_L(\kappa)}\eta_+(\underline{f},\underline{g},\underline{h}))\leqq c(\underline{f},\underline{g},\underline{h})\exp(atL)$ where $c(\underline{f},\underline{g},\underline{h})$ is a constant depending on $\underline{f},\underline{g}$, and $\underline{h}$, but independent of κ. It follows that independently of κ:

$$H_L(\kappa)\geqq -aL. \tag{6.6}$$

(6.6) contains the Glimm and Schrader bounds.

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