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The Power Counting Theorem for Feynman Integrals with Massless Propagators

J. H. Lowenstein

Physics Department, New York University, New York, N. Y., USA

W. Zimmermann

Max-Planck-Institut für Physik und Astrophysik, München, Federal Republic of Germany

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Abstract. Dyson's power counting theorem is extended to the case where some of the mass parameters vanish. Weinberg's ultraviolet convergence conditions are supplemented by infrared convergence conditions which combined are sufficient for the convergence of Feynman integrals.

1. Introduction

In the theory of renormalization Dyson's power counting theorem plays a decisive part [1–3]. The contribution of a proper Feynman diagram to a Green's function has the form

$$J=\int dkR(k,p)$$
$$R=\frac{P}{\prod_{j=1}^{n}(l_j^2-m_j^2+i\varepsilon(\vec{l}_j^2+m_j^2))^{n_j}} \tag{1.1}$$

where

$$\begin{aligned} &k=(k_1\dots k_m)\,, && p=(p_1\dots p_N)\,,\\ &k_j=(k_{j0}k_{j1}k_{j2}k_{j3}), && p_j=(p_{j0}p_{j1}p_{j2}p_{j3})\,,\\ &dk=dk_1\dots dk_m\,, && dk_j\dots dk_{j0}dk_{j1}dk_{j2}dk_{j3}\,,\\ &m_j\geqq 0, \quad n_j>0\,. \end{aligned} \tag{1.2}$$

k_j and p_j are Minkowski vectors with the metric $(+1,-1,-1,-1)$. The vectors l_j are linear combinations

$$l_j=K_j(k)+P_j(p) \tag{1.3}$$

of the vectors $k_1,\dots,k_m$ and $p_1,\dots,p_N$ with $K_j\neq 0$. P is a polynomial in the components of k and p. The denominator of R is the common denominator of the unrenormalized integrand and the subtraction terms.

If all masses are non-zero Weinberg's version of the power counting theorem can be used to prove that the integral (1.1) is absolutely convergent provided the renormalized integrand R has been constructed according to Bogoliubov's subtraction rules [3, 4]. It can further be shown that the limit $\varepsilon\to+0$ exists as a covariant tempered distribution.

So far the power counting theorem has only been stated for non-vanishing masses. In the present paper Weinberg's ultraviolet convergence conditions are supplemented by infrared convergence conditions which will be shown to be sufficient for the convergence of integrals (1.1). The limit $\varepsilon \to +0$ [5] as well as the application to field theoretic models [6–9] are discussed in separate papers.

Our results are consistent with recent work by Bergère and Lam, as well as by Trute and Pohlmeyer, on the asymptotic behavior of parametrized Feynman integrals for small mass values [10–12].

Some general definitions are given in Section 2. Section 3 contains the statement and proof of the power counting theorem. The concept of reduced integrals, which is useful for some application of the theorem, is introduced in Section 4.

2. General Definitions

We consider integrals of the form (1.1). L denotes the space of the linear forms

$$l = \sum_{j=1}^{n} a_j k_j + \sum_{j=1}^{N} b_j p_j \tag{2.1}$$

which will be interpreted as inhomogeneous linear forms in the integration variables $k_1, \ldots, k_m$. Elements of L are called linerarly (in)dependent if their homogeneous parts (in k) are linearly (in)dependent. A set of elements in L is called a basis of L if their homogeneous parts form a basis for the space of the homogeneous forms in k.

We observe that always for an absolutely convergent integral (1.1) a basis

$$l_{j_1} \ldots, l_{j_m} \tag{2.2}$$

exists consisting of linear forms which occur in the denominators of (1.1). Otherwise there would be at most $m' < m$ linearly independent forms

$$l_{j_1}, \ldots, l_{j_{m'}} \tag{2.3}$$

with the remaining l_j being linear combinations of vectors (2.3) and p_j. Extending (2.3) to a basis

$$l_{j_1}, \ldots, l_{j_{m'}}, w_1, \ldots, w_c \qquad m' + c = m$$

of L with Jacobian one (relative to $k_1, \ldots, k_m$) we find

$$J = \int dl_{j_1} \ldots dl_{j_{m'}} dw_1 \ldots dw_c R$$

with the divergent subintegral

$$\int dw_1 \ldots dw_c R = \frac{1}{\prod (l_j^2 - m_j^2 + \ldots)^{n_j}} \int dw_1 \ldots dw_c P\,.$$

Therefore, a basis (2.2) of L must always exist if the integral (1.1) is to be absolutely convergent.

For the formulation of the power counting theorem we will need certain subintegrals which we set up as follows. Let

$$u_1 = l_{i_1}, \ldots, u_a = l_{i_a}, v_1 = l_{j_1}, \ldots, v_b = l_{j_b} \tag{2.4}$$

be a basis of L with Jacobian one (relative to $k_1, \ldots, k_m$). Using (2.4) as new integration variables for (1.1) we obtain

$$\begin{aligned} &J = \int du dv R\,, \\ &u = (u_1 \ldots u_a), \qquad v = (v_1 \ldots v_b)\,, \\ &du = du_1 \ldots du_a, \qquad dv = dv_1 \ldots dv_b \end{aligned} \tag{2.5}$$

where P and l_j are expressed in terms of u, v and p through

$$k = k(u, v, p).$$

We consider a hyperplane H defined by the condition that the linear forms

$$v_1 = l_{j_1}, \ldots, v_b = l_{j_b}$$

have constant values. The subintegral of (1.1) along H is then given by

$$J(H) = \int du R. \tag{2.6}$$

We distinguish two different definitions for the dimension of a subintegral (2.6).

The upper dimension $\overline{\dim}$ refers to the behavior for large values of the integration variables. The lower dimension $\underline{\dim}$ refers to the behavior for small values of the integration variables. We define

$$\overline{\dim} J(H) = \overline{\deg}_u R + 4a, \tag{2.7}$$

$$\underline{\dim} J(H) = \underline{\deg}_u R + 4a. \tag{2.8}$$

The upper degree $\overline{\deg}_u$ (or lower degree $\underline{\deg}_u$) denotes the leading power of ϱ in the limit $\varrho \to \infty$ (or $\varrho \to 0$) if $u_j = \varrho \hat{u}_j$ is substituted into R. More precisely,

$$\bar{v} = \overline{\deg}_u R, \quad \underline{v} = \underline{\deg}_u R, \tag{2.9}$$

if

$$\lim_{\varrho \to \infty} \frac{R}{\varrho^{\bar{v}}} \neq 0, \infty, \quad \lim_{\varrho \to 0} \frac{R}{\varrho^{\underline{v}}} \neq 0, \infty \tag{2.10}$$

for almost all values of $u_1, \ldots, u_a$ and the remaining parameters $v_1, \ldots, v_b, p_1, \ldots, p_N$.

We quote some rules for the upper and lower degree. Let $N, D, F, F_1, \ldots, F_r$ be complex-valued functions of real four-vectors $u_1, \ldots, u_a, v_1, \ldots, v_b, p_1, \ldots, p_N$ to which the definitions $\overline{\deg}_u$ and $\underline{\deg}_u$ may be applied. Then the following rules hold

$$\overline{\deg}_u F^n = n \overline{\deg}_u F, \tag{2.11}$$

$$\underline{\deg}_u F^n = n \underline{\deg}_u F, \tag{2.12}$$

$$\overline{\deg}_u \frac{N}{D} = \overline{\deg}_u N - \overline{\deg}_u D, \tag{2.13}$$

$$\underline{\deg}_u \frac{N}{D} = \underline{\deg}_u N - \underline{\deg}_u D, \tag{2.14}$$

$$\overline{\deg}_u \prod_{j=1}^{r} F_j = \sum_{j=1}^{r} \overline{\deg}_u F_j, \tag{2.15}$$

$$\underline{\deg}_u \prod_{j=1}^{r} F_j = \sum_{j=1}^{r} \underline{\deg}_u F_j, \tag{2.16}$$

$$\overline{\deg}_u \sum_{j=1}^{r} F_j \leqq \max_j \{\overline{\deg}_u F_j\}, \tag{2.17}$$

$$\underline{\deg}_u \sum_{j=1}^{r} F_j \geqq \min_j \{\underline{\deg}_u F_j\}. \tag{2.18}$$

Let F be a polynomial of $u=(u_1,\ldots,u_a)$, $v=(v_1,\ldots,v_b)$ and $p=(p_1,\ldots,p_N)$ with vectors u_i, v_j, p_r. Then we may write

$$F=\sum_\alpha Q_\alpha M_\alpha, \qquad Q_\alpha \not\equiv 0 \tag{2.19}$$

where M_α are independent monomials in u and Q_α are polynomials in v, p which are not identically zero. The upper and lower degrees of F are given by

$$\overline{\deg}_u F=\max_\alpha\{\deg M_\alpha\}, \tag{2.20}$$

$$\underline{\deg}_u F=\min_\alpha\{\deg M_\alpha\}. \tag{2.21}$$

3. Convergence Theorem

In this section the power counting theorem will be formulated for integrals J of type (1.1) assuming that a basis (2.2) of L can be formed. Weinberg's hypothesis of the power counting theorem may be stated as follows:

Ultraviolet Convergence Condition. *The inequality*

$$\overline{\dim} J(H)=\overline{\deg}_u R+4a<0 \tag{3.1}$$

holds for any basis (2.4) *and for any hyperplane H defined by constant values of* $v_1,\ldots,v_b$.

In particular, the upper dimension of the full integral J should be negative. Weinberg's condition (3.1) is sufficient for the absolute convergence of J provided all masses are different from zero

$$m_j>0, \qquad j=1,\ldots,n.$$

In the general case we propose in addition the following

Infrared Convergence Condition. *The inequality*

$$\underline{\dim} J(H)=\underline{\deg}_u R+4a>0 \tag{3.2}$$

holds for any basis (2.4) *satisfying*

$$m_{i_1}=\ldots=m_{i_a}=0 \tag{3.3}$$

and for any hyperplane H defined by constant values of $v_1,\ldots,v_b$.

The ultraviolet and infrared convergence conditions combined form the hypothesis of the

Power Counting Theorem. *Let J be an integral of the form* (1.1) *for which a basis* (2.2) *of L can be formed. J is absolutely convergent if the ultraviolet convergence condition* (3.1) *and the infrared convergence condition* (3.2–3) *hold.*

Due to the inequality

$$\frac{l_0^2+\vec{l}^2+m^2}{|l_0^2-\vec{l}^2-m^2+i\varepsilon(\vec{l}^2+m^2)|}\leqq\left(1+\frac{4}{\varepsilon^2}\right)^{\frac{1}{2}}$$

the absolute convergence of (1.1) is implied by the absolute convergence of the corresponding Euclidean integral

$$J=\int dkR(k,p)$$

$$R=\frac{P}{\prod_{j=1}^{n}(l_j^2+m_j^2)^{n_j}} \tag{3.4}$$

where now

$$l_j^2=l_{j0}^2+\vec{l}_j^2 .$$

Therefore, we may restrict ourselves to proving the absolute convergence of (3.4) under the conditions (3.1–3).

We begin proving a lemma on the infrared convergence of certain integrals which are homogeneous in the integration variables.

Lemma. *Consider integrals of the form*

$$F=\int_{u_i^2\leq 1} du_1\dots du_a\frac{M}{\prod_j(U_j^2)^{n_j}} \tag{3.5}$$

where the U_j are linear combinations of the Euclidean four-vectors $u_1,\dots,u_a$ and M is a monomial in the components of $u_1,\dots,u_a$. M may be factorized as

$$M=\prod_{i=1}^{a}M_i \tag{3.6}$$

where M_i is a monomial of u_i. For any subset

$$u_{i_1},\dots,u_{i_c} \tag{3.7}$$

of the integration variables we form the integral

$$F_{i_1\dots i_c}=\int_{u_i^2\leq 1} du_{i_1}\dots du_{i_c}\frac{M_{i_1}\dots M_{i_c}}{\prod_j^{i_1\dots i_c}(U_j^2)^{n_j}} \tag{3.8}$$

where the product $\prod_j^{i_1\dots i_c}$ extends over all U_j which are linear combinations of vectors (3.7) *only. The integrals* (3.8) *are called sections of* (3.5).

The statement is that the integral (3.5) *is absolutely convergent if the dimension $d_{i_1\dots i_c}$ of each section* (3.8) *is positive:*

$$\dim F_{i_1\dots i_c}=d_{i_1\dots i_c}>0 . \tag{3.9}$$

This condition includes the dimension of the full integral which we denote by d,

$$d=\dim F=d_{1\dots a}>0 .$$

Proof. We decompose the integral (3.5) into

$$F=\sum_P F_P$$

$$F_P=\int_{u_{i_1}^2\leq\dots\leq u_{i_a}^2\leq 1} du_1\dots du_a\frac{M}{\prod_j(U_j^2)^{n_j}} \tag{3.10}$$

with the sum extending over all permutations

$$P=\begin{pmatrix}1\ldots a\\ i_1\ldots i_a\end{pmatrix}.$$

We will check the convergence of each term F_P. In order to simplify the notation we rename the integration variables and monomials by

$$w_1=u_{i_1},\ \ldots, w_a=u_{i_a},$$
$$N_1=M_{i_1},\ldots,N_a=M_{i_a}.$$

Moreover, we denote the momenta U_j and exponents n_j of the denominators by

$$W_{11},\ldots,W_{21},\ldots,\ldots,W_{a1},\ldots$$
$$n_{11},\ldots,n_{21},\ldots,\ldots,n_{a1},\ldots$$

such that each W_{ij} is a linear combination of $w_1,\ldots,w_i$ with non-vanishing coefficient of $w_{i'}$

$$W_{ij}=\sum_{i'=1}^{i} c_{iji'}w_{i'}, \qquad c_{iji}\neq 0.$$

In this notation F_P may be written in the form

$$F_P=\int_{w_1^2\leqq 1} dw_1\frac{N_1}{\prod_j(W_{1j}^2)^{n_{1j}}}\int_{w_1^2\leqq w_2^2\leqq 1} dw_2\frac{N_2}{\prod_j(W_{2j}^2)^{n_{2j}}}\cdots\int_{w_{a-1}^2\leqq w_a^2\leqq 1} dw_a\frac{N_a}{\prod_j(W_{aj}^2)^{n_{aj}}}. \tag{3.11}$$

According to the hypothesis of the Lemma the dimension d_c of each section

$$F_c=F_{i_1\ldots i_c}=\int_{w_1^2\leqq 1} dw_1\frac{N_1}{\prod_j(W_{1j}^2)^{n_{1j}}}\cdots\int_{w_c^2\leqq 1} dw_c\frac{N_c}{\prod_j(W_{cj}^2)^{n_{cj}}} \tag{3.12}$$

is positive,

$$\dim F_c=d_c>0.$$

d_c satisfies the recursion formula

$$d_c=4+\deg N_c-\deg\prod_j(W_{cj}^2)^{n_{cj}}+d_{c-1}. \tag{3.13}$$

We choose a number δ with

$$d_c>\delta>0 \quad \text{for} \quad c=1,\ldots,a \tag{3.14}$$

and form the integral

$$G=\int_{w_1^2\leqq 1} dw_1\frac{|N_1|}{\prod_j(W_{1j}^2)^{n_{1j}}}\int_{w_1^2\leqq w_2^2\leqq 1} dw_2\frac{|N_2|}{\prod_j(W_{2j}^2)^{n_{2j}}}\cdots\int_{w_{a-1}^2\leqq w_a^2\leqq 1} dw_a\frac{|N_a|}{|w_a|^{d-\delta}\prod_j(W_{aj}^2)^{n_{aj}}} \tag{3.15}$$

Since $|w_a|\leqq 1$ and $d-\delta=d_a-\delta>0$ the integral F_P is majorized by G,

$$|F_P|\leqq G. \tag{3.16}$$

We will prove the convergence of G by recursively estimating the integrals

$$G_c = \int_{w_c^2 \leqq w_{c+1}^2 \leqq 1} dw_{c+1} \frac{|N_c|}{\prod_j (W_{c+1,j}^2)^{n_{c+1,j}}} \cdots \int_{w_{a-1}^2 \leqq w_a^2 \leqq 1} dw_a \frac{|N_a|}{|w_a|^{d-\delta} \prod_j (W_{aj}^2)^{n_{aj}}} . \qquad (3.17)$$

The dimension of

$$G_{a-1} = \int_{w_{a-1}^2 \leqq w_a^2 \leqq 1} dw_a \frac{|N_a|}{|w_a|^{d-\delta} \prod_j (W_{aj}^2)^{n_{aj}}}$$

is [see Eq. (3.13)]

$$\dim G_{a-1} = 4 + \deg N_a - \deg \prod_j (W_{aj}^2)^{n_{aj}} + \delta - d .$$

Now, by a change of integration variable,

$$G_{a-1} = \frac{1}{|u_{a-1}|^{\delta_{a-1} - d}} \int_{1 \leqq w_a^2 \leqq \frac{1}{w_{a-1}^2}} dw_a \frac{|N_a|}{|w_a|^{d-\delta} \prod_j W_{aj}^2} .$$

In the last line the limit $1/|u_{a-1}^2| \to \infty$ could be performed since the dimension of the integral is negative. Hence

$$G_{a-1} \leqq \frac{\gamma_{a-1}}{|u_{a-1}|^{d_{a-1} - \delta}}$$

where γ_{a-1} is a constant.

Repeating this argument recursively we obtain

$$G_c \leqq \frac{\gamma_c}{|w_c|^{d_c - \delta}}$$

by Eq. (3.13–14). Finally

$$G \leqq \gamma \int_{w_1^2 \leqq 1} dw_1 \frac{|N_1|}{|w_1|^{d_1 - \delta} \prod_j (W_{1j}^2)^{n_{1j}}} .$$

The integral on the right hand side exists since its dimension

$$\dim F_1 - d_1 + \delta = \delta > 0$$

is positive. By (3.16) each term of the decomposition (3.10) is absolutely convergent which implies the absolute convergence of (3.5). This completes the proof of the lemma.

We now turn to the

Proof of the Power Counting Theorem. Let S_0 be the set of all momenta l_j with $m_j = 0$. Let S be any subset

$$S \subseteqq S_0$$

T denotes the complementary set

$$T = S_0 \setminus S .$$

We require that with a momentum l_j the set S should contain any l_i which satisfies

$$l_i^2 \equiv l_j^2, \quad m_i = 0.$$

We decompose the integral (3.4) into

$$J = \sum_S A_S \tag{3.18}$$

where

$$A_S = \int\limits_{\substack{l_i^2 \leqq r^2 \text{ in } S \\ l_i^2 \geqq r^2 \text{ in } T}} dk \frac{P}{\prod\limits_j (l_j^2 + m_j^2)^{n_j}}. \tag{3.19}$$

For studying A_S we select momentum vectors

$$u_1 = l_{i_1}, \ldots, u_a = l_{i_a} \tag{3.20}$$

in S which form a basis of S. Then $l_j \in S$ is a linear combination of $u_1, \ldots, p_1, \ldots,$

$$l_i = U_i + Q_i$$

$$U_i = \sum_{j=1}^{a} c_{ij} u_j, \quad Q_i = \sum_{j=1}^{N} d_{ij} p_j.$$

We say that S or the integral A_S has zero external momenta if $Q_i = 0$ for all $l_i \in S$.

For r small enough the term A_S vanishes unless all external momenta vanish. For the proof we observe that $u_\alpha^2 \leqq r^2$ since the u_α occur among the $l_i \in S$. The U_j are of the form

$$U_j = \sum_{\alpha=1}^{a} \eta_\alpha u_\alpha$$

where $|\eta_\alpha| \leqq \eta$ with η being characteristic number of the integral. Now

$$|Q_j| \leqq |l_j| + |U_j| \leqq (1+\eta) r$$

implies

$$r \geqq \frac{|Q_j|}{1+\eta} \quad \text{for any} \quad Q_j,$$

if the domain of integration is not empty. If at least one $Q_j \neq 0$ we may choose r such that

$$0 < r < \frac{|Q_j|}{1+\eta}. \tag{3.21}$$

But then the domain of integration is empty and $A_S = 0$. Hence for r small enough we find

$$J = \sum_S A_S \tag{3.22}$$

where S is restricted to those subsets for which $Q_j = 0$ for any $l_j \in S$.

In each integral A_S we introduce new variables of integration as follows. By adding suitable vectors

$$v_1 = l_{j_1}, \ldots, v_b = l_{j_b}, \quad a + b = m, \tag{3.23}$$

we extend (3.20) to a basis

$$u_1, \ldots, u_a, v_1, \ldots, v_b \tag{3.24}$$

of L with Jacobian one (relative to $k_1, \ldots, k_m$). Then each $l_j \in S$ is a linear combination of $u_1, \ldots, u_a$. The remaining l_j are linear combinations of $u_1, \ldots, u_a, v_1, \ldots, v_b$ $p_1, \ldots, p_N$. We next write the numerator P as a polynomial in u

$$\begin{aligned} &P = \sum C_{S\alpha} M_\alpha\,, \\ &M_\alpha = \prod_{i=1}^{a} M_{i\alpha_i}\,, \qquad \alpha = (\alpha_1, \ldots, \alpha_a)\,, \\ &M_{i\alpha_i} = u_{i0}^{\alpha_{i0}}\, u_{i1}^{\alpha_{i1}} u_{i2}^{\alpha_{i2}} u_{i3}^{\alpha_{i3}}\,, \qquad \alpha_i = (\alpha_{i0}, \ldots, \alpha_{i3})\,. \end{aligned} \tag{3.25}$$

with the coefficients being polynomials in $v_1, \ldots, v_b$, $p_1, \ldots, p_N$. Then

$$\begin{aligned} &A_S = \sum_\alpha A_{S\alpha}\,, \\ &A_{S\alpha} = \int_{l_j^2 \leqq r^2 \text{ in } S} du \frac{M_\alpha}{\prod_S (l_j^2)^{n_j}} \int dv \frac{C_{S\alpha}}{\prod_T (l_j^2)^{n_j} \prod_U (l_j^2 + m_j^2)^{n_j}}\,, \\ &U = (l_1, \ldots, l_n) \setminus S_0\,. \end{aligned} \tag{3.26}$$

We now estimate the v-integrals

$$\int_{l_i^2 \geqq r^2 \text{ in } T} dv \frac{C_{S\alpha}}{\prod_T (l_j^2)^{n_j} \prod_U (l_j^2 + m_j^2)^{n_j}}\,. \tag{3.27}$$

To this end we consider the integral

$$\int dv \frac{P}{\prod_T (l_j^2 + M^2)^{n_j} \prod_U (l_j^2 + m_j^2)^{n_j}} \tag{3.28}$$

with $M > 0$. In (3.28) all masses are different from zero. Because of the ultraviolet convergence conditions the integral (3.28) is absolutely convergent. Each l_j in (3.28) is of the form

$$l_j = V_j(v) + R_j(u, p)\,.$$

Using

$$\frac{l_j^2 + m^2}{V_j^2 + m^2} \leqq 1 + \frac{|l_j - V_j|}{m} + \frac{|l_j - V_j|^2}{m^2}$$

we find

$$\begin{aligned} &\int dv \frac{|P|}{\prod_T (V_j^2 + M^2)^{n_j} \prod_U (V_j^2 + M^2)^{n_j}} \\ &\quad \leqq c \int dv \frac{|P|}{\prod_T (l_j^2 + M^2)^{n_j} \prod_U (l_j^2 + M^2)^{n_j}}\,. \end{aligned}$$

Hence

$$\int dv \frac{P}{\prod_T (V_j^2 + M^2)^{n_j} \prod_U (V_j^2 + m_j^2)^{n_j}}$$

is absolutely convergent. The denominator does not depend on u while the numerator is a polynomial in u. Applying Lemma 3 of Ref. [13] we find that

$$\int dv \frac{C_\alpha(v,p)}{\prod_T (V_j^2+M^2)^{n_j} \prod_U (V_j^2+m_j^2)^{n_j}}$$

is absolutely convergent. With this we can estimate (3.27):

$$\int_{l_j^2 \geqq r^2} dv \frac{|C_{S\alpha}|}{\prod_T (l_j^2)^{n_j} \prod_U (l_j^2+m_j^2)^{n_j}}$$

$$\leqq \left(1+\frac{M^2}{r^2}\right)^\tau \int dv \frac{|C_\alpha|}{\prod_T (l_j^2+M^2)^{n_j} \prod_U (l_j^2+M^2)^{n_j}}$$

$$\left(\text{using} \frac{l_j^2+M^2}{l_j^2} \leqq 1+\frac{M^2}{r^2}\right)$$

$$\leqq d \int dv \frac{|C_\alpha|}{\prod_T (V_j^2+M^2)^{n_j} \prod_U (V_j^2+m_j^2)^{n_j}}$$

using

$$\frac{V_j^2+M_j^2}{l_j^2+M_j^2} \leqq 1+\frac{|l_j-V_j|}{M_j}+\frac{|l_j-V_j|^2}{M_j^2}$$

The integral of the last line only depends on p and the masses. Hence

$$|J| \leqq \sum_{S\alpha} D_{S\alpha} \int_{l_j^2 \leqq r^2} du \frac{|M_\alpha|}{\prod_S (l_j^2)^{n(j)}} . \tag{3.29}$$

The integrals on the right hand side can further be estimated by

$$\int_{l_j^2 \leqq r^2 \text{ in } S} du \frac{|M_\alpha|}{\prod_S (l_j^2)^{n_j}} = r^d \int_{l_j^2 \leqq 1 \text{ in } S} du \frac{|M_\alpha|}{\prod_S (l_j^2)^{n_j}}$$

$$\leqq r^d \int_{u_i^2 \leqq 1} du \frac{|M_\alpha|}{\prod_S (l_j^2)^{n_j}}$$

where d is the dimension of the integral. Thus

$$|J| \leqq \sum_{S\alpha} C_{S\alpha} \int_{u_i^2 \leqq 1} du \frac{|M_\alpha|}{\prod_S (l_j^2)^{n_j}} \tag{3.30}$$

with the sum restricted to those M_α which occur in (3.25) with non-vanishing coefficient. According to the Lemma (page 8 of this paper) we have convergence of the integrals on the right hand side if the dimension of any section is positive. In order to check the dimension d of the full integral

$$I = \int_{u_i^2 \leqq 1} du \frac{M_\alpha}{\prod_S (l_j^2)^{n_j}} \tag{3.31}$$

we form the subintegral

$$J(H) = \int du_1 \dots du_a \frac{P}{\prod (l_j^2+m_j^2)^{n_j}} \tag{3.32}$$

of (3.4) along a hyperplane H defined by constant values of $v_1,\ldots,v_b$. By the hypothesis (3.2–3) the lower dimension δ of $J(H)$ is positive,

$$\delta = 4a + \underline{\deg}_u P - \underline{\deg}_u \prod_j (l_j^2 + m_j^2)^{n_j} > 0\,. \tag{3.33}$$

This implies

$$\begin{aligned} 0 < \delta &= 4a + \underline{\deg}_u P - \deg \textstyle\prod_S (l_j^2)^{n_j} - \deg_u \prod_T (l_j^2)^{n_j} \\ &\leqq 4a + \underline{\deg}_u P - \deg \textstyle\prod_S (l_j^2)^{n_j} \\ &\leqq 4a + \deg M_\alpha - \deg \textstyle\prod_S (l_j^2)^{n_j} = d\,. \end{aligned}$$

Hence the dimension of (3.32) is positive.

We further have to verify that the dimension $d_{i_1\ldots i_c}$ of each section

$$I_{i_1\ldots i_c} = \int du_{i_1}\ldots du_{i_c} \frac{M_{\alpha i_1\ldots i_c}}{\prod_{i_1\ldots i_c}(l_j^2)^{n_j}} \tag{3.34}$$

is positive. Here $M_{\alpha i_1\ldots i_c}$ is the restriction of the product (3.25) to factors depending on $u_{i_1}\ldots u_{i_\alpha}$. $\prod_{i_1\ldots i_c}$ denotes the product over all factors for which $l_j \in S$ is a linear combination of the vectors $u_{i_1}\ldots u_{i_c}$ only. Useful information is obtained by comparing the expansion

$$P = \sum C_\alpha M_\alpha \tag{3.35}$$

of P with respect to monomials M_α in $u_1,\ldots,u_a$ with the expansion

$$P = \sum C'_{\alpha'} M'_{\alpha'} \tag{3.36}$$

with respect to independent monomials $M'_{\alpha'}$ in $u_{i_1}\ldots u_{i_c}$ only. We know that $M_{\alpha i_1\ldots i_c}$ occurs as a factor of at least one monomial M_α with $C_\alpha \neq 0$. Since the monomials M_α are linearly independent the factor $C'_{\alpha'}$ of $M'_{\alpha'} = M_{\alpha i_1\ldots i_c}$ in (3.36) must also be different from zero. This implies the inequality

$$\underline{\deg}_{u'} P \leqq \deg M_{\alpha i_1\ldots i_c} \tag{3.37}$$

which will be crucial for the proof of the theorem. $\underline{\deg}_{u'}$ denotes the lower degree with respect to the variables $u' = (u_{i_1},\ldots,u_{i_c})$. We now form the subintegral

$$J(H') = \int du_{i_1}\ldots du_{i_c} \frac{P}{\prod_j (l_j^2 + m_j^2)^{n_j}} \tag{3.38}$$

along a hyperplance H' defined by constant values of $v_1,\ldots,v_b$ and the momenta u_j which do not belong to u'. The lower dimension δ' of (3.38) is positive by hypothesis (3.2–3),

$$\begin{aligned} 0 < \delta' &= 4c + \underline{\deg}_{u'} P - \underline{\deg}_{u'} \prod_j (l_j^2 + m_j^2)^{n_j}\,, \\ &\leqq 4c + \underline{\deg}_{u'} P - \deg \textstyle\prod_{i_1\ldots i_c} (l_j^2)^{n_j}\,. \end{aligned}$$

With (3.37)

$$\begin{aligned} d_{i_1\ldots i_c} &= 4c + \deg M_{\alpha i_1\ldots i_c} - \deg \textstyle\prod_{i_1\ldots i_c} (l_j^2)^{n_j} \\ &\geqq 4c + \underline{\deg}_{u'} P - \deg \textstyle\prod_{i_1\ldots i_c} (l_j^2)^{n_j} \end{aligned}$$

follows. Hence the dimension $d_{i_1 \dots i_c}$ of (3.38) is positive. According to the lemma each integral on the right hand side of (3.30) converges. This completes the proof of the theorem.

4. Reduced Integrals

In this section we discuss integrals of the form

$$I = \int dk \prod_{j=1}^{n} \Delta_j(l_j) \tag{4.1}$$

in the notation of (1.1) and

$$\Delta_j(l_j) = \frac{M_j}{(l_j^2 - m_j^2 + i\varepsilon(\vec{l}_j^2 + m_j^2))^{n_j}}, \qquad n_j > 0, \tag{4.2}$$

where M_j is a monomial in k and p. For integrals of this type we introduce the concept of the reduced integral. Let

$$S = (l_{j_1}, \dots, l_{j_c}) \tag{4.3}$$

be any subset of the momenta $l_1, \dots, l_n$. From the elements of S we select a basis, i.e. we choose linearly independent forms $u_1, \dots, u_a$ of L such that each $l_j \in S$ is a linear combination of $u_1, \dots, u_a$ and $p_1, \dots, p_N$. With respect to S we form the reduced integral

$$\begin{aligned} &I_{\mathrm{red}}(S) \propto \int du_1 \dots du_a \textstyle\prod_S \Delta_j(l_j) \\ &l_j = l_j(u, p), \qquad u = (u_1, \dots, u_a) \end{aligned} \tag{4.4}$$

where the product $\prod_S$ extends over the $l_j \in S$ only. The reduced integral (4.4) is defined up to a factor which depends on the chosen basis.

Of special interest are reduced integrals of vanishing masses and vanishing external momenta, i.e.

$$m_j = 0 \quad \text{if} \quad l_j \in S,$$

$$l_j = l_j(u), \quad \text{independent of } p, \quad \text{if} \quad l_j \in S.$$

In this case each factor $\Delta_j(l_j)$ occurring in the reduced integral is homogeneous in u.

In case that (4.1) represents an unrenormalized Feynman integral the reduced integrals have a simple graphical interpretation: $I_{\mathrm{red}}(S)$ is the Feynman integral which corresponds to the reduced diagram $S' = S/T$ where all lines of T have been contracted to a point.

With the concept of the reduced integral we can give an equivalent formulation of the infrared convergence condition for integrals of type (4.1)[1]. Consider a basis

$$u_1 = l_{i_1}, \dots, u_a = l_{i_a}, \qquad v_1 = l_{j_1}, \dots, v_b = l_{j_b} \tag{4.5}$$

of L with

$$m_{i_1} = \dots = m_{i_a} = 0. \tag{4.6}$$

[1] This formulation was used by Mack [14] to study infrared convergence of integrals like (4.1) in the context of conformally in variant theorems.

For any such basis the infrared convergence condition reads

$$\underline{\deg}_u \prod \Delta_j(l_j) + 4a > 0 . \tag{4.7}$$

We now form the reduced integral

$$I_{\text{red}}(S) \propto \int du \prod_S \Delta_j(l_j) \tag{4.8}$$

with respect to the set S of all momenta l_j with $m_j = 0$ and $l_j = 0$ at $u = 0$. Then for any $l_j \notin S$ we have

$$\left.\begin{array}{l} m_j \neq 0 \\ \text{or} \quad l_j \neq 0 \quad \text{at} \quad u = 0 \end{array}\right\} \quad \text{if} \quad l_j \notin S .$$

Therefore,

$$\underline{\deg}_u \prod \Delta_j(l_j) = \deg \prod_S \Delta_j(l_j) \tag{4.9}$$

and

$$\underline{\deg}_u \prod \Delta_j(l_j) + 4a = \dim I_{\text{red}}(S) . \tag{4.10}$$

Hence an equivalent formulation of the infrared convergence condition for the integral (4.1) is

$$\dim I_{\text{red}}(S) > 0 \tag{4.11}$$

for any set S of momenta l_j with

$$\left.\begin{array}{l} m_j = 0 \\ \text{and} \quad l_j = 0 \quad \text{at} \quad u = 0 \end{array}\right\} \quad \text{if} \quad l_j \in S . \tag{4.12}$$

With this result, we are able to formulate the infrared convergence condition for an integral of the type [notation of (1.1) and (4.1)]

$$\int dk Q \prod_{j=1}^{n} \Delta_j(l_j) \tag{4.13}$$

where Q is a polynomial in k and p, in terms of a power counting criterion involving the formal integral

$$\int dk \prod_{S_0} \Delta_j(l_j) \tag{4.14}$$

where the product is restricted to the set S_0 of momenta with $m_j = 0$. In particular, we have the following

Corollary to the Power Counting Theorem. *The integral* (4.13) *is absolutely convergent if the ultraviolet convergence condition* (3.1) *holds and if any reduced integral of* (4.14) *with vanishing external momenta has positive dimension.*

Proof.

$$\begin{aligned} \underline{\deg}_u Q \prod_{j=1}^{n} \Delta_j(l_j) &= \underline{\deg}_u Q + \underline{\deg}_u \prod_{j=1}^{n} \Delta_j(l_j) \\ &\geq \underline{\deg}_u \prod_{j=1}^{n} \Delta_j(l_j) . \end{aligned}$$

Hence the infrared convergence condition of (4.13) is implied by that of (4.1). Any reduced integral of (4.1) with (4.12) is also a reduced integral of (4.14). This completes the proof.

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J. H. Lowenstein
Physics Department
New York University
New York, N. Y., USA

W. Zimmermann
Max-Planck-Institut
für Physik und Astrophysik
D-8000 München 4
Föhringer Ring 6
Federal Republic of Germany