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When is a Field Theory a Generalized Free Field?

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Abstract. We show within a scalar relativistic quantum field theory that if either some even truncated n-point-function vanishes or some multiple commutator of the field operators is a c-number then the field is necessarily a generalized free field.

I.

The generalized free fields are well known examples for relativistic quantum field theories. Introduced in 1961 by Greenberg [1] they have been extensively studied since then. Of special interest was the question under what conditions a field theory is necessarily a generalized free field. As shown by Dell'Antonio [2], Robinson [3] and Greenberg [4] such a sufficient condition is that the support of the field in momentum space excludes certain domains. Robinson [5] gave another criterion namely the vanishing of the truncated n-point-functions beyond some N. In this note we shall strengthen this result and prove – with entirely different methods than Robinson – that if an arbitrary even truncated Wightman function vanishes the field must be a generalized free field.

II.

We consider a relativistic scalar field $A(x)$ which we assume to fulfill Wightman's axioms [6, 7]. We denote the vacuum state by Ω and the n-point-functions by

$$\mathscr{W}_n(x_1, \ldots, x_n) = (\Omega, A(x_1), \ldots, A(x_n)\Omega)\,.$$

Without restriction we can assume $(\Omega, A(x)\Omega) = 0$. The truncated n-point-functions [7] are recursively defined by

$$\mathscr{W}_1(x) = \mathscr{W}_1^T(x) \qquad [\text{and therefore } \mathscr{W}_1^T(x) = 0]$$

$$\mathscr{W}_n(x_1, \ldots, x_n) = \sum\nolimits_{\text{partitions}} \mathscr{W}_{r_1}^T(x_{l_1(1)} \ldots x_{l_1(r_1)}) \times \mathscr{W}_{r_2}^T(x_{l_2(1)} \ldots x_{l_2(r_2)}) \ldots \mathscr{W}_{r_s}^T(x_{l_s(1)} \ldots x_{l_s(r_s)})\,.$$

Now we are able to formulate

Theorem 1. *If* $\mathscr{W}_{2n}^T(x_1 \ldots x_{2n}) \equiv 0$ *for some* $n \geqq 2$ *then*

$$[\ldots[A(x_1), A(x_2)] \ldots A(x_n)] = (\Omega, [\ldots[A(x_1), A(x_2)] \ldots A(x_n)]\Omega)^T \cdot \mathbb{1}\,.$$

Proof. One can easily prove by induction that

$$\begin{aligned}&(\Omega, [\dots[A(x_1), A(x_2)]\dots A(x_n)]\\ &\quad\times[A(x_{n+1})[A(x_{n+2})\dots[A(x_{2n-1}), A(x_{2n})]\dots]\Omega)\\ &=(\Omega, [\dots[A(x_1), A(x_2)]\dots A(x_n)][A(x_{n+1})\dots[A(x_{2n-1}), A(x_{2n})]\dots]\Omega)^T\\ &\quad+(\Omega, [\dots[A(x_1), A(x_2)]\dots A(x_n)]\Omega)^T\\ &\quad\times(\Omega, [A(x_{n+1})\dots[A(x_{2n-1}), A_{2n})]\dots]\Omega)^T .\end{aligned}$$

But $\mathscr{W}^T_{2n}(x_1, \dots, x_{2n})\equiv 0$ and therefore we get

$$\begin{aligned}&\|[\dots[A(x_1), A(x_2)]\dots A(x_n)]\Omega\|^2\\ &=|(\Omega, [\dots[A(x_1), A(x_2)]\dots A(x_n)]\Omega)^T|^2 .\end{aligned}$$

This implies

$$\begin{aligned}&[\dots[A(x_1), A(x_2)]\dots A(x_n)]\Omega\\ &=(\Omega, [\dots[A(x_1), A(x_2)]\dots A(x_n)]\Omega)^T\cdot\Omega\end{aligned}$$

and by locality (see Jost [7], p. 99) we get Theorem 1.

III.

Our main result is contained in

Theorem 2. *If* $[\dots[A(x_1), A(x_2)]\dots A(x_n)]=c\cdot\mathbb{1}$ *then* $A(x)$ *is a generalized free field.*

Proof. a) By a theorem of Robinson [3] and Greenberg [4] it is sufficient to show that $\tilde{A}(p)\equiv 0$ for $p^2<0$ (i.e. for all spacelike momentum p). For this purpose let us introduce the following space of test functions: Define

$S := \{x\in\mathbb{R}^4 | x^2<0\}$ "spacelike region"

$\tilde{\mathscr{D}}(S) := \{f\in\mathscr{S}(\mathbb{R}^4) | \tilde{f}(p)=\int e^{ipx}f(x)d^4x\in\mathscr{D}(S)\}$

"space of test functions whose Fourier transforms have support only in the spacelike region".

We note that with f also $\bar{f}$ is in $\tilde{\mathscr{D}}(S)$. The spectrum condition [6, 7] implies

$$A(f)\Omega=0 \quad \text{if} \quad f\in\tilde{\mathscr{D}}(S) .$$

b) For a vector Φ in the dense domain of the field operators $A(f)$ we have the inequality

$$\|A(f)\Phi\|\leqq\|\Phi\|^{1-\frac{1}{2^m}}\,\|\underbrace{A(\bar{f})A(f)\dots A(\bar{f})A(f)}_{2^m\text{ operators}}\Phi\|^{\frac{1}{2^m}}$$

By this $A(f_1)\dots A(f_n)\Phi=0$ for all $f_1, \dots, f_n\in\tilde{\mathscr{D}}(S)$ implies $A(f)\Phi=0$ for all $f\in\tilde{\mathscr{D}}(S)$.

c) **Proposition.** $A(f)A(g_1)\dots A(g_l)\Omega=0$ *for all* l *and all test functions* $f\in\tilde{\mathscr{D}}(S)$ *and* $g_1, \dots, g_l\in\mathscr{S}(\mathbb{R}^4)$.

Proof. By induction:

$$A(f)\Omega=0 \quad \text{for} \quad f\in\tilde{\mathscr{D}}(S) \quad \text{by the spectrum condition.}$$

Now assume $A(f)A(g_1)\dots A(g_l)\Omega=0$ for all $f\in\tilde{\mathscr{D}}(S)$ and all $g_1,\dots,g_l\in\mathscr{S}(\mathbb{R}^4)$ then for $f_1,\dots,f_n\in\tilde{\mathscr{D}}(S)$

$$\begin{aligned}&A(f_1)\dots A(f_n)A(g_1)\dots A(g_{l+1})\Omega\\ &=A(f_1)\dots A(f_{n-1})[A(f_n),A(g_1)]A(g_1)\dots A(g_{l+1})\Omega\\ &=A(f_1)\dots[A(f_{n-1})[A(f_n),A(g_1)]]A(g_2)\dots A(g_{l+1})\Omega\\ &\ \vdots\\ &=\underbrace{[A(f_1)[A(f_2)\dots[A(f_n),A(g_1)]}_{=0}\dots]A(g_2)\dots A(g_{l+1})\Omega=0\end{aligned}$$

because by assumption every multiple commutator with $n+1$ field operators vanishes.

d) The vectors $A(g_1)\dots A(g_l)\Omega$ for all l and $g_1,\dots,g_l\in\mathscr{S}(\mathbb{R}^4)$ form a dense set in the domain of $A(f)$ and therefore $A(f)\equiv 0$ for $f\in\tilde{\mathscr{D}}(S)$. This proves our Theorem 2.

Remark. It one looks into the proofs of the above mentioned theorem of Robinson [3] and Greenberg [4] one immediately realizes that it is already sufficient to know that

$$A(f)A(g)\Omega=0 \quad \text{for all} \quad f\in\tilde{\mathscr{D}}(S) \quad \text{and} \quad g\in\mathscr{S}(\mathbb{R}^4).$$

But by the inequality given in part (b) we can easily formulate this as a condition imposed on an arbitrary even truncated Wightman function, namely

$$\mathscr{W}^T_{2n}(g,f_1,\dots,f_{2n-2},h)=0 \quad \text{for all} \quad g,h\in\mathscr{S}(\mathbb{R}^4) \quad \text{and} \quad f_1,\dots,f_{2n-2}\in\tilde{\mathscr{D}}(S).$$

Conclusion. By combining these two theorems we have shown that in an interacting relativistic quantum field theory every even truncated Wightman function must not vanish identically.

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