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Euclidean Field Theory

I. The Moment Problem

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Abstract. The extension of the Schwinger functions to various positive linear functionals on the Borchers algebra is discussed. In one case, we construct a measure on $\mathscr{S}'$ and give criteria for uniqueness as well as for the homogeneous chaos to lead to an $\mathscr{L}_2$-space.

1. Introduction

In recent years there has been much development in constructive quantum field theory involving the formal framework of generalized random processes; see for example [5, 9, 19, 20]. Still unanswered is the question as to those Wightman quantum field theories admitting such a representation. In this paper we wish to begin an examination of this question and its consequences. More generally one may ask whether Nelson's sharp time euclidean framework [9] may be modified to encompass all Wightman theories or failing this, can restrictions on the latter be given for a reasonably broad equivalence theorem between relativistic and euclidean fields? The success of euclidean methods in constructive field theory amply warrants such an investigation in spite of the lack of a four dimensional example.

The first task of placing the relativistic theory within a euclidean framework has been completely solved by Osterwalder and Schrader [1, 5] in terms of Schwinger functions. The ideas presented here develop the point of view that the probabilistic euclidean field theory arises when these Schwinger functions admit certain extensions to the Borchers algebra over the underlying test function space. As a model, we study the Schwartz space on R^4 though it is clear any other complete nuclear *-algebra will serve equally well. Within this paper, we assume and examine two classes of extensions, the positive and strongly positive ones [10]. For the first case, a euclidean field theory results without necessarily requiring the existence of an infinite volume probability measure. Such a measure arises upon formulating a moment problem using the second notion of positivity. In the study of the $P(\Phi)_2$ model, extended Schwinger functions are defined through the infinite volume limit and positivity is an immediate consequence of the form for the finite volume measure. General mathematical conditions allowing existence of these extensions may be written down, but whether these require further restrictions beyond the Wightman axioms is not yet known.

Conditions on the relativistic theory which lead to most of Nelson's sharp time framework have been given by Simon [19]. More recently, starting with positive extensions of the Schwinger functions satisfying a growth condition which implies strong positivity, Frölich [20] has studied the related euclidean theory within the context of the $P(\Phi)_2$ model. For the considerable development on these questions which has taken place in recent years as well as for earlier literature see [21]. In relation to these investigations, we consider a general moment problem appropriate when sharp time euclidean fields do not exist. Various aspects of the infinite dimensional moment problem which are needed in developing further the resulting generalized random process are presented fully and in a form that we hope will be useful to the reader. Some of these are probably known to some workers in the field. These techniques have been used to describe the embedding of the relativistic Hilbert space as a subspace of a euclidean Hilbert space [7]. It seems to us that the most subtle part of this enlarged framework will be an appropriate generalization of Nelson's Markov property to the non-sharp time theory. Already one knows it is more general than needed [19, 20].

The first part of this paper, Section 2, deals with the existence of positive extensions and reduces the task to that of simultaneously satisfying positivity and symmetry. The relation between reality properties and time reversal invariance has also been noted by Simon [19]. Section 3 describes the extended Schwinger functional moment problems together with uniqueness and invariance of the measure. In Section 4, natural conditions on the moment problem are given which allow polynomials (homogeneous chaos) in the euclidean field to be dense in the $\mathscr{L}_2$-space with respect to the measure derived in Section 3. When such is the case, the euclidean Hilbert space of Section 2 is the entire $\mathscr{L}_2$-space. Throughout, our discussion is confined to the case of a neutral, scalar quantum field.

2. Extension Problems

In this section we formulate the extension problem for constructing a euclidean field theory from the euclidean framework for quantum field theory given by Osterwalder and Schrader [1]. The reader is referred to this work for detailed definitions and properties of the spaces of test functions we shall use, as well as for notation.

Let $\mathscr{S}(R^{4n})$ denote the Schwartz space of test functions on R^{4n} with rapid decrease. Then $\mathscr{S}=\oplus_n \mathscr{S}(R^{4n})$ will be the locally convex direct sum of these spaces in which a typical element is a finite sequence

$$f=\{f_0, f_1, f_2, \ldots\}$$

with $f_0 \in C$ and $(f)_n = f_n \in \mathscr{S}(R^{4n})$, $n=1,2,\ldots$. When equipped with the product

$$(f \times g)_n (x_1, \ldots, x_n) = \sum_{k=0}^{n} f_k(x_1, \ldots, x_k)\, g_{n-k}(x_{k+1}, \ldots, x_n)$$

and the involution ($\overline{}$ means complex conjugate)

$$(f^*)_n (x_1, \ldots, x_n) = \overline{f_n(x_n, \ldots, x_1)},$$

$\mathscr{S}$ becomes a topological $*$-algebra (the Borchers algebra) with unit $\mathbf{1}=\{1,0,0,\ldots\}$ [2]. Important structure properties for the algebraic dual, $\mathscr{S}^*$, and topological dual, $\mathscr{S}'$, of $\mathscr{S}$ are obtained in terms of the positive cone

$$\mathscr{S}^+ = \overline{\mathrm{sp}}\left\{\sum_{k=1}^{N} f_k^* \times f_k \mid \{f_k\} \subset \mathscr{S},\ N \text{ a positive integer}\right\}.$$

The notation $\overline{\mathrm{sp}}$ indicates the closed linear span. A linear functional $\boldsymbol{T} = \{T_0, T_1, T_2, \ldots\}$ is an infinite sequence of linear functionals T_k on $\mathscr{S}(R^{4k})$ and is said to be positive if

$$\boldsymbol{T}(f) = \sum_{n=0}^{\infty} \langle T_n, f_n \rangle \geqq 0 \qquad \forall f \in \mathscr{S}^+ .$$

Such elements of $\mathscr{S}^*$ are also continuous; namely, in $\mathscr{S}'$ [3, 4].

The euclidean theory is obtained in terms of the subspace $\mathscr{S}_0 = \oplus_n {}^0\mathscr{S}(R^{4n})$ where the 0-component is again C and for $n = 1, 2, \ldots, {}^0\mathscr{S}(R^{4n})$ consists of those functions in $\mathscr{S}(R^{4n})$ which together with all partial derivatives vanish unless $x_i \neq x_j$ for all $1 \leqq i < j \leqq n$. A linear functional $\mathscr{S} \in \mathscr{S}_0'$ satisfying the euclidean axioms E0–E4 of Osterwalder and Schrader [1, p. 88] with E0 modified as in [5, p. 80] will be called a *Schwinger state.* Its components S_n are the Schwinger functions. As is well known, the relativistic theory is completely characterized by a positive linear functional $\mathscr{W} \in \mathscr{S}^*$ [2, 3, 6] satisfying the cluster property along with other linear conditions. Such a $\mathscr{W}$ will be termed a Wightman state. The main result of the euclidean theory is the relation between these two functionals.

Theorem (Osterwalder, Schrader [1, 5]). *To each Wightman state on $\mathscr{S}$ there corresponds a unique Schwinger state on $\mathscr{S}_0'$ and conversely.*

The connection between the two states is quite explicit and is given [5] by the relations

$$\mathscr{S}(f) = \boldsymbol{S}_R(f_R) = \tilde{\boldsymbol{W}}_R(\check{f}_R) \qquad \forall f \in \mathscr{S}_< \tag{2.1}$$

and

$$\mathscr{W}(g) = \boldsymbol{W}_R(g_R) = \tilde{\boldsymbol{W}}_R(\tilde{g}_R) \quad \forall g \in \mathscr{S} . \tag{2.2}$$

The test function space $\mathscr{S}_< = \oplus_n \mathscr{S}_<(R^{4n})$ with 0-component C and $\mathscr{S}_<(R^{4n})$ those test functions in $\mathscr{S}(R^{4n})$ which with all derivatives vanish unless $-\infty < x_1^0 < x_2^0 < \cdots < x_n^0 < \infty$; where $x = (x^0, \vec{x}) \in R^4$ and x^0 plays the role of the "time" component. $\boldsymbol{S}_R$ and $\boldsymbol{W}_R$ are the reduced functionals arising from translation invariance without alteration of the 0-component. $(\tilde{\ })$ is given by Fourier transformation in each component and $(\check{\ })$ likewise denotes a linear mapping from $\mathscr{S}(R_+^4)$ onto a dense subset of $\mathscr{S}(\overline{R}_+^4)$ [1, Lemma 4.1, p. 95].

As both $\mathscr{S}_0$ and $\mathscr{S}_<$ are closed subspaces of $\mathscr{S}$ but not subalgebras, a Schwinger state does not allow us to form a euclidean field theory by reconstruction. This will however be the case if we can extend this state to a positive linear functional on $\mathscr{S}$. As we are dealing with an extension, the underlying relativistic theory will be unchanged. For a characterization of the extensions of interest to us we must recall several homeomorphisms of $\mathscr{S}$ [1]. The time inversion operator is defined by

$$(\theta f)_n(x_1, \ldots, x_n) = f_n(\theta x_1, \ldots, \theta x_n) \quad \text{with} \quad \theta(x^0, \vec{x}) = (-x^0, \vec{x}),$$

while the euclidean group, $iSO(4)$, on R^4 acts according to

$$(\alpha(a, R)f)_n(x_1, \ldots, x_n) = f_n(R^{-1}(x_1 - a), \ldots, R^{-1}(x_n - a))$$

for $a \in R^4$, $R \in SO(4)$. For translations alone we put $\alpha(a, I) = \alpha(a)$. For the infinite symmetric group $\boldsymbol{P} = \prod_{n=0}^{\infty} P_n$ with P_n being the symmetric group on n objects we define for $\pi \in \boldsymbol{P}$

$$(\pi f)_n(x_1, \ldots, x_n) = f_n(x_{\pi_n(1)}, \ldots, x_{\pi_n(n)}) \quad \text{with} \quad (\pi)_n = \pi_n \in P_n .$$

Throughout this paper, ext $\boldsymbol{S}$ will denote an appropriate linear extension to $\mathscr{S}$ of a Schwinger state $\boldsymbol{S}$. Among the many which exist by the Hahn-Banach theorem the following subclass appears most profitable.

Definition 2.1. An extension, ext $\boldsymbol{S}$, will be called an extended Schwinger state if the euclidean axioms E0–E4 are widened to include

Ext 0: ext $\boldsymbol{S} \in \mathscr{S}^*$ and is positive;

Ext 1: ext $\boldsymbol{S}(f) =$ ext $\boldsymbol{S}(\alpha(a, R)f)$ for all $(a, R) \in iSO(4)$ and ext $\boldsymbol{S}(\theta f) =$ ext $\boldsymbol{S}(f)$;

Ext 3: ext $\boldsymbol{S}(f) =$ ext $\boldsymbol{S}(\pi f)$ for all $\pi \in \boldsymbol{P}$;

valid for any $f \in \mathscr{S}$.

When such an extension exists, the reconstruction theorem [2, 6] will produce a euclidean field theory over $\mathscr{S}(R^4)$ corresponding to the underlying relativistic field theory by means of the Osterwalder-Schrader theorem. To fix the notation for Section 4, the euclidean Hilbert space will be $\mathscr{H}_E = \overline{\mathscr{S}/N_E}^E$ where the euclidean null space is the set

$$N_E = \{f \in \mathscr{S} \mid \text{ext } \boldsymbol{S}(f^* \times f) = 0\}$$

and the euclidean topology is derived from the following sesquilinear form on the cosets $\psi(f) \in \mathscr{S}/N_E$,

$$(\psi(f), \psi(g))_E = \text{ext } \boldsymbol{S}(f^* \times g) .$$

In the usual manner [2] with $\Omega_E = \psi(\mathbf{1})$, the euclidean field B is a continuous linear map from $\mathscr{S}$ into closed linear operators in $\mathscr{H}_E$ defined by $B(f)\,\psi(g) = \psi(f \times g)$. These are commutative on $\mathscr{S}/N_E$ as a consequence of Ext 3. The relation between E2, Ext 1 and an embedding of the relativistic Hilbert space reconstructed over the Wightman state as a closed subspace of $\mathscr{H}_E$ is taken up elsewhere [7].

It is to be expected that there are Wightman states for which no extended Schwinger states exist. In this direction the following necessary conditions must apply.

Proposition 2.2. *For related Schwinger and Wightman states, $\mathscr{S}$ and $\mathscr{W}$ respectively, the following equivalent conditions are necessary for the existence of a positive extension to $\mathscr{S}$ of $\boldsymbol{S}$:*

(a) $\boldsymbol{S}(f) = \overline{\boldsymbol{S}(\bar{f})}$ $\quad \forall f \in \mathscr{S}_0$;

(b) $\boldsymbol{S}(\theta f) = \boldsymbol{S}(f)$ $\quad \forall f \in \mathscr{S}_0$;

(c) $\mathscr{W}(f) = \overline{\mathscr{W}(\theta \bar{f})}$ $\quad \forall f \in \mathscr{S}$.

Proof. The necessity of (a) is immediate since when Ext 0 is true, ext $S(f)$ $=\text{ext}\,\overline{S(f^*)}$ and due to the symmetry of S for $f\in\mathscr{S}_0$ this becomes (a).

Suppose S is real then so is the restriction to $\mathscr{S}_<$. For $f\in\mathscr{S}_<$, the relation (2.1) implies

$$\tilde{W}_R(\check{f}_R)=\overline{\tilde{W}_R(\check{\bar{f}}_R)}=\overline{\tilde{W}_R(\theta_s\check{\bar{f}}_R)}=\overline{\theta_s\tilde{W}_R}(\check{f}_R),$$

in which θ_s is the space inversion. The functionals $\tilde{W}_R$ and $\overline{\theta_s\tilde{W}_R}$ are equal on a dense subset of $\mathscr{S}(\bar{R}^4_+)$ [1, Lemma 4.1, p. 95]. Since θ_s is a symmetry and continuous, this equality extends to all of $\mathscr{S}(\bar{R}^4_+)$ whereupon we conclude from (2.2) the relation

$$\mathscr{W}(g)=\tilde{W}_R(\tilde{g}_R)=\overline{\tilde{W}_R(\theta_s\bar{\tilde{g}}_R)}=\overline{\mathscr{W}(\theta\bar{g})}$$

for all $g\in\mathscr{S}$. Reversing these steps establishes the equivalence of (a) with (c).

The equivalence of (b) with (c) follows in the same way using the symmetry of S to write (b) as

$$S(f)=S(\theta\tilde{\pi}f),\qquad \tilde{\pi}f=\bar{f}^*,\ \forall f\in\mathscr{S}_0\,.$$

Then if $f\in\mathscr{S}_<$ so is $\theta\tilde{\pi}f$ and the Osterwalder-Schrader relations become

$$\tilde{W}_R(\check{f}_R)=\tilde{W}_R(\theta_s\tilde{\pi}\check{f}_R)$$

or by continuity

$$\mathscr{W}(g)=\mathscr{W}(\theta\tilde{\pi}g)=\overline{\mathscr{W}(\theta\bar{g})},\qquad \forall g\in\mathscr{S}\,.$$

Next let us turn to the question of extensions for S which are positive. Extensions which are required to be only symmetric abound upon averaging any Hahn-Banach extension over the compact group P with an invariant mean for this group. However, it should be remarked that P does not preserve the positive cone $\mathscr{S}^+$, hence Ext 3 does not follow automatically once Ext 0 has been achieved. It is a remarkable fact that with a technical assumption, $S\restriction_{\mathscr{S}_<}$ always has at least one positive extension to $\mathscr{S}$.

Proposition 2.3. *Let S be a real Schwinger state with $S_1\equiv 0$. Then $S\restriction_{\mathscr{S}_<}$ has a positive extension to $\mathscr{S}$.*

Proof. For the restriction $S_n\in\mathscr{S}'_<(R^{4n})$, hence there exists a positive integer $m(n)$ and a positive constant $c(n,m)$ such that $|S_n(f_n)|\leqq c(n)\,|f_n|_{m(n)}$ for all $f_n\in\mathscr{S}_<(R^{4n})$ with $|\cdot|_{m(n)}$ one of the Schwartz norms on $\mathscr{S}(R^{4n})$. Set $\bigcup\limits^{m(n)}=\{f_n\in\mathscr{S}(R^{4n})\,|\,|f_n|_{m(n)}<1\}$ and define sequences $\{m(n)\}$, $\{c(n)\}$ for $n=0,1,2,\ldots$, with $c(0)=\frac{1}{2}$, $m(0)=0$. Then $U=\bigoplus\limits_n(1/2^{n+1}c(n))\bigcup\limits^{m(n)}$ is a convex, balanced neighborhood of zero in $\mathscr{S}$.

Suppose $f+p\in U$ with $f\in\mathscr{S}_<$ and $p\in\mathscr{S}^+$. Then for $n\geqq 2$

$$|f_n+p_n|_{m(n)}=\sup_{\substack{x\in R^{4n}\\ |\alpha|\leqq m(n)}}(1+\|x\|)^{m(n)}\,|(f_n+p_n)^{(\alpha)}(x_1,\ldots,x_n)|<1/2^{n+1}c(n)\,.$$

For convenience denote $\Omega_< = \{x \in R^{4n} | x_1^0 < x_2^0 < \cdots < x_n^0\}$ so that $\operatorname{supp} f_n \subset \Omega_<$. This means

$$\sup_{\substack{x \in R^{4n} \setminus \Omega_< \\ |\alpha| \leqq m(n)}} (1 + \|x\|)^{m(n)} |p_n^{(\alpha)}(x_1, \ldots, x_n)| < 1/2^{n+1} c(n),$$

or subsequently

$$\sup_{\substack{x \in \Omega_< \\ |\alpha| \leqq m(n)}} (1 + \|x\|)^{m(n)} |p_n^{(\alpha)}(x_1, \ldots, x_n)| = \sup_{\substack{x \in \Omega_< \\ |\alpha| \leqq m(n)}} (1 + \|x\|)^{m(n)} |p_n^{(\alpha)}(x_n, \ldots, x_1)| < 1/2^{n+1} c(n).$$

Combining the two estimates produces $|p_n|_{m(n)} < 1/2^{n+1} c(n)$ and accordingly $|f_n|_{m(n)} \leqq |f_n + p_n|_{m(n)} + |p_n|_{m(n)} < 1/2^n c(n)$ when $n \geqq 2$. For the 0-component, $|f_0 + p_0| < 1$ with $p_0 \geqq 0$ requires $f_0 < 1$.
Finally for such f, the value of the Schwinger functional is bounded by

$$S(f) \leqq f_0 + \sum_{n \geqq 2} |S_n(f_n)| \leqq 3/2.$$

The Bauer-Namioka extension theorem [8, V, 5.4] will then guarantee the existence of the desired extension. Whether or not every Schwinger state has an extension which is positive remains open.

The matter of an extension with both symmetry and positivity appears more delicate. It should be remarked that a symmetric, positive extension of $S{\restriction}_{\mathscr{S}_<}$ is not necessarily an extension of $\boldsymbol{S}$ unless it is euclidean invariant when restricted to $\mathscr{S}_0$. For $\boldsymbol{S}$ itself, jointly positive and symmetric extensions are governed by the Bauer-Namioka theorem. As a base for the subspace invariant under $\boldsymbol{P}$ define

$$\mathscr{N} = \operatorname{sp}\{\boldsymbol{f} - \pi \boldsymbol{f} \,|\, \boldsymbol{f} \in \mathscr{S}, \pi \in \boldsymbol{P}\}$$

and extend $\boldsymbol{S}$ to $\mathscr{S}_0 + \mathscr{N}$ by

$$\operatorname{ext}_1 \cdot \boldsymbol{S}(\boldsymbol{f} + \boldsymbol{g}) = \boldsymbol{S}(\boldsymbol{f}) \qquad \forall \boldsymbol{f} \in \mathscr{S}_0, \boldsymbol{g} \in \mathscr{N} \rceil \mathscr{S}_0.$$

This extension is well defined and leads immediately to the result,

Proposition 2.4. *A real Schwinger state $\boldsymbol{S}$ has a positive, symmetric extension to $\mathscr{S}$ if and only if there is a convex, balanced neighborhood U of zero in $\mathscr{S}$ such that $\operatorname{ext}_1 \boldsymbol{S}$ is bounded above on $(\mathscr{S}_0 + \mathscr{N}) \cap (U - \mathscr{S}^+)$.*

Alternative criteria involving the kernel of $\boldsymbol{S}$ may be readily found [4]. One may pursue the necessary and sufficient condition of Proposition 2.4 through the relations (2.1), (2.2). In geometrical terms the result is that the underlying Wightman state should be positive on a second closed cone in addition to $\mathscr{S}^+$. It would be of interest to establish usable sufficient conditions on the Wightman functionals which lead to Proposition 2.4.

Before turning to another type of positivity for $\operatorname{ext} \boldsymbol{S}$ in the next section let us note that once an extension of $\boldsymbol{S}$ has been found for which Ext 0 and Ext 3 hold, then Ext 1 follows generally. Suppose $\operatorname{ext}_1 \boldsymbol{S}$ is the assumed extension then

$$\operatorname{ext}_2 S(\boldsymbol{f}) = \tfrac{1}{2}[\operatorname{ext}_1 \boldsymbol{S}(\boldsymbol{f}) + \operatorname{ext}_1 \boldsymbol{S}(\theta \boldsymbol{f})]$$

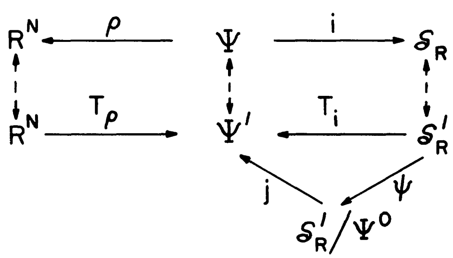
is a θ-invariant extension of the same type. For the remainder, let M denote an invariant mean on $\mathscr{L}_\infty(iSO(4))$ and consider the continuous bounded function on $iSO(4)$ defined by

$$F_f : (a, R) \rightsquigarrow \mathrm{ext}_2 S(\alpha(a, R) f) \qquad \forall f \in \mathscr{S}.$$

Clearly, $\mathrm{ext}\, S(f) = M[F_f(\cdot)]$ is the desired extension since the Schwinger functional satisfies the euclidean invariance axiom, E1.

3. Construction of a Measure

For the most part, our terminology governing measures on nuclear spaces is that of Gelfand and Vilenkin [11]; however, for convenience we recall certain of these notions. The real-valued Schwartz functions will be denoted as $\mathscr{S}_R(R^4)$, or more briefly $\mathscr{S}_R$, whereupon $\mathscr{S}(R^4) = \mathscr{S}_R + i\mathscr{S}_R$. Similarly with a slight abuse of notation $\mathscr{S}'_R$ indicates continuous real linear functionals on $\mathscr{S}_R$ so that $T \in \mathscr{S}'$ if and only if $T(f) = \mathrm{Re}\, T(f) - i\, \mathrm{Re}\, T(if)$ for all $f \in \mathscr{S}_R$ with $\mathrm{Re}\, T \in \mathscr{S}'_R$. Suppose $\Psi \subset \mathscr{S}_R$ is a finite dimensional (real) subspace with (absolute) polar $\Psi^0 \subset \mathscr{S}'_R$. Then the canonical coset mapping ψ factors the transpose of the injection $i : \Psi \to \mathscr{S}_R$ so that the following diagram is commutative. Dotted lines show dualities.



A coordinate representation for Ψ consists in selecting a basis $\{f_1, f_2, \ldots, f_N\}$ where $\dim \Psi = N$ and with respect to this basis $\Psi \overset{\varrho}{\cong} R^N$. A cylinder set in $\mathscr{S}'_R$ is then a collection of equivalence classes modulo Ψ^0 which may be equivalently represented as

$$Z = \psi^{-1}(A) = Z_{\Psi^0}(A) = Z_{\{f_1, f_2, \ldots, f_N\}}(B),$$

where B is a Borel set in R^N and A the "Borel" set in $\mathscr{S}'_R | \psi^0$ defined by ${}^T i(Z) = {}^T \varrho(B) = j(A)$. If μ is a measure defined on the σ-algebra $\mathscr{A}$ generated by the algebra $\mathscr{A}_0$ of cylinder sets, the induced or conditional measure on cylinder sets is written

$$\mu(Z) = \mu_{\Psi^0}(A) = \nu_{f_1, f_2, \ldots, f_N}(B).$$

3.1. M. Riesz and M. G. Krein Extensions

Consider the polynomial algebra $\mathscr{P}(\mathscr{S}_R)$ over the complex numbers generated by $\mathscr{S}_R$. It is not hard to see that this is dense in $\mathscr{S}$. Suppose $P(t_1, t_2, \ldots, t_N)$ is a non-negative polynomial defined on R^N. A positive (non-negative) polynomial in $\mathscr{P}(\mathscr{S}_R)$ then corresponds to the formal substitution

$$P(f_1, f_2, \ldots, f_N) = \sum_{\substack{j_k=0 \\ 1 \leqq k \leqq N}}^{n_k} c(j_1, \ldots, j_N) f_1^{j_1} \times f_2^{j_2} \times \cdots \times f_N^{j_N},$$

where the algebra product in $\mathscr{S}$ replaces the ordinary product of real numbers. The fact that $P(f_1, \ldots, f_N)$ is no longer independent of the order in which the products of its factors are taken is of no consequence due to the symmetry of the extended Schwinger functionals. Further, there is no loss of generality in requiring that the vectors $\{f_1, f_2, \ldots, f_N\} \subset \mathscr{S}_R$ be linearly independent. Certainly for different choices of vectors in $\mathscr{S}_R$ the same polynomial on R^N produces different elements of $\mathscr{P}(\mathscr{S}_R)$, say $\mathscr{P}_+$; which defines an order relation.

There is an alternative way to view $\mathscr{P}(\mathscr{S}_R) \equiv \mathscr{P}$. A function $F: \mathscr{S}_R' \to C$ will be said to have finitely many variables if there is a finite collection of linearly independent functions in $\mathscr{S}_R$, say $\{f_1, f_2, \ldots, f_N\}$, such that

$$F: T \rightsquigarrow F(\langle T, f_1 \rangle, \langle T, f_2 \rangle, \ldots, \langle T, f_N \rangle) \qquad \forall T \in \mathscr{S}_R',$$

where $F(\cdot)$ is a complex-valued function on R^N. In this way, $\mathscr{P}$ is just the subset of polynomials in finitely many variables and $P \in \mathscr{P}_+$ if and only if $P(\langle \cdot, f_1 \rangle, \ldots$ $\ldots, \langle \cdot, f_N \rangle) \geqq 0$ on $\mathscr{S}_R'$. The subclass of real polynomially bounded functions will be written as $\mathscr{E}(\mathscr{S}_R') \equiv \mathscr{E}$, which one readily verifies is a real linear space.

It is now appropriate to introduce the second type of positive extension from the Osterwalder-Schrader theory that we will examine. Criteria for the existence of such extensions are to be had upon modifying Proposition 2.4.

Definition 3.1. Let ext $\boldsymbol{S}$ be an extension to $\mathscr{P}(\mathscr{S}_R)$ of a Schwinger state for which
(a) ext $\boldsymbol{S}$ is continuous;
(b) ext $\boldsymbol{S}$ is symmetric;
(c) ext $\boldsymbol{S}(\mathscr{P}_+) \geqq 0$.

After Powers [10], such a state will be called strongly positive. From a strongly positive state, we may define a real linear functional I on $\mathscr{P}(\mathscr{S}_R) \cap \mathscr{E}(\mathscr{S}_R)$ by

$$I(P) \equiv \operatorname{ext} \boldsymbol{S}(P(f_1, f_2, \ldots, f_N)).$$

The functional I then satisfies the conditions of the Riesz-Krein extension theorem [17, Theorem 2.6.2, p. 69] whereby it may be extended to a positive real linear functional ext I on $\mathscr{E}$. When $F \in \mathscr{E}$ has the form $F(\langle \cdot, f_1 \rangle, \ldots, \langle \cdot, f_N \rangle)$ we shall also write for this extension

$$\operatorname{ext} I(F) = \operatorname{ext} I_{f_1, f_2, \ldots, f_N}(F).$$

Upon extending further by linearity, ext I is then defined on the complex functions in $\mathscr{E} + i\mathscr{E}$ and agrees with ext $\mathscr{S}$ on $\mathscr{P}$. The virtue of such an extension lies in the fact that it corresponds to a measure on the cylinder sets $\mathscr{A}_0$ in $\mathscr{S}_R'$ and solves an infinite dimensional moment problem for ext $\boldsymbol{S}$.

Definition 3.2. Suppose Z is a cylinder set in $\mathscr{S}_R'$ with characteristic function χ_Z. Then

$$\mu(Z) = \operatorname{ext} I(\chi_Z)$$

defines a positive finitely additive set function on $\mathscr{A}_0$.

We claim that μ is actually a cylinder set measure. μ is certainly well defined, for suppose $Z_{\{f_1,f_2,\ldots,f_n\}}(B)$ is a cylinder set for some Borel set B in R^N. Then

$$\mu(Z) = \operatorname{ext} I_{f_1,f_2,\ldots,f_N}(\chi_B) = \nu_{f_1,f_2,\ldots,f_N}(B)$$

defines a normalized, regular, Borel measure $\nu_{f_1,f_2,\ldots,f_N}$ on R^N.

It remains to verify certain compatibility conditions that ensure μ depends only upon the cylinder set and not the manner in which it is represented. Suppose $Z = Z_{\Psi^0}(A)$ and $\{f_1, f_2, \ldots, f_N\} = \vec{f}$, $\{g_1, g_2, \ldots, g_N\} = \vec{g}$ are two bases for Ψ related by a real non-singular $N \times N$ matrix C, $\vec{g} = C\vec{f}$. $\mu(Z)$ is independent of which basis is used as

$$\operatorname{ext} I_{f_1,\ldots,f_N}(\chi_B) = \operatorname{ext} I_{g_1,\ldots,g_N}(\chi_{C(B)}).$$

The second compatibility requirement concerns two subspaces $\Psi_1, \subset \Psi_2, \subset \mathscr{S}'_R$ with bases $\{f_1, \ldots, f_N\}$ and $\{f_1, \ldots, f_N, f_{N+1}, \ldots, f_{N+M}\}$ respectively. For $F \in \mathscr{E}$ of the form $F(\langle \cdot, f_1 \rangle, \ldots, \langle \cdot, f_N \rangle)$ we may multiply by the function which is identically 1 on R^M so that

$$\operatorname{ext} I_{f_1,\ldots,f_N}(F) = \operatorname{ext} I_{f_1,\ldots,f_{N+M}}(F \times 1).$$

A short calculation ascertains the compatibility of the measures $\mu_{\Psi_1^0}$ and $\mu_{\Psi_2^0}$. It then follows that μ is a cylinder set measure [11, Eq. (4), p. 309].

The relation between μ and $\operatorname{ext} \boldsymbol{S}$ is a classical proposition from the theory of the moment problem. One may actually do a little better without any further work.

Proposition 3.3. *Suppose $F \in \mathscr{E}(\mathscr{S}'_R)$ and is measurable with respect to the Borel sets in $\mathscr{S}'_R \setminus \Psi^0$ for some finite dimensional subspace $\Psi \subset \mathscr{S}_R$. Then*

$$\operatorname{ext} I(F) = \int_{\mathscr{S}'_R/\Psi^0} F \, d\mu_{\Psi^0}$$

and in particular

$$\operatorname{ext} \boldsymbol{S}(P(f_1, f_2, \ldots, f_N)) = \int_{R^N} P(t_1, t_2, \ldots, t_N) \, d\nu_{f_1,\ldots,f_N}(t_1, t_2, \ldots, t_N).$$

Remarks. This representation for $\operatorname{ext} I$ does not require the countable additivity of μ on $\mathscr{A}_0$.

Proof. For F continuous, the validity of the above integral representation is a standard result in the moment problem [12, Theorem 1.1, p. 3].

Next suppose $F \geqq 0$ and $0 \leqq s_n \leqq F$ is a sequence of simple functions increasing pointwise to F. By positivity $0 \leqq \operatorname{ext} I(s_n) \leqq \operatorname{ext} I(F)$ and by monotone convergence

$$\int F \, d\mu_{\Psi^0} = \lim_{n \to \infty} \int s_n \, d\mu_{\Psi^0} \leqq \operatorname{ext} I(F).$$

For the converse inequality recall that since F is polynomially bounded there exists a fixed polynomial P such that for any $\varepsilon > 0$, there exists a compact set $K \subset R^N$ depending only on ε for which

$$0 \leqq F(1 - \chi_K) \leqq \varepsilon P / I(P).$$

Since P is chosen to dominate F we may assume $I(P)>0$ otherwise there is nothing to prove. The function $F\chi_K$ has bounded range, so $0 \leqq F\chi_K \leqq b < \infty$. In fact, if $\mathscr{Q}$ is a positive polynomial which dominates F then $b \leqq \sup \mathscr{Q}\chi_K < \infty$. Following Rudin's proof of the F. Riesz representation theorem [13, Step X, p. 46] one may deduce the estimate

$$\operatorname{ext} I(F\chi_K) \leqq \int F\, d\mu_{\Psi^0} + 3\varepsilon + \varepsilon^2 .$$

Hence

$$\operatorname{ext} I(F) = \operatorname{ext} I(F\chi_K) + \operatorname{ext} I(F(1-\chi_K)) \leqq \int F\, d\mu_{\Psi^0} + 4\varepsilon + \varepsilon^2 .$$

When F is a general measurable function in $\mathscr{E}$, the representation results upon decomposing F into positive and negative parts. The representation also extends to complex functions.

Remark. Under suitable growth conditions on $\operatorname{ext} \boldsymbol{S}(f^p)$ as a function of p, positivity of ext $\boldsymbol{S}$ alone will imply strong positivity [18].

3.2. Uniqueness of μ

As we shall see in Section 4, the circumstance in which the measure μ is unique provides good technical control over the resulting euclidean field on $(\mathscr{S}_R', \mathscr{A}, \mu)$. This question is clearly that of uniqueness for the Riesz-Krein extension to $\mathscr{E}$ which requires

$$\sup_{z \in \mathscr{P} \cap \mathscr{E} : z \leqq x} I(z) = \inf_{y \in \mathscr{P} \cap \mathscr{E} : x \leqq y} I(y) \qquad \forall x \in \mathscr{E} .$$

In this form, this condition is not particularly useful and is best exploited in terms of the one dimensional measures $v_f, f \in \mathscr{S}_R$.

First we shall need to extend a result due to Riesz for the power moment problem on the real line. A solution, say v, to this problem is called N-extremal if either of the following conditions are valid [17, Definition 2.3.3, p. 45]:

(a) v is unique,

(b) v is the limit circle case.

The required generalization is the next lemma.

Lemma 3.4. *Let v be a normalized N-extremal solution to the one dimensional moment problem. Then the polynomials are dense in $\mathscr{L}_p(R, dv)$ for $1 \leqq p < \infty$.*

Proof. That the polynomials are dense in $\mathscr{L}_1$ and $\mathscr{L}_2$ respectively under these hypotheses is due to Naimark and Riesz [17, pp. 43–49]. Suppose $\omega_k(t)$, $k=0,1,2,\dots$ denote the orthonormalized polynomials of degree k on $\mathscr{L}_2(R, dv)$; namely,

$$\int dv(t)\, \omega_i(t)\, \omega_j(t) = \delta_{ij}$$

with $\omega_k \in \mathscr{L}_p$ for $1 \leqq p < \infty$. Suppose χ_A is the characteristic function of a Borel set $A \subset R$ and $\{\alpha_k\}$ are its Fourier coefficients with respect to the $\{\omega_k\}$. Then Hölder's inequality for $1 \leqq p < 2$ and Riesz's theorem give the relation

$$\lim_{N\to\infty} \left\| \chi_A - \sum_{k=0}^{N} \alpha_k \omega_k \right\|_p \leqq \lim_{N\to\infty} \left\| \chi_A - \sum_{k=0}^{N} \alpha_k \omega_k \right\|_2 = 0 .$$

The polynomials are therefore dense in these $\mathscr{L}_p$.

Suppose the polynomials are not dense in $\mathscr{L}_{\frac{p}{p-1}}$, $1<p<2$. There then exists a non-trivial $f \in \mathscr{L}_p$ for which

$$\langle f, \omega_k \rangle = \int d\nu f \omega_k = 0 \qquad k=0,1,2,\ldots.$$

Now as $\mathrm{sp}\{\omega_k\}$ is dense in $\mathscr{L}_p$, there exist coefficients $\{c_k\}$ such that

$$\lim_{N\to\infty} \left\| f - \sum_{k=0}^{N} c_k \omega_k \right\|_p = 0\,;$$

or rather that

$$\lim_{N\to\infty} \left\langle f - \sum_{k=0}^{N} c_k \omega_k, \omega_i \right\rangle = 0 \qquad i=0,1,2,\ldots.$$

This requires $c_k = \langle f, \omega_k \rangle = 0$, which is a contradiction. The following criterion for the uniqueness of μ now holds.

Theorem 3.5. *The measure μ is unique if and only if each of the one dimensional cylinder set measures ν_f, $f \in \mathscr{S}_R$, is unique.*

Proof. The condition is certainly necessary so let us suppose each ν_f is unique. Suppose for some finite linearly independent set $\{f_1, f_2, \ldots, f_N\} = \vec{f}$ we have two solutions of the corresponding moment problem; say for each Borel set $B \subset R^N$, there are measures

$$\nu_{\vec{f}}^{(1)}(B) = \mathrm{ext}\, I_{\vec{f}}^{(1)}(\chi_B), \nu_{\vec{f}}^{(2)}(B) = \mathrm{ext}\, I_{\vec{f}}^{(2)}(\chi_B)$$

for two Riesz-Krein extensions.

Consider a set $A = A_1 \times A_2 \cdots \times A_N$ in which each $A_k \subset R$ is a Borel set. For a given integer $m=1,2,3,\ldots$ choose a polynomial $P_{m_k}(t_k)$ by Lemma 3.4 such that

$$\|\chi_{A_k} - P_{m_k}\|_{\mathscr{L}_{2^N}(R, d\nu_{f_k})} \leqq (1/Nm2^N)\,.$$

Then $\|P_{m_k}\|_{2^N} \leqq 2$. Define a polynomial in N-variables by

$$\mathscr{Q}_m(t_1, \ldots, t_N) = P_{m_1}(t_1)\, P_{m_2}(t_2) \ldots P_{m_N}(t_N)\,.$$

After repeated use of Hölder's inequality we derive the following estimate valid for either $i=1$ or 2,

$$\begin{aligned}\mathrm{ext}\, I^{(i)}(|\chi_A - \mathscr{Q}_m|) &= \int_{R^N} |\chi_{A_1}(t_1) \ldots \chi_{A_N}(t_N) - \mathscr{Q}_m(t_1, \ldots, t_N)|\, d\nu_{f_1,\ldots,f_N}^{(i)}{}^{(t_1,\ldots,t_N)} \\ &\leqq \sum_{e=1}^{N} \|\chi_{A_e} - P_{m_e}\|_2\, \|P_{m_1}\|_4 \ldots \|P_{m_{e-1}}\|_{2^e} \leqq 1/2m\,.\end{aligned}$$

Consequently for each $m=1,2,\ldots$,

$$\begin{aligned}|\nu_{\vec{f}}^{(1)}(A) - \nu_{\vec{f}}^{(2)}(A)| &\leqq |\mathrm{ext}\, I^{(1)}(\chi_A) - \mathrm{ext}\, \boldsymbol{S}(\mathscr{Q}_m(f_1, \ldots, f_N))| \\ &\quad + |\mathrm{ext}\, \boldsymbol{S}(\mathscr{Q}_m(f_1, \ldots, f_N)) - \mathrm{ext}\, I^{(2)}(\chi_A)| \leqq 1/m\,.\end{aligned}$$

Standard arguments imply that $\nu_{\vec{f}}^{(1)}$ and $\nu_{\vec{f}}^{(2)}$ agree on the Borel sets in R^N.

A most useful criterion for the uniqueness of μ is due to Carleman [12, Theorem 1.10].

Corollary. *The measure μ is unique if* $\sum_{p=1}^{\infty} [\operatorname{ext} S(f^{2p})]^{\frac{-1}{2p}} = \infty$ *for each* $f \in \mathscr{S}_R$.

The literature on the moment problem provides a number of necessary and sufficient conditions for the uniqueness of v_f in terms of the moments $\operatorname{ext} \mathbf{S}(f^p)$. It is worth pointing out that Theorem 3.5 actually proves that even if the v_f are not unique but only limit circle solutions to the moment problem, nevertheless the measure μ is uniquely determined by a specific choice among these.

3.3. Continuity of μ

Following Gelfand and Vilenkin [11] consider $a > 0$ and the function $F(t) = 1$ for $|t| \geqq a$ and t^2/a^2 for $|t| < a$. Suppose $\{f_j\}$ is a sequence of functions tending to zero in $\mathscr{S}_R$ and μ is a measure constructed from a strongly positive state. Then

$$0 \leqq \mu[\{T \in \mathscr{S}'_R | |\langle T, f_j \rangle| \geqq a] \leqq \operatorname{ext} I(F) \leqq \operatorname{ext} \mathbf{S}_2(f_j^2/a^2)$$

and

$$\lim_{j \to \infty} \mu[\{T \in \mathscr{S}_{R'} | |\langle T, f_j \rangle| \geqq a] = 0 .$$

This implies that the measure μ is continuous [11, Chapter 4, § 1.4] and hence by a theorem of Minlos [11, Chapter 4, Theorem 3] countably additive on $\mathscr{A}_0$. General arguments allow a unique countably additive extension to $\mathscr{A}$ which we shall again denote by μ. A little more may be obtained from these remarks.

Proposition 3.6. *Each strongly positive state on $\mathscr{P}(\mathscr{S}_R)$ has a unique, continuous, positive extension to $\mathscr{S}$.*

Proof. As part of Definition 3.1 we have required a strongly positive state to be continuous on $\mathscr{P}(\mathscr{S}_R)$ with respect to the topology of $\mathscr{S}$. As $\mathscr{P}$ is dense in $\mathscr{S}$, such a state extends by continuity to a unique element of $\mathscr{S}'$. Let us denote both the state and its extension by $\operatorname{ext} \mathbf{S}$.

For the case when $f_1, \ldots, f_N$ are functions in $\mathscr{S}$, decomposing each f_k into its real and imaginary parts leads to a decomposition of a polynomial $P(f_1, \ldots, f_N) = P_1 + iP_2$ where P_1, P_2 are real polynomials in functions from $\mathscr{S}_R$. The symmetry of $\operatorname{ext} \mathbf{S}$ implies the relation

$$\operatorname{ext} \mathbf{S}(P^* \times P) = \operatorname{ext} \mathbf{S}(P_1^2 + P_2^2) \geqq 0$$

by strong positivity. Finally consider $\boldsymbol{f} \in \mathscr{S}$ and a sequence of polynomials $\{\boldsymbol{P}_j\} \subset \mathscr{P}(\mathscr{S}_R)$ such that $\boldsymbol{P}_j \to \boldsymbol{f}$ in $\mathscr{S}$. Then

$$|\operatorname{ext} \mathbf{S}(\boldsymbol{f}^* \times \boldsymbol{f} - \boldsymbol{P}_j^* \times \boldsymbol{P}_j)| \leqq |\operatorname{ext} \mathbf{S}(\boldsymbol{f}^* \times (\boldsymbol{f} - \boldsymbol{P}_j))| + |\operatorname{ext} \mathbf{S}((\boldsymbol{f}^* - \boldsymbol{P}_j^*) \times \boldsymbol{P}_j)|$$
$$\to 0 \quad \text{as} \quad j \to \infty$$

since multiplication in $\mathscr{S}$ is jointly continuous on $\bigoplus_{n=0}^{M} \mathscr{S}(R^{4n}) \times \bigoplus_{n=0}^{N} \mathscr{S}(R^{4n})$, [3, Theorem 1.3.5].

3.4. Invariant Measures

As we have seen in § 2, we may assume without loss of generality that there exist strongly positive satisfying all the extension axioms of Definition 2.1. A consequence of this is that the class of measures constructed in § 3.1 contains ones which are invariant with respect to the flow on $\mathcal{S}'_R$ induced by the euclidean group. To see how this comes about let us notice first of all the following structure.

Proposition 3.7. *For any* $(a,R)\in iSO(4)$ *define a mapping* $\eta^{-1}_{(a,R)}:\mathcal{S}'_R\to\mathcal{S}'_R$ *by* $\langle T,\alpha(a,R)f\rangle=\langle\eta^{-1}_{(a,R)}T,f\rangle$ *for all* $f\in\mathcal{S}_R$, $T\in\mathcal{S}'_R$. *Then*

(a) η^{-1} *is a group homomorphism;*

(b) $\eta_{(a,R)}$ *is a homeomorphism of* $\mathcal{S}'_R$ *with respect to* $\sigma(\mathcal{S}'_R,\mathcal{S}_R)$;

(c) $\eta^{-1}_{(a,R)}\mathcal{A}_0=\mathcal{A}_0$;

(d) $\eta^{-1}_{(a,R)}\mathcal{A}=\mathcal{A}$;

(e) $\eta^{-1}_{(a,R)}$ *is measure preserving if and only if* $\mu[\eta^{-1}_{(a,R)}Z]=\mu[Z]$ *for each cylinder set* Z.

None of these facts are hard to verify. For (e), for each $(a,R)\in iSO(4)$ define a measure $\mu_{(a,R)}$ on $\mathcal{A}$ by $\mu_{(a,R)}[A]=\mu[\eta^{-1}_{(a,R)}A]$ with a measurable set A. That $\mu_{(a,R)}=\mu$ is a consequence of invariance on cylinder sets and the uniqueness of extensions from $\mathcal{A}_0$ to $\mathcal{A}$. The measure preserving nature of the euclidean flow on $\mathcal{A}_0$ returns us to the Riesz-Krein extension and the relation

$$\operatorname{ext} I_{f_1,\ldots,f_N}(\chi_B)=\operatorname{ext} I_{\alpha^{-1}(a,R)f_1,\ldots\alpha^{-1}(a,R)f_N}(\chi_B)$$

for all $(a,R)\in iSO(4)$ and Borel sets $B\subset R^N$. An extension of I which achieves this is to be obtained upon averaging on the euclidean group.

Theorem 3.8. *Let* M *be an invariant mean on* $\mathcal{L}_\infty(iSO(4))$ *and for each* (a,R) $\in iSO(4)$ *define*

$$\Lambda_B(a,R)=\operatorname{ext} I_{\alpha^{-1}(a,R)f_1,\ldots,\alpha^{-1}(a,R)f_N}(\chi_B)\,.$$

Then $\operatorname{inv.ext} I_{f_1,\ldots,f_N}(\chi_B)=M[\Lambda_B(\cdot)]$ *is a euclidean invariant Riesz-Krein extension whose related measure,* μ_{inv}, *is invariant with respect to the euclidean flow.*

Proof. The existence of an invariant mean on $\mathcal{L}_\infty(iSO(4))$ follows on general grounds [14, Theorem 2.2.1], and as $\Lambda_B(\cdot)$ is bounded we need only show that it is measurable. For this suppose $\varrho\geqq 0$ is a C^∞-function on R^N with compact support and $\int dt\,\varrho(t)=1$. Define $\varrho_\varepsilon(t)=\varepsilon^{-N}\varrho(t/\varepsilon)$ with $\varepsilon>0$ and put $\chi_{B,\varepsilon}=\chi_B*\varrho_\varepsilon(t)$. Some properties of these mollifiers follow:

(i) $\chi_{B,\varepsilon}\in C^\infty(R^N)$ for all $\varepsilon>0$ and Borel sets B;

(ii) $0\leqq\chi_{B,\varepsilon}\leqq 1$;

(iii) $\lim_{\varepsilon\downarrow 0}\chi_{B,\varepsilon}=\chi_B$ pointwise.

Now consider on $iSO(4)$ the functions

$$\Lambda_{B,\varepsilon}(a,R)=\int_{R^N}\chi_{B,\varepsilon}(t)\,d\nu_{\alpha^{-1}(a,R)f_1,\ldots,\alpha^{-1}(a,R)f_N}(t)$$

for $\varepsilon=1/n$; $n=1,2,3,\ldots$. From the continuity of μ, we conclude that $\{\Lambda_{B,1/n}\}$ is a sequence of continuous functions for which by dominated convergence satisfies

$$\lim_{n\to\infty}\Lambda_{B,1/n}(a,R)=\Lambda_B(a,R)\,.$$

$\Lambda_B(\cdot)$ is then measurable with respect to Haar measure on the euclidean group.

It is readily verified that inv. ext$I$ is a Riesz-Krein extension of I due to the euclidean invariance of ext$\boldsymbol{S}$.

As a final comment; an amusing property of invariant measures when restricted to cylinder sets is their ergodicity.

Proposition 3.9. *There exist no non-trivial cylinder sets invariant with respect to the translation homeomorphism.*

Proof. The first step is to show that for $\{f_1, f_2, \ldots, f_N\}$ linearly independent functions from $\mathscr{S}_R$, there exists $a_0 > 0$ such that the functions $\{f_1, f_2, \ldots, f_N, \alpha^{-1}(a) f_1, \ldots, \alpha^{-1}(a) f_N\}$ are also linearly independent for $\|a\| > a_0$. For suppose we apply the Gram test for linear independence to this second collection of functions. Typically, the Gram matrix has the following form in $N \times N$ blocks

$$G' = \left(\begin{array}{c|c} g_{ij} & h_{ij}(a) \\ \hline h_{ij}(a) & g_{ij} \end{array} \right)$$

where the entries are $g_{ij} = \int dt\, f_i(t) f_j(t)$, $h_{ij}(a) = \int dt\, f_i(t) f_j(t+a)$. Expanding $\det G'$ by a Laplace expansion on the first N rows produced the relation

$$\det G' = (\det G)^2 + H(a)\,.$$

G is the Gram matrix for the first collection of functions and has $\det G \neq 0$, while $H(a)$ is in $\mathscr{S}(R^4)$ as a function of a.

Secondly, let $Z_{\{f_1, \ldots, f_n\}}(B)$ be a non-trivial invariant cylinder set with a sufficiently large that $\{f_1, \ldots, f_N, \alpha^{-1}(a) f_1, \ldots, \alpha^{-1}(a) f_N\}$ form a basis for a subspace $\Psi \subset \mathscr{S}_R$ of dimension $2N$. Suppose $x \in B$ and $y \in R^N \setminus B$ and define $T \in \Psi'$ by the relations $\langle T, f_k \rangle = x_k$, $\langle T, \alpha^{-1} f_k \rangle = y_k$ for $k = 1, 2, \ldots, N$. The Hahn-Banach theorem provides for an extension of T to $\mathscr{S}'_R$, also written T, for which

$$T \in Z_{\{f_1, \ldots, f_N\}}(B),\ \eta_{(a,1)} T \in Z_{\{f_1, \ldots, f_N\}}(R^N \setminus B)\,.$$

So, Z cannot be invariant under all translation homeomorphisms $\eta_{(a,1)}$, $a \in R^4$.

4. The Euclidean Field

The primary objective for our investigation was to learn which extensions of Schwinger functionals would lead to a euclidean field theory of random variables. This is now accomplished by one of the standard constructions in probability theory applied to the measure space $(\mathscr{S}'_R, \mathscr{A}, \mu)$ corresponding to a strongly positive state. Throughout this section, μ will denote a measure for which the euclidean flow is measure preserving; i.e., μ is invariant.

Definition. 4.1. Define a mapping $\varphi : \mathscr{S}_R \to \mathscr{E}(\mathscr{S}'_R)$ by $\varphi(f)(T) = \langle T, f \rangle$ $\forall f \in \mathscr{S}_R$, $T \in \mathscr{S}'_R$; subject to $[\eta_{(a,R)} \varphi(f)](T) \equiv \varphi(f)(\eta^{-1}_{(a,R)} T)$ for each $(a, R) \in iSO(4)$.

The generalized random process φ will be the euclidean field appropriate to the underlying Osterwalder-Schrader theory. A number of properties for φ are immediate:

(a) φ is linear;

(b) $\varphi(f)$ is measurable for each $f \in \mathscr{S}_R$;

(c) $f_j \to 0$ in $\mathscr{S}_R$ implies $\varphi(f_j) \to 0$ pointwise on $\mathscr{S}'_R$;

(d) if $f_1, \ldots, f_N$ are linearly independent then $\nu_{f_1, f_2, \ldots, f_N}$ is the joint probability distribution for $\varphi(f_1), \ldots, \varphi(f_N)$;

(e) for a polynomial $P(f_1, f_2, \ldots, f_N)$,

$$\operatorname{ext} S(P(f_1, f_2, \ldots, f_N)) = \int P(\varphi(f_1), \varphi(f_2), \ldots, \varphi(f_N))\, d\mu\,;$$

(f) $\varphi(f) \in \mathscr{L}_p(\mathscr{S}_R, \mathscr{A}, \mu)$ for each $0 < p < \infty$ and for such p when $f_j \to 0$ in $\mathscr{S}_R$, $\|\varphi(f_j)\|_p \to 0$. In particular, $\varphi(f_j) \to 0$ in measure.

(g) φ is euclidean covariant; namely, $\eta_{(a,R)}\varphi(f) = \varphi(\alpha_{(a,R)} f)$ for each $(a, R) \in iSO(4)$.

4.1. Maximal Measures

Let us re-examine the euclidean field theory of § 2 for a strongly positive state. The relations

$$W\Omega_E = 1,\ WB(f_1^{j_1} \times f_2^{j_2} \times \cdots \times f_N^{j_N})\,\Omega_E = \varphi(f_1)^{j_1}\varphi(f_2)^{j_2} \ldots \varphi(f_N)^{j_N}$$

for $\{f_1, f_2, \ldots, f_N\} \subset \mathscr{S}_R$ and elsewhere on $\mathscr{P}(\mathscr{S}_R)$ by linearity define an isometry between $\mathscr{H}_E$ and the $\mathscr{L}_2(\mathscr{S}_{R'}, \mathscr{A}, \mu)$ closure of polynomials in $\varphi(f)$. A convenient representation for this closure, first due to Wiener for Brownian motion, has been given as follows [15]. If $\mathscr{P}_n$ denotes the complex linear span of all polynomials in $\varphi(f)$ with degree $\leqq n$, form the mutually orthogonal subspaces (homogeneous chaos)

$$\mathscr{Q}_0 = \mathscr{P}_0 .\ \mathscr{Q}_1 = \mathscr{P}_1 \ominus \mathscr{P}_0, \ldots, \mathscr{Q}_n = \mathscr{P}_n \ominus \mathscr{P}_{n-1}, \ldots .$$

Then $\mathscr{P}(\mathscr{S}_R) = \bigcup_{n=0}^{\infty} \mathscr{P}_n$ and $\mathscr{H}_E = \bigoplus_{n=0}^{\infty} \overline{\mathscr{Q}_n}$ ($\mathscr{L}_2$-closure). One of the interesting questions for generalized random processes concerns the nature of $\mathscr{H}_E$ as a closed subspace of $\mathscr{L}_2(\mathscr{S}'_R, \mathscr{A}, \mu)$. For a generalized free field, the extension process leads to a Gaussian measure μ for which the homogeneous chaos is dense in $\mathscr{L}_2$ [16]. We next give conditions on the measure μ for this to be true in the generality discussed here.

Definition 4.2. Let μ be a measure corresponding to a strongly positive state on $\mathscr{S}$. μ is called maximal if each of the one dimensional measures ν_f, $f \in \mathscr{S}_R$, is N-extremal.

Theorem 4.3. *Suppose μ is maximal. Then polynomials in $\varphi(f)$, $f \in \mathscr{S}_R$, are dense in $\mathscr{L}_p(\mathscr{S}_{R'}, \mathscr{A}, \mu)$ for $1 \leqq p < \infty$. In particular, $\mathscr{H}_E = \mathscr{L}_2(\mathscr{S}'_R, \mathscr{A}, \mu)$.*

Proof. The proof is similar to the one for Theorem 3.5 after two remarks. It is easily verified that $\mathscr{A}$ is the σ-algebra generated by $\{\varphi(f) \mid f \in \mathscr{S}_R\}$. Moreover $\mathscr{A}_0$ is dense in $\mathscr{A}$ in the sense that to each $A \in \mathscr{A}$, $\varepsilon > 0$, there exists a cylinder set Z such that for the symmetric difference $A \Delta Z$, one has $\mu(A \Delta Z) < \varepsilon^2$. If $Z = Z_{\{f_1, f_2, \ldots, f_N\}}(B)$, then there is a finite family $\{B_{\alpha(1)} \times \cdots \times B_{\alpha(N)}\}$, $\alpha = 1, \ldots, P$; of disjoint Borel sets in R^N with each $B_{\alpha(k)}$ a Borel set in R for which

$$\nu_{f_1, \ldots f_N}\left(B \Delta \bigcup_{\alpha=1}^{P} B_{\alpha(1)} \times \cdots \times B_{\alpha(N)}\right) < \varepsilon^2 .$$

The method used in Theorem 3.5 may now be modified to approximate each characteristic function $\chi_{B_{\alpha(k)}}$ by a polynomial $P_{\alpha(k)}(\varphi(f_k))$ in $\mathscr{L}_p$-norm for $2<p<\infty$.

Corollary. *When the measure μ is unique, $\mathscr{H}_E=\mathscr{L}_2(\mathscr{S}'_R,\mathscr{A},\mu)$.*

A partial converse to Theorem 4.3 is apparently more subtle, Theorem 4.4.

4.2. Ergodicity

Our discussion of the measure μ has not so far involved any euclidean cluster property. For physical applications, it should be expected that this property will be extremely important. Let us suppose that the polynomials in the euclidean field are dense in $\mathscr{L}_2$; for example, μ is maximal. Then given two measurable sets $A, B\in\mathscr{A}$ with characteristic functions χ_A,χ_B; we may determine for given $\varepsilon>0$ polynomials $P(\varphi(f_1),\ldots,\varphi(f_N))$, $Q(\varphi(g_1),\ldots,\varphi(g_M))$ such that

$$\|\chi_A-P\|_2<\varepsilon,\qquad \|\chi_B-Q\|_2<\varepsilon\,.$$

As a consequence, for any translation homeomorphism

$$\begin{aligned}|\mu(A\cap\eta_a^{-1}B)-\mu(A)\,\mu(B)|\leqq 4\varepsilon(1+\varepsilon)+|\operatorname{ext}\boldsymbol{S}(P\times\alpha(a)\,Q)\\ -\operatorname{ext}\boldsymbol{S}(P)\operatorname{ext}\boldsymbol{S}(Q)|\,.\end{aligned}$$

The conclusion is that the translation flow, or loosely speaking μ; is ergodic, weakly mixing, mixing if and only if the following conditions hold on the strongly positive Schwinger state

$$\text{ergodic} - \lim_{T\to\infty}\frac{1}{T}\int_0^T dt\operatorname{ext}\boldsymbol{S}(P\times\alpha(t)Q)=\operatorname{ext}\boldsymbol{S}(P)\operatorname{ext}\boldsymbol{S}(Q)\,;$$

$$\text{weakly mixing} - \lim_{T\to\infty}\frac{1}{T}\int_0^T dt\,|\operatorname{ext}\boldsymbol{S}(P\times\alpha(t)Q)-\operatorname{ext}\boldsymbol{S}(P)\operatorname{ext}\boldsymbol{S}(Q)|=0\,;$$

$$\text{mixing} - \lim_{T\to\infty}\operatorname{ext}\boldsymbol{S}(P\times\alpha(T)Q)=\operatorname{ext}\boldsymbol{S}(P)\operatorname{ext}\boldsymbol{S}(Q)\,;$$

for any two polynomials P and Q. Here $\alpha(t)=\alpha((t,\vec{0}))$ which is sufficient for the whole transition group due to euclidean and θ-invariance. The latter is an implication of Propositions 2.2 and 3.6.

As an illustration of properties for a mixing translation flow, we examine a converse to Theorem 4.3.

Theorem 4.4. *Let μ be a measure constructed from a strongly positive state which has the mixing cluster property. Then polynomials in the euclidean field are dense in $\mathscr{L}_2(\mathscr{S}'_R,\mathscr{A},\mu)$ if and only if μ is maximal.*

Proof. It is sufficient to look at real functions, so consider $\alpha^{-1}(a)f\in\mathscr{S}_R$ and denote the sub σ algebra of $\mathscr{A}$ generated by $\varphi(\alpha^{-1}(a)f)$ by $\mathscr{F}_a$. For each $a\in R^4$, $\mathscr{F}_a$ is the Borel sets on R. If F is a positive or integrable random variable on $(\mathscr{S}'_R,\mathscr{A},\mu)$ indicate the conditional expectation of F with respect to $\mathscr{F}_a$ by $\mathscr{F}_a[F]$.

Suppose $B\subset R$ is a Borel set with characteristic function χ_B and $\{f,f_1,\ldots,f_N\}$ are linearly independent functions from $\mathscr{S}_R$. Then using the invariance of μ and the mixing property for the flows $\eta_a : \mathcal{S}'_R \to \mathcal{S}'_R$, $a \in R^4$,

$$\begin{aligned}
&\int \chi_B \mathcal{F}_0 [\varphi(f_1)^{j_1} \dots \varphi(f_N)^{j_N}] \, d\nu_f \\
&= \lim_{\|a\| \to \infty} \int \chi_B \mathcal{F}_a [\varphi(f_1)^{j_1} \dots \varphi(f_N)^{i_N}] \, d\nu_{\alpha^{-1}(a)f} \\
&= \lim_{\|a\| \to \infty} \int \chi_B \varphi(f_1)^{j_1} \dots \varphi(f_N)^{j_N} d\nu_{\alpha^{-1}(a) f f_1 \dots f_N} \\
&= c \int \chi_B d\nu_f \,.
\end{aligned}$$

The constant c is the expectation of $\varphi(f_1)^{i_1} \dots \varphi(f_N)^{j_N}$.

This implies that $\mathcal{F}_0 [\varphi(f)^j \varphi(f_1)^{j_1} \dots \varphi(f_N)^{j_N}] = c\varphi(f)^j$ or rather since $\mathcal{F}_0$ is the orthogonal projection from $\mathcal{L}_2(\mathcal{S}'_R, \mathcal{A}, \mu)$ onto $\mathcal{L}_2(R, d\nu_f)$ that $\mathcal{F}_0$ maps polynomials in the euclidean field onto polynomials in $\varphi(f)$. Thus, the latter polynomials are dense in $\mathcal{L}_2(R, d\nu_f)$ and by Riesz's result ν_f is N-extremal.

Remark. The mixing assumption in Theorem 4.4 could equally well be replaced by ergodicity.

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