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Sommerfeld-Watson Representation for Double-Spectral Functions

II. Crossing Symmetric Pion-Pion Scattering Amplitude without Regge Poles

J. S. Frederiksen

Research School of Physical Sciences, The Australian National University, Canberra, Australia 2600

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Abstract. We discuss, for the case of pion-pion scattering, a closed system of equations which may be used for a self-consistent calculation of partial-wave amplitudes. It is shown that, for a given sufficiently small input function, the equations have a locally unique solution in a particular Banach space of doubly Hölder continuous partial wave amplitudes. At a fixed point, the scattering amplitude is shown to satisfy both a crossing symmetric unsubtracted Mandelstam representation and the elastic unitarity condition. In this initial study the partial-wave amplitudes are holomorphic in the right half complex angular-momentum plane.

1. Introduction

Since Mandelstam [1] in 1958, proposed the double spectral representation for the two-particle scattering amplitude there have been numerous attempts to solve the crossing-unitarity equations [2]. While dispersion theory has led to some fruitful phenomenological correlations of experimental data, these attempts have on the whole met with limited success.

Several years ago, Atkinson [3] proposed a list of remedies for the then diseased state of strong-interaction dynamics; parts of this programme were subsequently implemented [3–5]. In particular, it was shown by means of fixed-point theorems that there exist crossing symmetric pion-pion scattering amplitudes satisfying the Mandelstam representation with elastic unitarity between the elastic and inelastic thresholds and the inelastic unitarity inequality above the latter. However, if more than one subtraction is needed in the double dispersion relation for the amplitude, it has not been possible to guarantee positivity and boundedness of the partial-wave amplitude $A(s, l)$ as $s\to\infty$. Now it is known that the f-meson resonance, with spin 2 occurs in the pion-pion spectrum and thus, at least three subtractions are expected in the dispersion relation.

The difficulty with the iteration equations for the double spectral function lies in the fact that, for the partial-waves to be bounded as $s\to\infty$, the double spectral function must have infinite oscillations. However, it is very difficult to incorporate suitable oscillations in the Banach space of double spectral functions. It was proposed in a previous paper [6] (referred to as I) that a likely way of overcoming this problem is to replace the iteration equations for the double spectral function by a closed system of equations for $A(s, l)$, the continuation of the partial-wave amplitude into the complex angular-momentum plane. At a fixed point of these equations, the double spectral function may, for the case when $A(s, l)$ contains Regge poles in the right half l-plane, be written as the sum of a Sommerfeld-Watson background integral and Regge terms. The contribution from the background integral should be the same as when $A(s, l)$ is holomorphic in the right half l-plane and thus would not necessitate subtractions in the dispersion relation. The Regge terms which lead to subtractions are given explicitly in terms of Legendre functions and the Regge trajectory and residue functions. For the sort of trajectory and residue functions that are expected on the basis of potential theory [2], the Regge terms will in general oscillate. Thus there is a real hope of overcoming the problem of guaranteeing boundedness of the partial-wave amplitudes; it seems that the positivity condition may have to be relegated to a numerical investigation.

In this paper, we examine the equations which define the mapping $A(s, l) \to A(s, l)$ for neutral pion-pion scattering in the case when $A(s, l)$ is holomorphic in the right half l-plane. The extension to charged pion-pion scattering is straightforward but it seems desirable to avoid the extra notational complications that the introduction of the isospin matrices involves. Section 2 describes the equations which consist of the Froissart-Gribov representation for $A(s, l)$, an unsubtracted dispersion relation for the t-discontinuity of the scattering amplitude and a Sommerfeld-Watson representation for the elastic contribution to the double spectral function. The Banach space within which solutions of these equations are sought is described and an explanation is given of how one may show that the equations have solutions if the input functions are sufficiently small. The technical aspects of this demonstration are given in Sections 3 and 4. Section 5 gives the constraints on various indices which must be satisfied if the proof is to work. It is also shown that, at a fixed point of the equations, one may construct a scattering amplitude which satisfies an unsubtracted Mandelstam representation.

2. Sommerfeld-Watson Transform

The Sommerfeld-Watson transform for the neutral pion-pion scattering amplitude is given by

$$F(s,t) = \frac{i}{2} \int_{-\frac{1}{2}+\varepsilon-i\infty}^{-\frac{1}{2}+\varepsilon+i\infty} \frac{dl(2l+1)\,A(s,l)}{\sin\pi l}\, \tfrac{1}{2}[P_l(-z)+P_l(z)]\,. \tag{2.1}$$

To ensure tu crossing symmetry, $\frac{1}{2}[P_l(-z)+P_l(z)]$ replaces the Legendre function which appeared in Eq. (I-2.1). [Equations from I are denoted by placing I- in front of the equation number.] The contour of integration is again a straight line parallel to the imaginary axis in the complex l-plane, and $A(s, l)$ is the partial wave amplitude which is a holomorphic function of l for $\mathrm{Re}\, l \geqq -\frac{1}{2}+\varepsilon$. The partial wave amplitudes will be constructed such that they have this property and such that $F(s, t)$ in Eq. (2.1) is well defined. The parameter ε satisfies

$$0<\varepsilon<\tfrac{1}{2} \tag{2.2}$$

and s and t are the usual Mandelstam variables.

We now change the integration variable to y, defined by

$$l = -\tfrac{1}{2} + \varepsilon + iy \tag{2.3}$$

and we use the reduced amplitude

$$B(s, y) = (s-4)^{-l} A(s, l) . \tag{2.4}$$

As discussed in Section 5 of I, one simplification that occurs in the case of relativistic pion-pion scattering is that one may insert a Hölder-continuous cut-off function $h(s)$ into the definition of $\varrho^{el}(s, t)$, the elastic contribution to the double spectral function; $h(s)$ is equal to unity in the elastic region $4 \leqq s \leqq 16$ and then decreases to zero for $s \geqq 16 + \Delta$ where $\Delta > 0$. This is possible since elastic unitarity does not now hold for $s > 16$ and thus $\varrho^{el}(s, t)$ is only constrained to be equal to the full double spectral function $\varrho(s, t)$ in the elastic region.

Consequently, we define

$$\varrho^{el}(s, t) = \theta(t-16)\, \theta(s-\tau(t))\, h(s) \frac{i}{2} q(s) \int_{-\infty}^{\infty} dy (y - i\varepsilon)\, P_{-\frac{1}{2}+\varepsilon+iy}(z)\, (s-4)^{-1+2\varepsilon+2iy} \cdot B(s_+, y)\, B(s_-, y) , \tag{2.5}$$

where

$$\tau(t) = \frac{4t}{t-16} , \tag{2.6}$$

$$q(s) = \left(\frac{s-4}{s}\right)^{\frac{1}{2}} , \tag{2.7}$$

and the suffices $\pm$ mean that the boundary values of the function must be taken respectively above and below the cut on the real s-axis. A suitable definition of the cut-off function is

$$h(s) = \begin{cases} 1 & : 4 \leqq s \leqq 16 , \\ \frac{1}{2} + \frac{1}{2} \cos \frac{\pi}{\Delta} (s-16) & : 16 < s \leqq 16 + \Delta, \ \Delta > 0 , \\ 0 & : s > 16 + \Delta . \end{cases} \tag{2.8}$$

The crossing symmetric double spectral function or double discontinuity of $F(s, t)$ can then be defined by

$$\varrho(s, t) = \varrho^{el}(s, t) + \varrho^{el}(t, s) + v(s, t) \tag{2.9}$$

for $t \geqq 4$, $s \geqq \Sigma(t)$ and otherwise it is equal to zero. The boundary curve $s = \Sigma(t)$ is given by

$$\Sigma(t) = \min \{\tau(t), \sigma(t)\} , \tag{2.10}$$

where

$$\sigma(t) = \frac{16t}{t-4} \tag{2.11}$$

and $v(s, t)$ is a symmetric input function which is supposed given and which vanishes for $s < 16$, $t < V(s)$ where

$$V(s) \geqq \max \{16, \Sigma(s)\} . \tag{2.12}$$

The explicit form for $V(s)$ that one would expect on the basis of perturbation theory was discussed in Ref. [3]; for our purposes however we shall only require that the above inequality is satisfied and we assume that $V(s)\to\infty$ as $s\downarrow 16$, $V(s)\to 16$ as $s\to\infty$.

The set of equations which replace the potential scattering equations studied in I and which define a mapping $B(s,y)\to\bar{B}(s,y)$ then consists of Eq. (2.5) together with the following equations:

$$D(s,t)=\frac{1}{\pi}\int_{\Sigma(t)}^{\infty} ds'[\varrho^{el}(s',t)+\varrho^{el}(t,s')]\left[\frac{1}{s'-s}+\frac{1}{s'-4+s+t}\right], \tag{2.13}$$

$$\bar{B}(s,y)=\frac{4}{\pi}(s-4)^{-\frac{1}{2}-\varepsilon-iy}\int_{4}^{\infty} dt\,Q_{-\frac{1}{2}+\varepsilon+iy}(z)\,D(s,t)+V(s,y), \tag{2.14}$$

$$\begin{aligned}V(s,y)=\frac{4}{\pi^2}(s-4)^{-\frac{1}{2}-\varepsilon-iy}\int_{16}^{\infty} ds'\int_{V(s')}^{\infty} dt\,Q_{-\frac{1}{2}+\varepsilon+iy}(z)\,v(s',t)\\ \cdot\left[\frac{1}{s'-s}+\frac{1}{s'-4+s+t}\right],\end{aligned} \tag{2.15}$$

where

$$z=1+\frac{2t}{s-4}. \tag{2.16}$$

The extra factor in Eq. (2.14), as compared with Eq. (I-2.6) is due to the presence of the u-channel.

Our object is now to find a suitable Banach space of functions and to show that if $B(s,y)$ and $V(s,y)$ belong to this space and if $\|V\|$ is sufficiently small then there exists a locally unique fixed point $\bar{B}(s,y)=B(s,y)$. We shall find that there are in fact classes of input functions $v(s,t)$ for which $V(s,y)$ belongs to a suitable space and which have appropriate large s and t behaviour such that, at a fixed point, an unsubtracted Mandelstam representation for $F(s,t)$ may be constructed.

We shall choose to look for solutions in a Banach space which is specified by a norm that is a generalization of that used in I and which allows one to extend the method of proof of I in a relatively straightforward way. After some experimentation, it was found satisfactory to look for solutions in a Banach space of functions, $f(s,y)$, specified by means of the following norm

$$\begin{aligned}\|f\|&=\sup\{s_2{}^{\lambda}|y_2+i|^{\frac{1}{2}+v}f(s_2,y_2)\}\\&+\sup\left\{s_2{}^{\lambda}|y_2+i|^{\frac{1}{2}+v}\frac{|f(s_1,y_1)-f(s_1,y_2)-f(s_2,y_1)+f(s_2,y_2)|}{\left|\frac{s_1-s_2}{s_1}\right|^{\mu}\left|\frac{y_1-y_2}{y_1+i}\right|^{\varrho}}\right\}\\&+\sup\left\{s_2{}^{\lambda}\left(\frac{s_2-4}{s_2}\right)|y_2+i|^{-\frac{1}{2}+v}|f_s(s_2,y_2)|\right\}\\&+\sup\left\{s_2{}^{\lambda}\left(\frac{s_2-4}{s_2}\right)^{1+\mu}|y_2+i|^{-\frac{1}{2}+v}\frac{|f_s(s_1,y_1)-f_s(s_1,y_2)-f_s(s_2,y_1)+f_s(s_2,y_2)|}{\left|\frac{s_1-s_2}{s_2}\right|^{\mu}\left|\frac{y_1-y_2}{y_1+i}\right|^{\varrho}}\right\}.\end{aligned} \tag{2.17}$$

The suprema are to be taken over $s_1 > s_2 \geqq 4$, $-\infty < y_2 < \infty$, $|y_1| > |y_2|$ and the indices are subject to a number of restrictions which we give in detail in Section 5. We note that the first two terms in Eq. (2.17) correspond to the norm used in I [Eq. (I-2.9)]. Moreover, if $\|f\| < \infty$ then, with $\lambda > 0$, $\mu > 0$, $\varrho > 0$, $v > \frac{1}{2}$,

$$|f_s(s, y)| \leqq \|f\| \, s^{-\lambda} \left(\frac{s-4}{s}\right)^{-1} |y+i|^{\frac{1}{2}-v}, \tag{2.18a}$$

$$|f_s(s_1, y) - f_s(s_2, y)| \leqq \|f\| \, s_2^{-\lambda} \left(\frac{s_2-4}{s_2}\right)^{-1-\mu} \left|\frac{s_1-s_2}{s_1}\right|^{\mu} |y+i|^{\frac{1}{2}-v}, \tag{2.18b}$$

$$|f_s(s, y_1) - f_s(s, y_2)| \leqq \|f\| \, s^{-\lambda} \left(\frac{s-4}{s}\right)^{-1} |y_2+i|^{\frac{1}{2}-v} \left|\frac{y_1-y_2}{y_1+i}\right|^{\varrho}, \tag{2.18c}$$

$$\begin{aligned} &|f_s(s_1, y_1) - f_s(s_1, y_2) - f_s(s_2, y_1) + f_s(s_2, y_2)| \\ &\quad \leqq \|f\| \, s_2^{-\lambda} \left(\frac{s_2-4}{s_2}\right)^{-1-\mu} \left|\frac{s_1-s_2}{s_1}\right|^{\mu} |y_2+i|^{\frac{1}{2}-v} \left|\frac{y_1-y_2}{y_1+i}\right|^{\varrho}. \end{aligned} \tag{2.18d}$$

A similar set of inequalities hold with the function, $f(s, y)$, replacing its derivative, $f_s(s, y)$, and with factors $\left(\frac{s-4}{s}\right)^{-1} |y+i|$ missing on the right hand sides; the relations were given explicitly in Eq. (I-2.10).

It may be of interest to briefly summarize the reasons for choosing the norm (2.17). We begin by recalling that there were two crucial points in the proof of I. First, there was a contribution to $\bar{B}(s, y)$ which contained in its integral representation a singular y'-integral as well as a singular s'-integral. This led us to choosing a Banach space of doubly Hölder-continuous functions. Secondly, we found in Section 4 of I that in order to obtain the required large y-behaviour for $|\bar{B}(s, y)|$ we needed to integrate by parts in the t-integral [Eq. (I-4.4)]. This was because $|Q_{-\frac{1}{2}+\varepsilon+iy}(z)|$ behaves only like $|y+i|^{-\frac{1}{2}}$ for large y whereas we needed $|\bar{B}(s, y)|$ to behave like $|y+i|^{-\frac{1}{2}-v}$, where v is the parameter occurring in Eq. (I-2.9). These same considerations apply to the relativistic case. However, since our equations now also involve $\varrho^{el}(t, s)$, the t-integral in Eq. (2.14) will contain the product of the partial wave amplitudes $B(t_+, y)\, B(t_-, y)$. On integrating by parts to obtain the required large y-behaviour of the crossed contribution to $\bar{B}(s, y)$ one finds that bounds on the derivative with respect to t of $B(t, y)$ are needed. Thus, a norm of the form given in (2.17) suggested itself.

Most of the rest of this paper will be devoted to demonstrating that if $\|B\| < \infty$ and $\|V\| < \infty$ then $\bar{B}(s, y)$ is well defined by Eqs. (2.5) and (2.13)–(2.15) and further, that a constant $\varkappa$ exists such that

$$\|\bar{B}\| \leqq \varkappa \|B\|^2 + \|V\|. \tag{2.19}$$

Because of the quadratic nature of the elastic unitarity relation used in deriving Eq. (2.5), it follows by an immediate extension of the proof that, if $B_a(s, y)$ and $B_b(s, y)$ are any two functions belonging to the space, the corresponding image functions satisfy

$$\|\bar{B}_a - \bar{B}_b\| \leqq \varkappa \{\|B_a\| + \|B_b\|\} \, \|B_a - B_b\|. \tag{2.20}$$

Then, from the Banach-Cacciopoli contraction mapping principle [3, 7], it can be shown that, for a given $V(s, y)$ such that $\|V\| < (4\varkappa)^{-1}$, there is a locally unique fixed point $\bar{B}(s, y) = B(s, y)$ in the ball

$$\|B\| \leqq \frac{1 - [1 - 4\varkappa \|V\|]^{\frac{1}{2}}}{2\varkappa}. \tag{2.21}$$

To begin the proof of (2.19) we combine Eqs. (2.5), (2.13), and (2.14), and write $\bar{B}(s, y)$ in the form

$$\bar{B}(s, y) = B^{(1)}(s, y) + B^{(2)}(s, y) + B^{(3)}(s, y) + B^{(4)}(s, y) + V(s, y). \tag{2.22}$$

Here $B^{(1)}(s, y)$, $B^{(2)}(s, y)$, $B^{(3)}(s, y)$, and $B^{(4)}(s, y)$ are the contributions to $\bar{B}(s, y)$ when the factors $\varrho^{el}(s', t)(s' - s)^{-1}$, $\varrho^{el}(s', t)(s' - 4 + s + t)^{-1}$, $\varrho^{el}(t, s')(s' - s)^{-1}$, and $\varrho^{el}(t, s')(s' - 4 + s + t)^{-1}$ respectively replace the integrand in Eq. (2.13). In the next three sections we shall study the boundedness and Hölder continuity of $B^{(1)}(s, y)$, $B^{(2)}(s, y)$, $B^{(3)}(s, y)$, $B^{(4)}(s, y)$ and $V(s, y)$.

3. Study of $B^{(1)}(s, y)$ and $B^{(2)}(s, y)$

To study the boundedness and Hölder-continuity of $B^{(1)}(s, y)$ it is convenient to proceed as in I by writing it as the sum of two terms:

$$B^{(1)}(s, y) = B_1^{(1)}(s, y) + B_2^{(1)}(s, y), \tag{3.1}$$

where

$$\begin{aligned} B_n^{(1)}(s, y) = \frac{2i}{\pi^2} \int_4^\infty \frac{ds'}{s' - s} h(s') q(s') \int_{-\infty}^{\infty} dy'(y' - i\varepsilon) B(s'_+, y') B(s'_-, y') \\ \cdot (s' - 4)^{-\frac{1}{2} + \varepsilon + 2iy'} \Lambda_n(s, s'; y, y'), \end{aligned} \tag{3.2}$$

$n = 1, 2$. Here, Λ_1 and Λ_2 are defined, in analogy with Eqs. (I-2.18) and (I-2.19), by

$$\begin{aligned} \Lambda_1(s', y, y') = \frac{\frac{1}{2}(s' - 4)^{-iy}(z_0'^2 - 1)}{(y' - y)(y' + y - 2i\varepsilon)} \\ \cdot [Q_{-\frac{1}{2}+\varepsilon+iy}(z_0') P'_{-\frac{1}{2}+\varepsilon+iy'}(z_0') - Q'_{-\frac{1}{2}+\varepsilon+iy}(z_0') P_{-\frac{1}{2}+\varepsilon+iy'}(z_0')], \end{aligned} \tag{3.3}$$

$$\begin{aligned} \Lambda_2(s, s', y, y') = \int_{\sigma(s')}^{\infty} dt \\ \cdot \left[(s - 4)^{-\frac{1}{2}-\varepsilon-iy} Q_{-\frac{1}{2}+\varepsilon+iy}\left(1 + \frac{2t}{s-4}\right) - (s' - 4)^{-\frac{1}{2}-\varepsilon-iy} Q_{-\frac{1}{2}+\varepsilon+iy}\left(1 + \frac{2t}{s'-4}\right)\right] \\ \cdot (s' - 4)^{-\frac{1}{2}+\varepsilon} P_{-\frac{1}{2}+\varepsilon+iy'}\left(1 + \frac{2t}{s' - 4}\right), \end{aligned} \tag{3.4}$$

where

$$z_0' = 1 + \frac{32s'}{(s' - 4)^2}. \tag{3.5}$$

The treatment of $B_1^{(1)}$ and $B_2^{(1)}$ can then be carried out in the manner described in Sections 3 and 4 of I respectively. The change in the definition of $\sigma(s')$ and z_0'

does not alter the proof, while the presence of the relativistic phase factor $q(s')$ and the cut-off function $h(s')$ means that one can obtain bounds on $B^{(1)}(s, y)$ which are better behaved for large s than was the case in I. We find, on using the fact that $h(s')$ is Hölder continuous:

$$|h(s'_1) - h(s'_2)| \leqq \varkappa h(s'_2) \left| \frac{s'_1 - s'_2}{s'_1} \right|^{\mu}, \tag{3.6}$$

that

$$|B^{(1)}(s, y)| \leqq \varkappa \|B\|^2 s^{-1+p(1-\eta)+\delta} |y+i|^{-\frac{3}{2}+\mu+\eta}, \tag{3.7a}$$

$$|B^{(1)}(s_1, y_1) - B^{(1)}(s_2, y_1) - B^{(1)}(s_1, y_2) + B^{(1)}(s_2, y_2)|$$
$$\leqq \varkappa \|B\|^2 s^{-1+p(1-\eta)+\mu+\delta} \left| \frac{s_1 - s_2}{s_1} \right|^{\mu} \left| \frac{y_1 - y_2}{y_1 + i} \right|^{\varrho} |y_2 + i|^{-\frac{3}{2}+\mu+\varrho+\eta+\delta}. \tag{3.7b}$$

Here

$$p = \begin{cases} \frac{1}{2} - \varepsilon & \text{if } 0 < \varepsilon \leqq \frac{1}{4} \\ \frac{1}{4} & \text{if } \frac{1}{4} < \varepsilon < \frac{1}{2} \end{cases}, \tag{3.8}$$

and, as in Eq. (I-4.16),

$$0 < \eta < \tfrac{1}{2} - 3\mu - 2\varrho, \tag{3.9}$$

where

$$0 < \mu < \min(\tfrac{1}{8}, \varepsilon), \tag{3.10}$$

$$0 < \varrho < \tfrac{1}{16}. \tag{3.11}$$

In Eq. (3.7), and in the following, $\varkappa$ is a generic constant which may change from one line to the next: the important point is that there exists such a number and that it depends only on the various indices. Similarly δ will be taken to be a generic small positive number which may be as small as one pleases, and which may change from one equation in which it occurs to the next. To ensure that the y' integral in Eq. (3.2) (with $n=2$) is absolutely convergent, we require that the index υ, appearing in Eq. (2.17), satisfies

$$\upsilon > 3/4 - \eta/2 + \mu/2. \tag{3.12}$$

Next we study the boundedness and Hölder-continuity of $B_s^{(1)}(s, y)$, the derivative with respect to s of $B^{(1)}(s, y)$. From Theorem 4 of Appendix B we have

$$B_s^{(1)}(s, y) = B_s^{(1a)}(s, y) + B_s^{(1b)}(s, y) + B_s^{(1c)}(s, y) \tag{3.13a}$$

where

$$B_s^{(1a)}(s, y) = B_{s1}^{(1a)}(s, y) + B_{s2}^{(1a)}(s, y), \tag{3.13b}$$

$$B_{sn}^{(1a)}(s, y) = \frac{2i}{\pi^2} \int_4^{\infty} \frac{ds'}{s'-s} \int_{-\infty}^{\infty} dy'(y' - i\varepsilon) \frac{\partial}{\partial s'}$$
$$\cdot [(h(s')\, q(s'))\, B(s'_+, y') \cdot B(s'_-, y')\, (s'-4)^{-\frac{1}{2}+\varepsilon+iy'}]\, (s'-4)^{iy'} \Lambda_n(s, s', y, y'), \tag{3.14}$$

$n = 1, 2$. Further,

$$B_s^{(1b)}(s, y) = \frac{2i}{\pi^2} \int_4^{\infty} \frac{ds'}{s'-s} \int_{-\infty}^{\infty} dy'(y' - i\varepsilon)\, h(s')\, q(s')\, B(s'_+, y')$$
$$\cdot B(s'_-, y')\, (s'-4)^{-\frac{1}{2}+\varepsilon+2iy'} \Lambda_{2s}(s, s's, y, y') \tag{3.15a}$$

and $B_s^{(1d)}(s, y)$ [referred to as Eq. (3.15b)] is defined by Eq. (3.15a) with $(s'-4)^{iy'} \cdot \Lambda_{2s}(s, s', y, y')$ replaced by $\Gamma(s, s', y, y')$, where

$$\Gamma(s, s', y, y') = \frac{\partial}{\partial s'} \int\limits_{\sigma(s')}^{\infty} dt \frac{P_{-\frac{1}{2}+\varepsilon+iy'}\left(1+\frac{2t}{s'-4}\right)}{(s'-4)^{\frac{1}{2}-\varepsilon-iy'}} \frac{Q_{-\frac{1}{2}+\varepsilon+iy}\left(1+\frac{2t}{s-4}\right)}{(s-4)^{\frac{1}{2}+\varepsilon+iy}}. \tag{3.16}$$

From Eqs. (3.2) and (3.14) we see that $B_{sn}^{(1a)}$ is given by the same expression as $B_n^{(1)}$ except that the function

$$h(s')\, q(s')\, B(s'_+, y')\, B(s'_-, y')\, (s'-4)^{-\frac{1}{2}+\varepsilon+iy'}$$

is replaced by its derivative with respect to s'. Thus the method of obtaining bounds and demonstrating the double Hölder continuity of $B_{sn}^{(1)}$ is essentially the same as that used for $B_n^{(1)}$. We note however, that an extra factor $(s'-4)^{-1}\,|y'+i|$ appears in the bound on the derivative with respect to s' of

$$h(s')\, q(s')\, B(s'_+, y')\, B(s'_-, y')\, (s'-4)^{-\frac{1}{2}+\varepsilon+iy'}$$

compared with the bound on this function itself. This means that the theorems in Appendix B which allow for singular behaviour of the weight functions at the end points of integration are needed to replace the corresponding theorems in Appendix B of I. Moreover, because of the introduction of the extra factor $|y'+i|$, the bound on Λ_2 given in Eq. (I-4.8) is unsuitable for deriving the required bound on $B_{s2}^{(1a)}$. However, instead of integrating by parts in Eq. (3.4) to obtain the alternative expression for Λ_2 in Eq. (I-44), we could have integrated by parts with the roles of the Legendre functions P and Q interchanged. Taking a compromise between the resultant bound and that given in Eq. (I-4.2) we have

$$h(s')\,|\Lambda_2(s, s', y, y')| \leqq \varkappa\, |y'+i|^{-\frac{1}{2}-\eta}\, |y+i|^{-\frac{1}{2}+\eta+\delta}\, s^{\frac{1}{4}(1-\eta)+\delta}, \tag{3.17}$$

where

$$0 < \eta < 1, \tag{3.18}$$

Here we have used the bounds on the Legendre and hypergeometric functions given in Appendix A and in Appendix A of I. Similar bounds for the double Hölder differences in the variables sy, $s'y$ can also be established by using the mean value theorems [cf. Eq. (I-3.8)]. Finally, using the fact that the derivative with respect to s' of $h(s')$ is a Hölder continuous cut-off function, we find that

$$|B_s^{(1a)}(s, y)| \leqq \varkappa\, \|B\|^2 \left(\frac{s}{s-4}\right)^{1-\varepsilon} s^{-\frac{3}{4}-\frac{\eta}{4}+\delta}\, |y+i|^{-\frac{1}{2}+\mu+\eta}, \tag{3.19a}$$

$$\begin{aligned} &|B_s^{(1a)}(s_1, y_1) - B_s^{(1a)}(s_2, y_1) - B_s^{(1a)}(s_1, y_2) + B_s^{(1a)}(s_2, y_2)| \qquad (3.19\text{b}) \\ &\leqq \varkappa\, \|B\|^2 \left(\frac{s_2}{s_2-4}\right)^{1+\mu-\varepsilon} s_2^{-\frac{3}{4}-\frac{\eta}{4}+\mu+\delta}\, |y_2+i|^{-\frac{1}{2}+\mu+\varrho+\eta+\delta} \left|\frac{s_1-s_2}{s_1}\right|^{\mu} \left|\frac{y_1-y_2}{y_1+i}\right|^{\varrho}, \end{aligned}$$

where η, μ, ϱ, and υ must satisfy inequalities (3.9), (3.10), (3.11), and (3.12) respectively.

We note that $B_s^{(1b)}$ is given by the same expression as $B_2^{(1)}$ except that Λ_{2s} replaces Λ_2; its boundedness and Hölder continuity can be studied by a method similar to that described in Section 4 of I. By starting with the expression for

Λ_2 in Eq. (3.4) and using the bounds on the Legendre functions given in Appendix A of I one can obtain a bound for Λ_{2s}. Similarly, from the expression for Λ_2 in Eq. (I-4.4) one can derive an alternative bound for Λ_{2s}. Compromising between these two bounds we have

$$h(s')\,|\Lambda_{2s}(s,s',y,y')| \leqq \varkappa\,|y'+i|^{\frac{1}{2}-\eta}\,|y+i|^{-\frac{1}{2}+\eta}\,s^{\frac{1}{4}(1-\eta)}, \tag{3.20}$$

where η satisfies inequality (3.18). In a similar way it can be shown that Λ_{2s} is doubly Hölder continuous in the variables sy, $s'y$. With $E(s,s',y)$ defined in Eq. (I-4.1) it then follows that $E_s(s,s',y)$ is bounded and Hölder continuous in $sy, s'y$. On applying Theorem 3 of Appendix B of I we find that Eq. (3.19) is also valid with the replacement $B^{(1a)} \to B^{(1b)}$. In fact, for $B^{(1b)}$ the factor $\left(\frac{s}{s-4}\right)^{1-\varepsilon}$ in Eq. (3.19a) and the factor $\left(\frac{s_2}{s_2-4}\right)^{1+\mu-\varepsilon}$ in Eq. (3.19b) could be removed.

From the expression for $\Gamma(s,s',y,y')$ given in Eq. (3.16), it is straightforward, using the bounds on the Legendre functions and their derivatives given in Appendix A and Appendix A of I, to obtain a bound on Γ. By integrating by parts in Eq. (3.16) an alternative estimate can be established. Compromising between the two we have

$$h(s')\,|\Gamma(s,s',y,y')| \leqq \varkappa\left(\frac{s'}{s'-4}\right)|y'+i|^{\frac{1}{2}-\eta}\,|y+i|^{-\frac{1}{2}+\eta}, \tag{3.21}$$

where η satisfies inequality (3.18). Moreover Γ is doubly Hölder continuous in the variables sy, $s'y$.

Defining

$$C(s,s',y) = \tfrac{1}{2}(s'-4)^{\varepsilon} \int\limits_{-\infty}^{\infty} dy'(y'-i\varepsilon)\,B(s'_+,y')\,B(s'_-,y')\,\Gamma(s,s',y,y') \tag{3.22}$$

we find that $C(s,s',y)$ is bounded and doubly Hölder continuous in sy, $s'y$. It is important to note that $C(s,s',y)$ behaves like $(s'-4)^{-1+\varepsilon}$ as $s'\downarrow 4$ while the double Hölder difference in $s'y$ behaves like $(s'_2-4)^{-1-\mu+\varepsilon}$ as $s'_2\downarrow 4$. Thus using Theorem 3 of Appendix A we find that Eq. (3.19) is also valid with the replacement $B_s^{(1a)} \to B_s^{(1c)}$. In fact, for $B^{(1c)}$ one can obtain estimates which are slightly better behaved at large s. Finally we see that Eq. (3.19) is valid with $B_s^{(1a)} \to B_s^{(1)}$.

The treatment of $B^{(2)}(s,y)$, defined in Eq. (2.22), is much simpler that that of $B^{(1)}(s,y)$ and can be carried out in the manner described in Section 5 of I. Because the denominator $(s'+t+s-4)$ replaces the singular Cauchy kernel $(s'-s)$ which occurs in the definition of $B^{(1)}(s,y)$, there is no singular integral in either s' or y' in the integral representation of $B^{(2)}(s,y)$. The method of showing that it is bounded and doubly Hölder-continuous is very similar to that used to treat $B_2(s,y)$ in Section 4 of I. We find that Eq. (3.7) is also valid with the replacement $B^{(1)} \to B^{(2)}$.

The treatment of $B_s^{(2)}(s,y)$ can be carried out in a manner similar to that used for $B_s^{(1b)}(s,y)$. It can be shown that Eq. (3.19) is also valid with $B_s^{(1a)} \to B_s^{(2)}$.

4. Treatment of $B^{(3)}(s, y)$ and $B^{(4)}(s, y)$

From Eqs. (2.22) and (2.5) we see that $B^{(3)}(s, y)$ takes the form

$$B^{(3)}(s,y)=\frac{2i}{\pi^2}\int_{16}^{\infty}\frac{ds'}{s'-s}\int_{-\infty}^{\infty}dy'(y'-i\varepsilon)\,\Omega(s,s',y,y'), \tag{4.1}$$

where

$$\Omega(s,s',y,y')=\int_{\tau(s')}^{\infty}dt\,h(t)\,q(t)\frac{Q_{-\frac{1}{2}+\varepsilon+iy}\left(1+\frac{2t}{s-4}\right)}{(s-4)^{\frac{1}{2}+\varepsilon+iy}}P_{-\frac{1}{2}+\varepsilon+iy'}\left(1+\frac{2s'}{t-4}\right) \cdot B(t_+,y')\,B(t_-,y')\,(t-4)^{-1+2\varepsilon+2iy'} \tag{4.2}$$

and $\tau(s')$ is defined in Eq. (2.6). In Eq. (4.2) the integrand vanishes for $t \geqq 16+\Delta$ because of the presence of the cut-off function $h(t)$. Thus, since $\tau(s')\to\infty$ as $s'\downarrow 16$ we find that $\Omega(s,16,y,y')=0$. A straightforward bound on the integrand in Eq. (4.2) would indicate that $|\Omega|$ and hence $|B^{(3)}|$ behave like $|y+i|^{-\frac{1}{2}}$ for large y. This estimate does not have the required large y behaviour, namely $|y+i|^{-\frac{1}{2}-\upsilon}$ [see Eq. (2.10)]. In order to improve on this bound we integrate by parts in Eq. (4.2) to obtain

$$\begin{aligned}\Omega(s,s',y,y')={}&-\tfrac{1}{2}(s-4)^{\frac{1}{2}-\varepsilon-iy}[(-\tfrac{1}{2}+\varepsilon+iy)(\tfrac{1}{2}+\varepsilon+iy)]^{-1}\left[\frac{2\tau(s')}{s-4}\left(2+\frac{2\tau(s')}{s-4}\right)\right]\\&\cdot Q'_{-\frac{1}{2}+\varepsilon+iy}\left(1+\frac{2\tau(s')}{s-4}\right)P_{-\frac{1}{2}+\varepsilon+iy'}\left(1+\frac{2s'}{\tau(s')-4}\right)B(\tau_+(s'),y')\,B(\tau_-(s'),y')\\&\cdot(\tau(s')-4)^{-1+2\varepsilon+2iy'}q(\tau(s'))\,h(\tau(s'))-\int_{\tau(s')}^{\infty}dt\tfrac{1}{2}(s-4)^{\frac{1}{2}-\varepsilon-iy}\\&\cdot[(-\tfrac{1}{2}+\varepsilon+iy)(\tfrac{1}{2}+\varepsilon+iy)]^{-1}\left[\frac{2t}{s-4}\left(2+\frac{2t}{s-4}\right)\right]\cdot Q'_{-\frac{1}{2}+\varepsilon+iy}\left(1+\frac{2t}{s-4}\right)\frac{\partial}{\partial t}\\&\cdot\left[P_{-\frac{1}{2}+\varepsilon+iy'}\left(1+\frac{2s'}{t-4}\right)B(t_+,y')\,B(t_-,y')\,(t-4)^{-1+2\varepsilon+2iy'}\cdot q(t)\,h(t)\right].\end{aligned} \tag{4.3}$$

We recall that it was precisely to do this integration by parts that we introduced the extra terms involving the derivative in the norm in Eq. (2.17). Taking a compromise between the bounds on Ω that one can derive from Eqs. (4.2) and (4.3), we find, on using the estimates on the Legendre functions given in Appendix A and Appendix A of I, that

$$|\Omega(s,s',y,y')| \leqq \varkappa\, s^{\frac{1}{2}-\varepsilon-\frac{3}{4}\eta}\, s'^{-\frac{1}{2}+\varepsilon}|y+i|^{-\frac{3}{2}+\eta}|y'+i|^{-\frac{1}{2}-2\upsilon-\eta}, \tag{4.4}$$

where η satisfies inequality (3.18). Similar bounds for the double Hölder differences in the variables sy, $s'y$ can then be established by using the mean value theorems. A slight adaptation of Theorem 3 of Appendix A of I is then needed to show that

$$|B^{(3)}(s,y)| \leqq \varkappa\,\|B\|^2 s^{-\frac{3}{4}\eta}|y+i|^{-\frac{3}{2}+\mu+\eta}, \tag{4.5a}$$

$$\begin{aligned}&|B^{(3)}(s_1,y_1)-B^{(3)}(s_2,y_1)-B^{(3)}(s_1,y_2)+B^{(3)}(s_2,y_2)|\\&\leqq \varkappa\,\|B\|^2 s_2^{-\frac{3}{4}\eta+\delta}|y_2+i|^{-\frac{3}{2}+\mu+\varrho+\eta+\delta}\left|\frac{y_1-y_2}{y_1+i}\right|^{\varrho}\left|\frac{s_1-s_2}{s_1}\right|^{\mu}.\end{aligned} \tag{4.5b}$$

Again, υ is required to satisfy inequality (3.12) to ensure that the y' integral in Eq. (4.1) is absolutely convergent. Further, since the inequalities (3.9), (3.10), and (3.11) were required in Section 3 we shall also impose them here.

From Theorem 4 of Appendix B we find that $B_s^{(3)}(s, y)$, the derivative with respect to s of $B^{(3)}(s, y)$ can be written in the form

$$B_s^{(3)}(s, y) = B_s^{(3a)}(s, y) + B_s^{(3b)}(s, y)\,, \tag{4.6}$$

where

$$B_s^{(3a)}(s, y) = \frac{2i}{\pi^2} \int_{16}^{\infty} \frac{ds'}{s' - s} \int_{-\infty}^{\infty} dy'(y' - i\varepsilon)\, \Omega_s(s, s', y, y')\,, \tag{4.7}$$

$$B_s^{(3b)}(s, y) = \frac{2i}{\pi^2} \int_{16}^{\infty} \frac{ds'}{s' - s} \int_{-\infty}^{\infty} dy'(y' - i\varepsilon)\, \Omega_{s'}(s, s', y, y')\,. \tag{4.8}$$

Taking the derivative with respect to s of the two expressions for $\Omega(s, s', y, y')$ given in Eqs. (4.2) and (4.3) we find on compromising between the resultant bounds that

$$|\Omega_s(s, s', y, y')| \leqq \varkappa \|B\|^2 s^{-\frac{1}{2} - \varepsilon - \frac{1}{4}\eta} s'^{-\frac{1}{2} + \varepsilon} |y + i|^{-\frac{1}{2} + \eta} |y' + i|^{-\frac{1}{2} - 2\upsilon - \eta}\,, \tag{4.9}$$

where η satisfies inequality (3.18). Again similar bounds can be established for the double Hölder differences in the variables sy, $s'y$. Finally a slight modification of Theorem 3 of Appendix A of I can be used to show that

$$|B_s^{(3a)}(s, y)| \leqq \varkappa \|B\|^2 s^{-1 - \frac{3}{4}\eta} |y + i|^{-\frac{1}{2} + \mu + \eta}\,, \tag{4.10a}$$

$$\begin{aligned} &|B_s^{(3a)}(s_1, y_1) - B_s^{(3a)}(s_2, y_1) - B_s^{(3a)}(s_1, y_2) + B_s^{(3a)}(s_2, y_2)| \\ &\quad \leqq \varkappa \|B\|^2 s_2^{-1 - \frac{3}{4}\eta + \delta} |y_2 + i|^{-\frac{1}{2} + \mu + \varrho + \eta + \delta} \left|\frac{y_1 - y_2}{y_1 + i}\right|^{\varrho} \left|\frac{s_1 - s_2}{s_1}\right|^{\mu}. \end{aligned} \tag{4.10b}$$

The restrictions on η, μ, ϱ, and υ given in Eqs. (3.8)–(3.12) are again assumed to hold.

Treating the term $B_s^{(3b)}(s, y)$ turns out to be somewhat more trouble-some. Differentiating the expression for $\Omega(s, s', y, y')$ in Eq. (4.2) with respect to s' leads one to the estimate

$$|\Omega_{s'}(s, s', y, y')| \leqq \varkappa \|B\|^2 s^{-\frac{1}{4} - \varepsilon} s'^{-\frac{3}{2} + \varepsilon} |y + i|^{-\frac{1}{2}} |y' + i|^{-\frac{1}{2} - 2\upsilon}\,. \tag{4.11}$$

To obtain an estimate with the same large y and y' behaviour as that given in Eq. (4.9) one might try the usual trick of integrating by parts. Now Eq. (4.3) is unsuitable and in order to get a suitable estimate for $\Omega_{s'}(s, s', y, y')$ one would need to be able to carry out explicitly the integral with respect to t of

$$P_{-\frac{1}{2} + \varepsilon + iy'}\left(1 + \frac{2s'}{t - 4}\right) B(t_+, y')\, B(t_-, y')\, (t - 4)^{-1 + 2\varepsilon + 2iy'}\,.$$

Clearly, we do not, from Eq. (2.17), have sufficient information about $B(t, y)$ to carry out this integral. We shall therefore be content with Eq. (4.11) and similar estimates on the double Hölder differences in the variables sy, $s'y$. A slight reformulation of Theorem 3 of Appendix A of I then shows that

$$|B_s^{(3b)}(s, y)| \leqq \varkappa \|B\|^2 s^{-\frac{5}{4} - \varepsilon + \delta} |y + i|^{-\frac{1}{2} + \mu}\,, \tag{4.12a}$$

$$\begin{aligned} &|B_s^{(3b)}(s_1, y_1) - B_s^{(3b)}(s_2, y_1) - B_s^{(3b)}(s_1, y_2) + B_s^{(3b)}(s_2, y_2)| \\ &\quad \leqq \varkappa \|B\|^2 s_2^{-\frac{5}{4} - \varepsilon + \delta} |y_2 + i|^{-\frac{1}{2} + \mu + \varrho + \delta} \left|\frac{y_1 - y_2}{y_1 + i}\right|^{\varrho} \left|\frac{s_1 - s_2}{s_1}\right|^{\mu}. \end{aligned} \tag{4.12b}$$

To ensure that the y' integral in Eq. (4.7) is absolutely convergent we require that

$$v > \tfrac{3}{4} + \mu/2 \tag{4.13}$$

which is a stronger constraint than that given in Eq. (3.12). Again we impose the restrictions in Eqs. (3.10) and (3.11) respectively on μ and ϱ.

Finally we come to the treatment of $B^{(4)}(s, y)$ defined in Eq. (2.22). Here the non singular denominator $(s' + t + s - 4)$ replaces the singular Cauchy kernel $(s' - s)$ which appears in the definition of $B^{(3)}(s, y)$ [Eq. (4.1)]. Thus the boundedness and double Hölder continuity of $B^{(4)}(s, y)$ can be established more simply than was the case for $B^{(3)}(s, y)$. However, once these properties have been established for $B^{(3)}(s, y)$ the simplest way of showing that Eq. (4.5) is also valid with the replacement $B^{(3)} \to B^{(4)}$ is probably to write

$$(s' + t + s - 4)^{-1} = (s' - s)^{-1} \left(\frac{s' - s}{s' + t + s - 4} \right)$$

and regard $(s' - s)^{-1}$ as the kernel and $\left(\frac{s' - s}{s' + t + s - 4} \right)$ as part of the weight function. Similarly the bound on $B_s^{(3)}(s, y)$ and on its double Hölder difference in the variables sy are also satisfied by $B_s^{(4)}(s, y)$.

5. Study of $V(s, y)$ and Construction of the Mandelstam Representation

We have yet to show that there are classes of symmetric input functions $v(s, t)$ for which $V(s, y)$, defined in Eq. (2.15), belongs to the Banach space with norm given in Eq. (2.17). Let us consider as an example the class of functions $v(s, t)$ for which

$$|v(s, t)| \leqq \varkappa s^{-a} t^{-a} \quad \text{or} \quad \leqq \varkappa (t^{-\frac{1}{2}+\varepsilon-\delta} s^{-a} + s^{-\frac{1}{2}+\varepsilon-\delta} t^{-a}), \tag{5.1}$$

where $a > \frac{3}{4}$ and for which $\frac{\partial}{\partial t} v(s, t)$, $\frac{\partial}{\partial s} v(s, t)$, $\frac{\partial}{\partial s} \frac{\partial}{\partial t} v(s, t)$, and $\frac{\partial^2}{\partial s^2} \frac{\partial}{\partial t} v(s, t)$ are all majorized by $\varkappa s^{-b} t^{-b}$ where $b > 1$. It may then be shown that $V(s, y)$ satisfies bounds which are no worse than those satisfied by $B^{(3)}(s, y)$ in Eq. (4.5). Similarly, the bound on $B_s^{(3)}(s, y)$ and on its double Hölder difference in the variables sy are also satisfied by $V_s(s, y)$. The method of deriving these results in similar to that outlined in Section 4, with the simplification that there is no y'-integral in Eq. (2.15).

We must now consider for what range of values of the indices Eq. (2.19) is valid. First we see from Eqs. (3.12) and (4.13) that v must satisfy inequality (4.13) in order to ensure that the y' integrals considered in Section 3 and 4 are all absolutely convergent. Also, for Eq. (2.19) to be valid we require that the powers of $|y + i|$ and s in Eqs. (3.7) and (4.5) are not greater than the corresponding powers in Eq. (I-2.10) and that the powers of $|y + i|$ and s in Eqs. (3.19), (4.10), and (4.12) are not greater than the corresponding powers in Eq. (2.18). Thus, v must satisfy the inequalities

$$\tfrac{3}{4} + \mu/2 < v < 1 - \mu - \varrho - \eta \tag{5.2}$$

which, with μ and ϱ satisfying inequalities (3.10) and (3.11) respectively, means that we must restrict η further to

$$0<\eta<\tfrac{1}{4}-\tfrac{3}{2}\mu-\varrho\,. \tag{5.3}$$

Moreover, the index λ which appears in Eq. (2.18) must satisfy

$$0<\lambda<\tfrac{3}{4}\eta\,. \tag{5.4}$$

It is in fact sufficient to take μ and ϱ very small and positive, which means that υ can range from slightly larger than $\frac{3}{4}$ to just less than 1, for sufficiently small η. Similarly η can range from just larger than zero to just smaller than $\frac{1}{4}$, which in turn means that λ may range from slightly larger than zero to just less than $\frac{3}{16}$. These ranges of υ and λ can in fact be extended but they shall suffice for our purposes. In particular, if in the second and fourth terms in Eq. (2.17) we replace s_2^λ by s_1^λ then our proof also works for

$$-\infty<\lambda\leqq 0\,, \tag{5.5}$$

provided we also add to the norm terms like the second and fourth terms but with μ replaced by zero and $f(s_1, y_1)-f(s_1, y_2)$ (resp. $f_s(s_1, y_1)-f_s(s_1, y_2)$) missing. This follows from the fact that the cut off function $h(s)$ appears in Eq. (2.5) and so it does not matter how rapidly $B(s, y)$ increases with s.

When the above restrictions on the indices hold we then see that there is a unique fixed point

$$\bar{B}(s, y)=B(s, y) \tag{5.6}$$

in the ball defined by Eq. (2.21), provided that $\|V\|<(4\varkappa)^{-1}$ [where $\varkappa$ now refers specifically to the constant appearing in Eq. (2.19)]. Furthermore, this solution of Eqs. (2.5), (2.13), (2.14), and (2.15) can be obtained by a convergent iteration procedure.

At this fixed point, $\varrho^{el}(s, t)$ is given by Eq. (2.5) and $A(s, l)$ can now be obtained for $\mathrm{Re}\, l\geqq -\frac{1}{2}+\varepsilon$ from the equations

$$A(s, l)=\frac{4}{\pi(s-4)}\int_4^\infty dt\, Q_l\left(1+\frac{2t}{s-4}\right)F_t(s, t)\,, \tag{5.7}$$

$$F_t(s, t)=\frac{1}{\pi}\int_{\Sigma(t)}^\infty ds'\,\varrho(s', t)\left[\frac{1}{s'-s}+\frac{1}{s'-4+s+t}\right] \tag{5.8}$$

with $\varrho(s, t)$ defined in Eq. (2.9). [We note that when $l=-\frac{1}{2}+\varepsilon+iy$, the above equations are equivalent to Eqs. (2.13)–(2.15) with $B(s, y)$ defined in Eq. (2.4).] By construction, $A(s, l)$ is a holomorphic function of l, for $\mathrm{Re}\, l\geqq -\frac{1}{2}+\varepsilon$, which vamishes as $s\to\infty$.

The Mandelstam representation for $F(s, t)$ may now be established by substituting the Froissart-Gribov representation for $A(s, l)$ [Eq. (5.6)] into the partial wave series

$$F(s, t)=\sum_{l=0}^\infty (2l+1)\tfrac{1}{2}[P_l(z)+P_l(-z)]\,A(s, l) \tag{5.9}$$

and then using Heine's identity [9] after interchanging the orders of summation and integration. Substituting Eq. (5.8) for $F_t(s, t)$ into the resultant expression then gives

$$F(s,t)=\frac{1}{\pi^2}\int_4^\infty dt'\int_{\Sigma(t')}^\infty ds'\,\varrho(s',t')\cdot\left[\frac{1}{(s'-s)(t'-t)}+\frac{1}{(s'-s)(t'-4+s+t)}+\frac{1}{(t'-t)(s'-4+s+t)}\right], \tag{5.10}$$

since $\varrho(s',t')$ is a symmetric function of s' and t'.

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Appendix A. Further Bounds for Legendre Functions

Here we establish some further properties of the Legendre functions which, together with the results of Appendix A of I, are needed in the main body of the paper. We note that Eqs. (I-A.1)–(I-A.5) are also valid when $m=3$. Also Eq. (I-A.7), with $n=2$, is a valid representation for $F(\frac{7}{2}, -\frac{5}{2}, 1\pm(\varepsilon+iy), \zeta)$ and it is then straight forward to obtain a suitable bound for this hypergeometric function from which one can show that

$$|Q'''_{-\frac{1}{2}\pm(\varepsilon+iy)}(z)|\leqq \varkappa|y+i|^{\frac{5}{2}}\left(\frac{z}{z-1}\right)^3 z^{-\frac{7}{2}\mp\varepsilon}, \tag{A.1}$$

$$|P'''_{-\frac{1}{2}+\varepsilon+iy}(z)|\leqq \varkappa|y+i|^{\frac{5}{2}}\left(\frac{z}{z-1}\right)^3 z^{-\frac{7}{2}+\varepsilon}. \tag{A.2}$$

Here Eqs. (I-A.1)–(I-A.5), with $m=3$, have also been used. As usual, $\varkappa$ is a generic constant. Moreover, since Eq. (I-A.25) is valid when $m=3$ we see that the derivatives with respect to y of the Legendre functions on the left-hand sides of Eqs. (A.1) and (A.2) can be bounded by the corresponding right-hand sides multiplied by $(1+\log z)$.

Appendix B. Properties of Singular Integrals

We now extend the theorems of Appendix B of I to allow for singular behaviour of the weight functions at the end points of integration. From a slight generalization of Pogorzelski's theorem [8] for singular integrals over a finite interval, the following theorem for singular integrals over the semi-infinite interval $[4,\infty)$ can be proved.

Theorem 1. *We consider a function $f(s)$, defined on the interval $4\leqq s<\infty$ of the real line and we define a norm*

$$\|f\|_1=\sup\left\{s_2{}^{\beta}\left(\frac{s_2-4}{s_2}\right)^{\alpha}|f(s_2)|\right\}+\sup\left\{s_2{}^{\beta}\left(\frac{s_2-4}{s_2}\right)^{\alpha+\mu}\frac{|f(s_1)-f(s_2)|}{\left|\frac{s_1-s_2}{s_1}\right|^{\mu}}\right\}, \tag{B.1}$$

where $\alpha > 0$, $\alpha + \mu < 1$ *and* $0 < \mu < \beta < 1$ *and the suprema are taken over* $4 \leqq s_2 < s_1 < \infty$. *If* $\|f\|_1 < \infty$, *and*

$$I_\pm(s) = P \int_4^\infty \frac{ds'}{s \pm s} f(s'), \tag{B.2}$$

where $s \geqq 4$, *then it may be shown that a constant,* $\varkappa$, *exists such that*

$$\|I_\pm\|_1 \leqq \varkappa \|f\|_1 . \tag{B.3}$$

Combining Theorem 1 with Theorem 2 of Appendix B of I, leads us to the following theorem.

Theorem 2. *Let us now consider a function of two variables* $f(s, y)$ *and we introduce the two-dimensional norm*

$$\|f\|_2 = \sup\left\{s_2{}^\beta \left(\frac{s_2 - 4}{s_2}\right)^\alpha |y_2 + i|^\gamma |f(s_2, y_2)|\right\} \tag{B.4}$$

$$+ \sup\left\{s_2{}^\beta \left(\frac{s_2 - 4}{s_2}\right)^{\alpha+\mu} |y_2 + i|^\gamma \frac{|f(s_1, y_1) - f(s_2, y_1) - f(s_1, y_2) + f(s_2, y_2)|}{\left|\frac{s_1 - s_2}{s_1}\right|^\mu \left|\frac{y_1 - y_2}{y_1 + i}\right|^\varrho}\right\},$$

where $\alpha > 0$, $\alpha + \mu < 1$, $0 < \mu < \beta < 1$, $0 < \varrho < \gamma < 1$ *and the suprema are taken over* $4 \leqq s_2 < s_1 < \infty$, $-\infty < y_2 < \infty$, $|y_1| > |y_2|$. *If* $\|f\|_2 < \infty$ *and*

$$I(s, y) = P \int_{-\infty}^\infty \frac{dy'}{y' - y} f(s, y'), \tag{B.5}$$

$$J_\pm(s, y) = P \int_4^\infty \frac{ds'}{s' \pm s} I(s, y), \tag{B.6}$$

where $s \geqq 4$, $-\infty < y < \infty$ *then a constant* $\varkappa$ *exists such that*

$$\|I\|_2 \leqq \varkappa \|f\|_2, \tag{B.7}$$

$$\|J_\pm\|_2 \leqq \varkappa \|f\|_2 . \tag{B.8}$$

The above results are also valid if y *is replaced by* $y + i\eta$ *and the* P *in Eq.* (B.5) *is dropped.*

For the case when both the Cauchy kernel and the weight function depend on s, we have the following theorem.

Theorem 3. *Consider a function* $g(s', s)$ *defined on* $4 \leqq s' < \infty$, $4 \leqq s < \infty$ *and introduce the norm*

$$\|g\|_3 = \sup\left\{s_2^{-a} s_2'^\beta \left(\frac{s_2' - 4}{s_2'}\right)^\alpha |g(s_2', s_2)|\right\}$$

$$+ \sup\left\{s_2^{-a} s_2'^\beta \left(\frac{s_2' - 4}{s_2'}\right)^{\alpha+\mu} \frac{|g(s_1', s_2) - g(s_2', s_2)|}{\left|\frac{s_1' - s_2'}{s_1'}\right|^\mu}\right\} \tag{B.9}$$

$$+ \sup\left\{s_1^{-b} s_2'^\beta \left(\frac{s_2' - 4}{s_2'}\right)^\alpha \frac{|g(s_2', s_1) - g(s_2', s_2)|}{|s_1 - s_2|^{\mu+\delta}}\right\},$$

where $\alpha > 0$, $\alpha + \mu < 1$, $\max\{a+\mu,\ b+\mu\} < \beta < 1$, $\mu > 0$, $a \geqq 0$, $b \geqq 0$ *and* δ *is a generic small positive number which can be chosen as small as one pleases. The suprema are taken over* $s_1' > s_2' \geqq 4$, $s_1 > s_2 \geqq 4$. *If* $\|g\|_3 < \infty$ *and*

$$G(s) = P \int_4^\infty \frac{ds'}{s'-s}\, g(s', s) \tag{B.10}$$

then it can be shown that $\|G_4\| < \infty$, *where*

$$\|G_4\| = \sup\{s_2^{\beta-a}|G(s_2)|\} + \sup\left\{ s_2^{\beta-c} \frac{|G(s_1)-G(s_2)|}{\left|\frac{s_1-s_2}{s_1}\right|^{\mu}} \right\} \tag{B.11}$$

and $c = \max\{a, b+\mu+\delta\}$. *The suprema in* (B.11) *are taken over* $s_1 > s_2 \geqq 4$.

Finally, we shall need the following theorem concerning the derivative of a principal value integral.

Theorem 4. *Consider a function* $f(s)$ *defined on* $4 \leqq s < \infty$ *and suppose that* $f(4) = 0$ *and* $\lim_{s\to\infty} f(s)\, s^{-1} = 0$.
Then if $s \geqq 4$

$$\frac{\partial}{\partial s} P \int_4^\infty \frac{ds'}{s'-s} f(s') = P \int_4^\infty \frac{ds'}{s'-s} \frac{\partial}{\partial s'} f(s'). \tag{B.12}$$

The theorem can be proved by using Leibniz's theorem and then integrating by parts.

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J. S. Frederiksen
Research School of Physical Sciences
The Australian National University
P.O.Box 4
Canberra, Australia 2600