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Collision Theory for Massless Fermions

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Abstract. Starting from the basic postulates of local relativistic quantum theory, the asymptotic incoming and outgoing collision states of massless Fermions are constructed. The corresponding Hilbert spaces have Fock structure and thus allow the usual definition of an S-matrix. In contrast to the massive case, there are geometric relations between the local nets of the underlying field algebra and the asymptotic fields.

1. Introduction

In this paper we establish the existence of collision states for massless Fermions in the framework of local relativistic quantum theory. It is amazing that a proof of this fact has not appeared before now – more than ten years after Haag and Ruelle developed their famous collision theory for massive particles [1, 2]. But it might be that their intuitively appealing ideas have turned away the attention of the experts from the simple facts allowing the construction also in the massless case.

The methods of Haag and Ruelle are based on two essential features of massive theories: absence of long-range forces and existence of almost local operators which create one-particle states from the vacuum. These facts make it possible to construct the spaces of incoming and outgoing collision states and to establish their Fock structure[1]. Only in a second step can one then define the asymptotic fields of the particles. But in order to be sure that they act as operators on the whole Hilbert space of states, one needs the additional assumption of asymptotic completeness of the theory.

It is very unlikely that this technique can successfully be carried over to the massless case. Therefore, we apply a completely different method which takes special care of the peculiar kinematics of massless particles. A basic ingredient of our proofs is the trivial fact that these particles move with the speed of light. So they have – loosely speaking – one degree of freedom less in configuration space than their massive counterparts. Imagine, for example, a massless particle which sits at the tip of a light cone in Minkowski space. This particle can never reach interior points of the cone. In fact all interior points of the cone become ultimately space-like to the position of the particle at asymptotic times. This naïve picture may be carried over to quantum theory if the number of space dimensions is odd. It is nothing else but the Huyghens principle [5].

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[1] Hepp [3] and Herbst [4] observed that actually only one of the above-mentioned properties is needed for a proof.

Using extensively this fact and microcausality, it is possible to construct directly the asymptotic field operators for the incoming and outgoing massless Fermions. These operators turn out to be bounded as a consequence of the Pauli principle and they have all the properties expected from a free field. They can therefore be used to construct in a canonical way the Fock spaces of the incoming and outgoing collision states.

The restriction to the case of massless Fermions has two reasons. The first is a purely technical one, stemming from the fact that in the Bose case the asymptotic field operators are unbounded. The difficulties connected with this are probably removable[2]. On the other hand, we start from the assumption that single particle states of the massless particles can be sharply defined. If the only massless particles in the theory are Fermions then one may expect, due to the Pauli principle, that no infinite numbers of them are produced in collisions and hence that all particles in the theory correspond to single particle states with precise mass. There should be no problems with infra particles. One then will have a usual collision theory and the present study shows how to construct the asymptotic fields and collision states for the massless Fermions. In the case of massless Bosons one may still hope to construct these quantities in the vacuum sector by the same method, but the infrared problem in other sectors remains and is not touched upon by this study.

Let us now briefly list our assumptions. Since we want to avoid all unnecessary complications, we shall formulate the postulates in terms of the field algebra $\mathfrak{F}$ instead of the field operators themselves. $\mathfrak{F}$ is assumed to be the global algebra of a net $\mathcal{O} \to \mathfrak{F}(\mathcal{O})$ of local algebras (attached to the open, bounded regions $\mathcal{O} \subset \mathbb{R}^4$) and to act irreducibly on the Hilbert space $\mathcal{H}$ of physical states[3]. In order to distinguish between Bose and Fermi operators, it is convenient to assume that there exists an automorphism γ of $\mathfrak{F}$ which acts like the identity on Bose operators and which changes the sign of Fermi operators, hence $\gamma^2 = \iota$. Each $F \in \mathfrak{F}(\mathcal{O})$ can then be decomposed into a Bose part $F_+ \in \mathfrak{F}(\mathcal{O})$ and a Fermi part $F_- \in \mathfrak{F}(\mathcal{O})$:

$$F_\pm = \tfrac{1}{2}(F \pm \gamma(F)). \tag{1}$$

We suppose that these operators have the usual commutation relations at space-like distances:

$$\begin{aligned} &F_+ F'_+ - F'_+ F_+ = [F_+, F'_+] = 0 \\ &F_+ F'_- - F'_- F_+ = 0 \qquad\qquad F \in \mathfrak{F}(\mathcal{O}_1), \quad F' \in \mathfrak{F}(\mathcal{O}_2) \quad \mathcal{O}_1 \subset \mathcal{O}'_2 . \\ &F_- F'_- + F'_- F_- = \{F_-, F'_-\} = 0 \end{aligned} \tag{2}$$

We furthermore assume that $\mathcal{H}$ carries a continuous unitary representation $L \to U(L)$ of the covering group of the Poincaré group $\mathcal{P}$. The operators $U(L)$ induce automorphisms of the field algebra $\mathfrak{F}$

$$U(L)\,\mathfrak{F}(\mathcal{O})\,U(L)^{-1} = \mathfrak{F}(L\mathcal{O}) \qquad L \in \mathcal{P}, \tag{3}$$

[2] As a matter of fact it turns out that the asymptotic fields exist in the Bose case as closable unbounded operators. With some technical assumptions on their extensions, one could easily prove that these operators are free fields.

[3] The reader who is not familiar with this approach may regard $\mathfrak{F}(\mathcal{O})$ as a set of bounded operators which are generated by the field operators localized in $\mathcal{O}$.

which leave the Bose and Fermi part of $\mathfrak{F}$ invariant

$$U(L)\mathfrak{F}_{\pm}U(L)^{-1}=\mathfrak{F}_{\pm}\,. \tag{4}$$

There is (up to a phase) exactly one unit vector Ω in $\mathscr{H}$, the vacuum, which is invariant under $U(L), L\in\mathscr{P}$. The spectrum of the generators of the translations $x\to U(x)$ is contained in the forward light cone and there is a subspace $\mathscr{H}_1\subset\mathscr{H}$ on which the $U(L)$ act like a representation of $\mathscr{P}$ with mass $m=0$. $\mathscr{H}_1$ is the subspace of massless one-particle states and we suppose that it exclusively contains Fermions; it is thus orthogonal to $[\mathfrak{F}_+\Omega]$.

2. The Asymptotic Fields and the Collision States

In order to establish the existence of asymptotic fields for massless Fermions we proceed in three steps. First we define certain sequences of operators which are suitable candidates for an approximation of the asymptotic fields. Then we show that these sequences are uniformly bounded and strongly convergent. Finally we prove that the limit operators are indeed free fields with all the required properties.

To begin with let us briefly repeat some simple facts about solutions of the wave equation [6]. It is well known that these functions can be represented as follows:

$$f(t\,|\,\boldsymbol{x})=(2\pi)^{-3/2}\int d^3p\,e^{i\boldsymbol{p}\boldsymbol{x}}\big(e^{i|\boldsymbol{p}|t}\tilde{f}_+(\boldsymbol{p})+e^{-i|\boldsymbol{p}|t}\tilde{f}_-(\boldsymbol{p})\big)\,. \tag{5a}$$

We are mainly interested in solutions which have compact support in $\mathbb{R}^3$ at finite times t and require

$$\tilde{f}_{\pm}(\boldsymbol{p})=\tilde{f}_1(\boldsymbol{p})\pm i|\boldsymbol{p}|\tilde{f}_2(\boldsymbol{p})\quad\text{with}\quad f_1(\boldsymbol{x}),\,f_2(\boldsymbol{x})\in\mathscr{D}(\mathbb{R}^3)\,. \tag{5b}$$

If $f_1(\boldsymbol{x})$ and $f_2(\boldsymbol{x})$ have compact support in $O\subset\mathbb{R}^3$, it follows from the Huyghens principle that $f(t\,|\,\boldsymbol{x})$ has support in $\{O+|t|\cdot\boldsymbol{n}:|\boldsymbol{n}|=1\}$ and this is again a compact set. The above ansatz also guarantees that the functions $f_{\pm}(\boldsymbol{x})$ are absolutely integrable.

Now let $\psi\in\mathfrak{F}_-$ be a local Fermi operator such that $x\to\psi(x)=U(x)\psi U(x)^{-1}$ is twice norm-continuously differentiable with respect to $x=(t,\boldsymbol{x})$. (Such operators exist and can easily be constructed by smearing any local Fermi operator with a suitable test function.) We then define, with $f(t\,|\,\boldsymbol{x})$ as above and $g(\boldsymbol{x})\in L^1(\mathbb{R}^3)$,

$$\psi_f(t)=(2\pi)^{-3/2}\int d^3x\,f(t\,|\,\boldsymbol{x})\psi(t,\boldsymbol{x})\quad\text{and}\quad\psi_g=(2\pi)^{-3/2}\int d^3x\,g(\boldsymbol{x})\psi(\boldsymbol{x})\,. \tag{6}$$

Since we want to use $\psi_f(t)$ for the construction of the asymptotic fields, we require that $\psi_{f_-}\Omega$ has a non-vanishing component in $\mathscr{H}_1$, the space of massless one-particle states. This is again no restrictive assumption. In fact the set $\mathscr{L}_1$ of vectors $P_1\psi_{f_-}\Omega$ (where P_1 denotes the projection onto $\mathscr{H}_1$ and ψ, f_- are, respectively, operators and functions as above) is dense in $\mathscr{H}_1$ as a consequence of the irreducibility of $\mathfrak{F}$. We want also to point out that $\mathscr{L}_1$ is invariant under Poincaré transformations.

To conclude this list of definitions, let us denote the set of non-negative functions $h\in\mathscr{D}(\mathbb{R}^1)$, which are normalized according to $\int dt\,h(t)=1$ by $\mathscr{D}^{\#}(\mathbb{R}^1)$.

In the course of our analysis we shall integrate $t \to \psi_f(t)$ with such functions:

$$\psi_f(h) = \int dt\, h(t)\psi_f(t), \quad h \in \mathscr{D}^\#(\mathbb{R}^1). \tag{7}$$

The following lemma is then a simple consequence of the commutation properties of Fermi fields at space-like distances and the fact that the integration with functions $h \in \mathscr{D}^\#(\mathbb{R}^1)$ is a completely positive mapping from the set of operator-valued functions into the set of operators on $\mathscr{H}$.

Lemma 1. *For ψ, f and h as above, we have*

$$0 \leqq \{\psi_f(h)^*, \psi_f(h)\} \leqq c \cdot (\|f_+\|^2 + \|f_-\|^2).$$

The constant c in this inequality neither depends on f nor on h. $\|f_\pm\|^2$ stands for $\int d^3x\, |f_\pm(\mathbf{x})|^2$.

Proof. Since h is a non-negative function, it is obvious that the operator

$$h(t)h(t') \cdot (\psi_f(t) - \psi_f(t'))^* \cdot (\psi_f(t) - \psi_f(t'))$$

is non-negative for arbitrary t and t'. One gets, therefore, after integration, bearing in mind that $\int dt\, h(t) = 1$,

$$0 \leqq \psi_f(h)^* \psi_f(h) \leqq \int dt\, h(t) \psi_f(t)^* \psi_f(t).$$

In the same way one shows

$$0 \leqq \psi_f(h) \psi_f(h)^* \leqq \int dt\, h(t) \psi_f(t) \psi_f(t)^*$$

and this gives altogether

$$0 \leqq \{\psi_f(h)^*, \psi_f(h)\} \leqq \int dt\, h(t)\, \|\{\psi_f(t)^*, \psi_f(t)\}\|.$$

Now one can exploit the fact that local Fermi operators anti-commute at space-like distances. Taking into account that $2|\bar{f}(t|\mathbf{x}) f(t|\mathbf{y})| \leqq |f(t|\mathbf{x})|^2 + |f(t|\mathbf{y})|^2$, one checks easily that

$$\|\{\psi_f(t)^*, \psi_f(t)\}\| \leqq \int d^3x\, |f(t|\mathbf{x})|^2 \cdot \int d^3z\, \|\{\psi^*, \psi(\mathbf{z})\}\|$$

and since $\|\{\psi^*, \psi(\mathbf{z})\}\|$ has compact support in $\mathbf{z}$ and $\int d^3x\, |f(t|\mathbf{x})|^2 \leqq 2(\|f_+\|^2 + \|f_-\|^2)$, the statement of the lemma follows. □

Let us next consider sequences of functions $h_T \in \mathscr{D}^\#(\mathbb{R}^1)$. If h is any element of $\mathscr{D}^\#(\mathbb{R}^1)$ we define

$$h_T(t) = |T|^{-\varepsilon} h(|T|^{-\varepsilon}(t - T)), \quad T \neq 0, \quad 0 < \varepsilon < 1. \tag{8}$$

h_T is obviously again an element of $\mathscr{D}^\#(\mathbb{R}^1)$. The support of h_T increases for large T like $|T|^\varepsilon$ and the distance of the support from the origin like $|T|$ since $\varepsilon < 1$. The Fourier transform of h_T has the form

$$\tilde{h}_T(\lambda) = (2\pi)^{-1/2} \int dt\, e^{i\lambda t} h_T(t) = e^{i\lambda T} \tilde{h}(|T|^\varepsilon \lambda). \tag{9}$$

It thus converges pointwise in the limit $|T| \to \infty$ to 0 for $\lambda \neq 0$ and to $(2\pi)^{-1/2}$ for $\lambda = 0$. We need h_T to define the approximating sequences $\psi_f(h_T)$ of the asymptotic fields. The next lemma tells us that $\psi_f(h_T)$ converges strongly in the limit of large T if applied to the vacuum vector Ω. For the proof we use standard measure theoretic arguments.

Lemma 2. *Let ψ, f, and h_T be as above. Then*

$$\underset{T\to\pm\infty}{\text{s-lim}}\ \psi_f(h_T)\Omega = P_1\,\psi_{f_-}\Omega \in \mathscr{H}_1\,.$$

Proof. Because of the translation invariance of Ω one gets

$$\psi_f(t)\Omega = e^{i(H+|\boldsymbol{P}|)^{\sim}t}\psi_{f_+}\,\Omega + e^{i(H-|\boldsymbol{P}|)^{\sim}t}\psi_{f_-}\,\Omega\,,$$

where H denotes the Hamiltonian and $\boldsymbol{P}$ the momentum operator. The operators $(H\pm|\boldsymbol{P}|)^{\sim}$ are self-adjoint with the states of finite energy as a core and we shall show in the Appendix that the discrete spectrum of both operators consists only of the single point 0. The eigenspaces of $(H\pm|\boldsymbol{P}|)^{\sim}$ corresponding to this point are $\{c\cdot\Omega\}$ and $\{c\cdot\Omega\}\oplus\mathscr{H}_1$, respectively. Both operators are non-negative and have therefore the spectral decompositions

$$(H+|\boldsymbol{P}|)^{\sim} = 0\cdot P_0 + \int_0^\infty \lambda E_+(d\lambda)$$

$$(H-|\boldsymbol{P}|)^{\sim} = 0\cdot (P_0+P_1) + \int_0^\infty \lambda E_-(d\lambda)\,.$$

P_0 projects onto Ω, P_1 onto $\mathscr{H}_1$ and the projection valued measures $E_\pm(d\lambda)$ are continuous (yet not necessarily absolutely continuous). Using these representations, Eq. (9), and the fact that $(\Omega,\psi\Omega)=0$, one arrives at

$$\psi_f(h_T)\Omega - P_1\,\psi_{f_-}\Omega$$

$$= (2\pi)^{1/2}\int_0^\infty e^{i\lambda T}\tilde{h}(|T|^\varepsilon\lambda)E_+(d\lambda)\psi_{f_+}\,\Omega + (2\pi)^{1/2}\int_0^\infty e^{i\lambda T}\tilde{h}(|T|^\varepsilon\lambda)E_-(d\lambda)\psi_{f_-}\,\Omega\,.$$

The right-hand side of this equation tends strongly to 0 in the limit of large T because

$$\|(2\pi)^{1/2}\int_0^\infty e^{i\lambda T}\tilde{h}(|T|^\varepsilon\lambda)E_\pm(d\lambda)\psi_{f_\pm}\Omega\|^2 = 2\pi\int_0^\infty |\tilde{h}(|T|^\varepsilon\lambda)|^2(\psi_{f_\pm}\Omega, E_\pm(d\lambda)\psi_{f_\pm}\Omega)$$

$$\leqq \int_0^{|T|^{-\varepsilon/2}} (\psi_{f_\pm}\Omega, E_\pm(d\lambda)\psi_{f_\pm}\Omega) + \sup_{\lambda>|T|^{\varepsilon/2}} 2\pi|\tilde{h}(\lambda)|^2\cdot\|\psi_{f_\pm}\Omega\|^2$$

and this finishes the proof of the lemma. □

After having established the convergence of $\psi_f(h_T)$ on the vacuum, it is now fairly simple to prove that $\psi_f(h_T)$ converges strongly itself. To abbreviate the argument we introduce some geometrical notions: we call the open cone of all points which have a positive time-like distance from a given compact region $\mathcal{O}\subset\mathbb{R}^4$ the *future tangent* of $\mathcal{O}$. The *past tangent* of $\mathcal{O}$ is defined analogously. Assume now that $\psi_f(t)$ is localized in $\mathcal{O}$ for small t. As a consequence of the support properties of $f(t\,|\,\boldsymbol{x})$ and $h_T(t)$, $\psi_f(h_T)$ is then localized in a region which is space-like separated from any given compact set in the future tangent of $\mathcal{O}$ for sufficiently large positive T. Because of the commutation relations of local operators at space-like distances, there exists therefore a natural domain for $\lim_{T\to\infty} \psi_f(h_T)$: the set of vectors $F\Omega$ which are created from the vacuum Ω by operators F localized in the future tangent of $\mathcal{O}$.

Lemma 3. *Let ψ, f and h_T be as above. Then*

a) $\operatorname*{s-lim}_{T\to\infty} \psi_f(h_T) = \psi_f^{\text{out}}$ *exists and* $\|\psi_f^{\text{out}}\| < \infty$.

If ψ, f, and h_T vary within the above restrictions ψ_f^{out} is uniquely determined by the one-particle state which it creates from the vacuum.

b) *For all $F_\pm$ localized in the future tangent of $\mathcal{O}$ the equations*

$$\psi_f^{\text{out}} F_\pm \mp F_\pm \psi_f^{\text{out}} = 0$$

hold. If $\hat{\psi}_f(t)$ is localized in $\hat{\mathcal{O}}$ for small t then we have also

$$\psi_f^{\text{out}} \hat{\psi}_{\hat{f}}^{\text{out}}(x) + \hat{\psi}_{\hat{f}}^{\text{out}}(x) \psi_f^{\text{out}} = 0$$

provided $\hat{\mathcal{O}} + x$ lies in the future or past tangent of $\mathcal{O}$.

c) *$\psi_f^{\text{out}}(x)$ is a solution of the wave equation:* $\Box_x \psi_f^{\text{out}}(x) = 0$.[4]

Proof.

a) Since $\psi_f(h_T)$ is uniformly bounded in T (Lemma 1) it suffices to establish the strong convergence on a dense set of vectors. We shall show in the Appendix that the set of all $F\Omega$, F being localized in the future tangent of $\mathcal{O}$, is dense in $\mathcal{H}$[5]. If one decomposes F into its Bose and Fermi parts and takes into account the preceding remarks as well as Lemma 2, one gets

$$\operatorname*{s-lim}_{T\to\infty} \psi_f(h_T) \cdot (F_+ + F_-)\Omega = \operatorname*{s-lim}_{T\to\infty} (F_+ - F_-) \cdot \psi_f(h_T)\Omega = (F_+ - F_-) \cdot P_1 \psi_{f_-} \Omega .$$

Therefore ψ_f^{out} exists and is bounded as a limit of uniformly bounded operators. Assume now that there is another operator $\hat{\psi}_{\hat{f}}^{\text{out}}$ which creates the same one-particle state out of the vacuum as ψ_f^{out}. If $\hat{\psi}_{\hat{f}}^{\text{out}}(t)$ is for small t localized in $\hat{\mathcal{O}}$ one concludes that for all F localized in the future tangent of $\mathcal{O} \cup \hat{\mathcal{O}}$

$$\begin{aligned}(\psi_f^{\text{out}} - \hat{\psi}_{\hat{f}}^{\text{out}}) \cdot F\Omega &= \operatorname*{s-lim}_{T\to\infty} (\psi_f(h_T) - \hat{\psi}_{\hat{f}}(\hat{h}_T)) \cdot (F_+ + F_-)\Omega \\ &= \operatorname*{s-lim}_{T\to\infty} (F_+ - F_-) \cdot (\psi_f(h_T) - \hat{\psi}_{\hat{f}}(\hat{h}_T))\Omega = 0\end{aligned}$$

and this proves $\psi_f^{\text{out}} = \hat{\psi}_{\hat{f}}^{\text{out}}$.

b) The first part of the statement needs no further explanation. The second part is a simple consequence of the first one, if one realizes that $\hat{\psi}_{\hat{f}}^{\text{out}}(x)$ can be approximated by operators F_- localized in the future tangent of $\mathcal{O}$ whenever $\hat{\mathcal{O}} + x$ is a subset of this cone.

c) The operator $\Box_x \psi_f^{\text{out}}(x)$ is bounded, since $\psi(x)$ is two times continuously differentiable with respect to x. It annihilates the vacuum and therefore all states $F\Omega$ with F localized in the future tangent of $\mathcal{O} + x$. $\Box$

[4] This and the subsequent propositions have been formulated for the out-operators. It needs no extra explanation that they hold in an analogous manner if out is replaced by in.

[5] This has to be verified since we did not assume the Reeh-Schlieder property for the vacuum.

Since the operators ψ_f^{out} are uniquely determined by the one-particle states $\mathscr{L}_1$ which they create from the vacuum, it is evident that the above construction of ψ_f^{out} does not depend on a special Lorentz frame. Moreover, the set of all ψ_f^{out} is invariant under Poincaré transformations. In the next lemma we shall show that the anticommutator of any two such operators is a c-number. For the proof we use a (slightly modified) argument due to Pohlmeyer [7].

Lemma 4. *Let ψ_f^{out} and $\hat{\psi}_{\hat{f}}^{\text{out}}$ be two operators with properties specified in the preceding lemma. Then*

$$\{\psi_f^{\text{out}}, \hat{\psi}_{\hat{f}}^{\text{out}}\} = (\Omega, \{\psi_f^{\text{out}}, \hat{\psi}_{\hat{f}}^{\text{out}}\}\Omega) \cdot \mathbb{1}\,.$$

Proof. Since $\{\psi_f^{\text{out}}, \hat{\psi}_{\hat{f}}^{\text{out}}\}$ commutes with all operators F localized in the future tangent of $\mathcal{O} \cup \hat{\mathcal{O}}$ is suffices to show that

$$\{\psi_f^{\text{out}}, \hat{\psi}_{\hat{f}}^{\text{out}}\}\Omega = c \cdot \Omega\,.$$

Now take any vector Φ which has in momentum space compact support K_Φ in the interior of the forward light cone and consider the function

$$F_\Phi(x, y) = (\Phi, \{\psi_f^{\text{out}}(x), \hat{\psi}_{\hat{f}}^{\text{out}}(y)\}\Omega)\,.$$

Since $\psi_f^{\text{out}}(x)$ and $\hat{\psi}_{\hat{f}}^{\text{out}}(y)$ are solutions of the wave equation and because of the restrictions on the energy-momentum spectrum of Φ, the Fourier transform $\tilde{F}_\Phi(p, q)$ has support (in the sense of distributions) in the compact set

$$\{p, q : p_0^2 - |\boldsymbol{p}|^2 = q_0^2 - |\boldsymbol{q}|^2 = 0, p + q \in K_\Phi\}$$

and therefore $F_\Phi(x, y)$ is an entire function. It follows from part (b) of Lemma 3 that $F_\Phi(x, y)$ vanishes in an open set of $\mathbb{R}^8$ and thus it vanishes for all x, y. Hence $\{\psi_f^{\text{out}}, \hat{\psi}_{\hat{f}}^{\text{out}}\}\Omega$ can only be a superposition of Ω and massless one-particle states. Yet the one-particle component of this vector must be zero since $\{\psi_f^{\text{out}}, \hat{\psi}_{\hat{f}}^{\text{out}}\}$ is a Bose operator and there are – by assumption – only massless Fermions in the model. □

Although it is not necessary for the construction of the collision states, it is worth mentioning that the set of operators ψ_f^{out} possesses a local structure which one may call *asymptotic locality* following Landau [8], who introduced this notion in the massive case.

Lemma 5. *If $\psi_f(t)$ and $\hat{\psi}_{\hat{f}}(t)$ are for small t localized in two space-like separated double cones $\mathcal{O}$ and $\hat{\mathcal{O}}$ respectively, then*

$$\{\psi_f^{\text{out}}, \hat{\psi}_{\hat{f}}^{\text{out}}\} = 0\,.$$

Proof. If follows from Eq. (5) and the fact that the Hamiltonian H acts like $|\boldsymbol{P}|$ on $\mathscr{H}_1$ that

$$P_1 \psi_{f_-}\, \Omega = P_1(\psi_{f_1} - i[H, \psi_{f_2}])\Omega = P_1 F_- \Omega$$

where $F_- = \psi_{f_1} - i[H, \psi_{f_2}]$ is a Fermi operator localized in $\mathcal{O}$. In the same way one shows

$$P_1 \hat{\psi}_{\hat{f}_-}\, \Omega = P_1(\hat{\psi}_{\hat{f}_1} - i[H, \hat{\psi}_{\hat{f}_2}])\Omega = P_1 \hat{F}_- \Omega$$

and $\hat{F}_-$ is localized in $\hat{\mathcal{O}}$. If one applies the techniques of the Jost-Lehmann-Dyson representation to $(\Omega, \{F_-(x), \hat{F}_-\}\Omega)$ one can conclude that

$$(\Omega, F_- P(m=0)\hat{F}_- \Omega) + (\Omega, \hat{F}_- P(m=0) F_- \Omega) = 0$$

where $P(m=0) = P_0 + P_1$ denotes the projection onto the zero mass states [8, 9]. From this relation, Lemma 4 and the fact that

$$(\Omega, \{\psi_f^{\text{out}}, \hat{\psi}_f^{\text{out}}\}\Omega) = (\Omega, F_- P(m=0)\hat{F}_- \Omega) + (\Omega, \hat{F}_- P(m=0) F_- \Omega)$$

the statement then follows. □

We are now in a position to construct the collision states for the massless Fermions with the help of the operators ψ_f^{out}. Since these operators have all the properties of a (smeared) free field, we can proceed as in the free field case. First we define the creation part $(\psi_f^{\text{out}})^{(+)}$ of ψ_f^{out}. For this purpose we take a uniformly bounded sequence of functions $\tilde{h}_n(p) \in \mathcal{D}(\mathbb{R}^4)$ which are zero in the half space of negative energy $p_0 \leqq 0$ and which converge uniformly to 1 on each compact subset of the half space of positive energy $p_0 > 0$. Then we integrate ψ_f^{out} with these functions

$$\psi_f^{\text{out}}(h_n) = (2\pi)^{-2} \int d^4x\, h_n(x) \psi_f^{\text{out}}(x).$$

Applying Lemma 4 we get

$$\|\psi_f^{\text{out}}(h_n - h_m)\| \leqq \|\psi_f^{\text{out}}(h_n - h_m)\Omega\|$$

and from this inequality and the properties of the functions h_n the existence of the uniform limit

$$(\psi_f^{\text{out}})^{(+)} = \lim_{n\to\infty} \psi_f^{\text{out}}(h_n) \tag{10}$$

follows at once. In the same way one establishes the existence of the destruction part $(\psi_f^{\text{out}})^{(-)}$ of ψ_f^{out}

$$(\psi_f^{\text{out}})^{(-)} = \lim_{n\to\infty} \psi_f^{\text{out}}(\bar{h}_n) = (\psi_f^{\text{out}*})^{(+)*}. \tag{11}$$

It is clear that the operators $(\psi_f^{\text{out}})^{(\pm)}$ do not depend on the special choice of the sequence h_n within the above restrictions. Bearing in mind that $\psi_f^{\text{out}}(x)$ is a solution of the wave equation and therefore has in momentum space its support on the forward and backward light cone, one can also easily verify that

$$U(L)(\psi_f^{\text{out}})^{(\pm)} U(L)^{-1} = (U(L)\psi_f^{\text{out}} U(L)^{-1})^{(\pm)}, \qquad L \in \mathcal{P}. \tag{12}$$

Finally it follows from Lemma 4 that the operators $(\psi_f^{\text{out}})^{(\pm)}$ have the commutation relations

$$\begin{aligned} \{(\psi_f^{\text{out}})^{(+)}, (\hat{\psi}_f^{\text{out}})^{(+)}\} &= \{(\psi_f^{\text{out}})^{(-)}, (\hat{\psi}_f^{\text{out}})^{(-)}\} = 0 \\ \{(\psi_f^{\text{out}})^{(-)}, (\hat{\psi}_f^{\text{out}})^{(+)}\} &= (\Omega, \psi_f^{\text{out}} \hat{\psi}_f^{\text{out}} \Omega) \cdot \mathbb{1}. \end{aligned} \tag{13}$$

Now let $\psi_1^{\text{out}}, \ldots, \psi_n^{\text{out}}$ be n operators of type ψ_f^{out} which create one-particle states $\Phi_1, \ldots, \Phi_n \in \mathcal{L}_1$ from the vacuum. We define the outgoing collision states of these particles by

$$\Phi_1 \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} \Phi_n = (\psi_1^{\text{out}})^{(+)} \ldots (\psi_n^{\text{out}})^{(+)}\Omega. \tag{14}$$

The following main theorem is then a simple consequence of the algebraic properties of $(\psi_k^{\text{out}})^{(\pm)}$ and $(\psi_k^{\text{out}})^{(-)}\Omega = 0$.

Theorem. *The states* $\overset{\text{out}}{\Phi_1} \times \cdots \times \overset{\text{out}}{\Phi_n}$ *have the properties:*

a) $\overset{\text{out}}{\Phi_1} \times \cdots \times \overset{\text{out}}{\Phi_n} = \sigma_P \cdot \overset{\text{out}}{\Phi_{p(1)}} \times \cdots \times \overset{\text{out}}{\Phi_{p(n)}}$, *where* $P = (p(1), \ldots, p(n))$ *is any permutation of the numbers* $(1, \ldots, n)$ *and* $\sigma_P = \pm 1$ *if* P *is an even or odd permutation, respectively.*

b) $U(L)(\overset{\text{out}}{\Phi_1} \times \cdots \times \overset{\text{out}}{\Phi_n}) = (U_1(L)\overset{\text{out}}{\Phi_1}) \times \cdots \times (U_1(L)\overset{\text{out}}{\Phi_n}), L \in \mathscr{P}$ *and* $U_1(L)$ *denotes the representation of the Poincaré transformations in* $\mathscr{H}_1$.

c) $(\overset{\text{out}}{\Phi_1} \times \cdots \times \overset{\text{out}}{\Phi_m}, \overset{\text{out}}{\Phi'_1} \times \cdots \times \overset{\text{out}}{\Phi'_n}) = \delta_{mn} \cdot \sum_P \sigma_P (\Phi_1, \Phi'_{p(1)}) \ldots (\Phi_n, \Phi'_{p(n)})$ *and the sum extends over all permutations* P *of the numbers* $(1, \ldots, n)$.

The theorem implies that the Hilbert space $\mathscr{H}^{\text{out}}$, which is generated by $\overset{\text{out}}{\Phi_1} \times \cdots \times \overset{\text{out}}{\Phi_n}, n \in \mathbb{N}$ and Ω is a Fock space over the one-particle space $\mathscr{H}_1$ of massless Fermions. Thus the vectors $\overset{\text{out}}{\Phi_1} \times \cdots \times \overset{\text{out}}{\Phi_n}$ can be interpreted as outgoing configurations of non-interacting particles $\Phi_1, \ldots, \Phi_n$ and this allows the usual definition and interpretation of an S-matrix for the massless Fermions.

3. Concluding Remarks

The Huyghens principle is not only responsible for the existence of collision states in the massless case, but it reflects itself also in some geometrical relations between the basic net $\mathfrak{F}$ and the net of the asymptotic fields. To illustrate this fact we construct the asymptotic algebra $\mathfrak{F}^{\text{out}}$ which is generated by the free fields ψ_f^{out}. Let $\psi_f(t)$ be any operator which is for small t localized in some double cone $\mathcal{O}_1$ and put

$$\psi_f^{\text{out}} = \underset{T \to \infty}{\text{s-lim}}\, \psi_f(h_T).$$

We denote by $\mathfrak{F}^{\text{out}}(\mathcal{O}_1)$ the von Neumann algebra which is generated by all such operators ψ_f^{out}. For arbitrary bounded regions we define $\mathfrak{F}^{\text{out}}(\mathcal{O})$ as the von Neumann algebra which is generated by the algebras $\mathfrak{F}^{\text{out}}(\mathcal{O}_1)$, $\mathcal{O}_1 \subset \mathcal{O}$. $\mathfrak{F}^{\text{out}}$ is then the global algebra of all $\mathfrak{F}^{\text{out}}(\mathcal{O})$. It can easily be checked using the results of the preceding section that $\mathcal{O} \to \mathfrak{F}^{\text{out}}(\mathcal{O})$ is a local, covariant net with all the properties usually required from a field algebra.

It follows now from part (*b*) of Lemma 3 that for all $F^{\text{out}} \in \mathfrak{F}^{\text{out}}(\mathcal{O})$ and arbitrary $F' \in \mathfrak{F}$ which are localized in the future tangent of $\mathcal{O}$ the following remarkable commutation relations hold:

$$\begin{aligned} [F_+^{\text{out}}, F'_+] &= [F_+^{\text{out}}, F'_-] = 0 \\ [F_-^{\text{out}}, F'_+] &= \{F_-^{\text{out}}, F'_-\} = 0 . \end{aligned} \tag{15}$$

These relations are the field theoretic version of Huyghens principle. They say, for example, that the influence coming from an asymptotic field $F^{\text{out}} \in \mathfrak{F}^{\text{out}}(\mathcal{O})$ does not disturb any measurement in the future tangent of $\mathcal{O}$. Analogously a field $F^{\text{in}} \in \mathfrak{F}^{\text{in}}(\mathcal{O})$ cannot be disturbed by any measurements in the past tangent of $\mathcal{O}$.

One might get the idea that in a model describing exclusively massless particles relation (15) should also hold for all F' localized in the past tangent of $\mathcal{O}$. (This would be, for example, the case if the commutation relations (2) for the basic net $\mathfrak{F}$

hold for space-like and time-like separated regions $\mathcal{O}_1$ and $\mathcal{O}_2$.) But this would imply that

$$[(\psi_f^{\text{out}} - \psi_f^{\text{in}}), F'_+] = \{(\psi_f^{\text{out}} - \psi_f^{\text{in}}), F'_-\} = 0$$

for $\psi_f^{\text{out}} \in \mathfrak{F}^{\text{out}}(\mathcal{O})$, $\psi_f^{\text{in}} \in \mathfrak{F}^{\text{in}}(\mathcal{O})$ and arbitrary F' localized in the past tangent of $\mathcal{O}$. Since $(\psi_f^{\text{out}} - \psi_f^{\text{in}})\Omega = 0$, one could then conclude that $\psi_f^{\text{out}} = \psi_f^{\text{in}}$. Hence there would be no scattering in the model.

Appendix

Here we give the proofs of two statements which were made in Section 2. First we calculate the point spectrum of the operators $(H \pm |\boldsymbol{P}|)^{\sim}$. As one expects from the continuity of the Poincaré transformations, the only possible eigenvalue of both operators is 0.

Lemma. *The discrete spectrum of the self-adjoint operators* $(H \pm |\boldsymbol{P}|)^{\sim}$ *which act like* $(H \pm |\boldsymbol{P}|)$ *on the states of finite energy consists of the single point* 0. *The corresponding eigen spaces are* $\{c \cdot \Omega\}$ *and* $\{c \cdot \Omega\} \oplus \mathcal{H}_1$, *respectively.*

Proof. Let Φ be an eigenstate of $(H + |\boldsymbol{P}|)^{\sim}$. Since $(H + |\boldsymbol{P}|)^{\sim}$ commutes with the spectral projections of $(H, \boldsymbol{P})$, one may assume that Φ is a state of finite energy, hence $(H + |\boldsymbol{P}|)\Phi = E\Phi$. If one multiplies this equation by $(H - |\boldsymbol{P}|)$ one gets (with $M^2 = H^2 - |\boldsymbol{P}|^2$)

$$M^2\Phi = E(H - |\boldsymbol{P}|)\Phi = E(2H - E)\Phi\,.$$

Now let Λ be a Lorentz boost in the Λ-direction and $\Phi_\Lambda = U(\Lambda)\Phi$. If follows from the preceding equation that

$$M^2\Phi_\Lambda = E(2(1 + |\Lambda|^2)^{1/2} \cdot H - 2(\Lambda \boldsymbol{P}) - E)\Phi_\Lambda$$

and this gives for the scalar product with Φ

$$E \cdot (\Phi, (2H - E)\Phi_\Lambda) = E \cdot (\Phi, (2(1 + |\Lambda|^2)^{1/2} \cdot H - 2(\Lambda \boldsymbol{P}) - E)\Phi_\Lambda)\,.$$

For $E \neq 0$ one gets therefore

$$((1 + |\Lambda|^2)^{1/2} - 1) \cdot (\Phi, H\Phi_\Lambda) = (\Phi, (\Lambda \boldsymbol{P})\Phi_\Lambda)\,.$$

If one now puts $\Lambda = \lambda \cdot \boldsymbol{n}$ and takes into account the continuity of Φ_Λ in λ it follows from this equation (after dividing by λ and going to the limit $\lambda \to 0$) that

$$(\Phi, (\boldsymbol{n}\boldsymbol{P})\Phi) = 0\,.$$

The same equation holds if Φ is replaced by $E(\Delta)\Phi$, where Δ is any Borel subset of the spectrum of $(H, \boldsymbol{P})$ and $E(\Delta)$ is the projection onto the corresponding subspace of $\mathcal{H}$. Consequently, $\Phi = c \cdot \Omega$ and this implies $\Phi = 0$ because of $E \neq 0$. Therefore the only eigenvalue of $(H + |\boldsymbol{P}|)^{\sim}$ is 0 and it is then obvious that $\{c \cdot \Omega\}$ is the corresponding eigenspace. The statement for $(H - |\boldsymbol{P}|)^{\sim}$ can be verified in the same way. □

In the second lemma of this Appendix, we shall show that the set of vectors $F\Omega$, F localized in the future tangent (or past tangent) of a compact set $\mathcal{O}$, is dense in $\mathcal{H}$. For the proof we exploit only the spectrum condition and the transformation properties of the net $\mathfrak{F}$ under translations. Hence the statement is also true in models where the vacuum does not have the Reeh-Schlieder property.

Lemma. *The vectors $F\Omega$, F localized in the future tangent of $\mathcal{O}$ are dense in $\mathcal{H}$.*

Proof. Let G be any local operator and Φ be any vector. As a consequence of the spectrum condition, the function $x \to (\Phi, G(x)\Omega)$ is analytic in the forward tube and thus cannot vanish in an open set of $\mathbb{R}^4$ unless it vanishes for all x. Since the vectors $G\Omega$, G local form a dense set in $\mathcal{H}$, and since every local operator G can be shifted by a time-like transformation into the future tangent of $\mathcal{O}$, it is obvious that there does not exist any vector $\Phi \neq 0$ which is orthogonal to all $F\Omega$, F localized in the future tangent of $\mathcal{O}$. □

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