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On the Perturbation of Gibbs Semigroups

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Abstract. The trace-norm convergence of the Hille-Phillips perturbation series is proved for the whole perturbation class of the generator of a Gibbs semigroup.

In Ref. [1], Uhlenbrock proposed the following terminology:

Definition. A selfadjoint semigroup $\{T(t)\}_{t \geqq 0}$ in a separable Hilbert space with the property:

$$\operatorname{tr} T(t) < \infty, \qquad \forall t > 0 \tag{1}$$

is called a Gibbs semigroup; and raised the problem of proving the trace-norm convergence of the Hille-Phillips perturbation series [2] for a conveniently large class of perturbations of the generator of a Gibbs semigroup. He gave also a proof of trace-norm convergence in the case of bounded perturbations, based on an inequality due to Ginibre and Gruber [3].

The aim of this note is to point out that a slight modification of this very argument allows to prove the trace-norm convergence of the series for the whole Hille-Phillips perturbation class.

Proposition. *Let $T(t)$ be a Gibbs semigroup and A its generator. Let B be A-bounded and such that:*

$$\int_0^1 \|B\,T(t)\|\, dt < \infty . \tag{2}$$

Then the series:

$$S(t) = \sum_{n=0}^{\infty} S_n(t) \tag{3}$$

with:

$$S_0(t) = T(t); \qquad S_n(t) = \int_0^t ds\, S_0(t-s)\, B S_{n-1}(s) \tag{4}$$

is $\|\cdot\|_1$-convergent uniformly for t in compact subsets of $(0, \infty)$. In particular, if B is moreover symmetric, then $S(t)$ is a Gibbs semigroup.

Proof. If B is A-bounded, then $B\,T(t) = [B\,R(\lambda, A)]\,[(\lambda - A)\,T(t)]$ is bounded and condition (2) makes sense. One can write $S_n(t)$ as a multiple (trace-norm) Bôchner integral:

$$S_n(t) = \int \cdots \int ds_1 \ldots ds_n\, \chi_n^t(s_0, s_1, \ldots, s_n)\, S_0(s_0)\, B S_0(s_1) \ldots B S_0(s_n), \tag{5}$$

where χ_n^t is the characteristic function of the set: $s_i \geqq 0$, $i=0, \ldots, n, \Sigma_{i=0}^n s_i = t$. Then:

$$\begin{aligned}\|S_n(t)\|_1 \leqq \int \cdots \int ds_1 \ldots ds_n \chi_n^t(s_0, s_1, \ldots, s_n) \\ \cdot \|S_0(s_0)\, B S_0(s_1) \ldots B S_0(s_n)\|_1 .\end{aligned} \tag{6}$$

We shall now use the inequality of Ginibre and Gruber [3]:

$$\left\| \prod_{i=0}^n A_i T(s_i) \right\|_1 \leqq \left(\prod_{i=0}^n \|A_i\| \right) \operatorname{tr} T\left(\sum_{i=0}^n s_i \right) \tag{7}$$

for every Gibbs semigroup $T(t)$ and bounded operators $A_0, \ldots, A_n$. We shall take $A_0 = S_0(s_0/2)$ and $A_i = B S_0(s_i/2)$, $i = 1, 2, \ldots, n$. Denoting $\varphi(t) = \|S_0(t)\| = \exp(\omega_0 t)$, and $\psi(t) = \|B S_0(t)\|$, we obtain from (6) and (7):

$$\|S_n(t)\|_1 \leqq 2^n \cdot \varphi * \psi^{*n}(t/2) \cdot \operatorname{tr} S_0(t/2) \tag{8}$$

wherefrom one can proceed as in [2], Theorem 13.4.1, showing that there exist constants $\omega > \omega_0$ and $\gamma < 1$, such that:

$$\|S_n(t)\|_1 \leqq \gamma^n \operatorname{tr} S_0(t/2) \exp(\omega t) \tag{9}$$

which finishes the proof.

Corollary. *Let $T(t)$ be a Gibbs semigroup and A its generator. Let $D \subset \mathbb{C}$ be a domain and, for every $z \in D$, let $B(z)$ be given such that:*

(i) *$B(z)$ is A-bounded and $z \rightsquigarrow B(z)\, T(t)$ is bounded-analytic on D for every $t > 0$.*

(ii) $\int_0^1 \sup_{z \in D} \|B(z)\, T(t)\|\, dt < \infty$.

If $S(z;t)$ is the semigroup generated by $A + B(z)$, then $z \rightsquigarrow S(z;t)$ is $\|\cdot\|_1$-analytic on D for every $t > 0$.

References

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