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On Completeness of Eigenfunctions of the One-Speed Transport Equation

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Abstract. It is shown that the set of Case's eigenfunctions of the one speed transport equation is complete in the rigged Hilbert space $W_2^1([-1,1]) \subset L_2(-1,1) \subset W_2^{-1}([-1,1])$.

1. Introduction

Case's method of singular eigenfunction expansions for solving the transport equation [1] seeks, by separation of variables, to construct a sufficiently rich set of solutions, called elementary solutions, which would enable one to expand an "arbitrary" solution of the equation into a Fourier series in terms of this set. An important point in this method is the completeness proof for the set of elementary solutions. Originally it was shown [2], by means of the theory of singular integral equations, that the expansion coefficients are uniquely determined for the class of Hölder-continuous functions, which, within this class, proves the completeness. An alternative to this constructive approach is the demonstration of the closure relation for the set of elementary solutions [1]. Unfortunately, either proof has to be carried out separately for each particular from of the transport equation under consideration, and moreover, there remains some doubt as to whether the obtained result is the strongest possible.

According to an idea by A. Skumanich, commented upon in Ref. [1], the completeness proof for Case's elementary solutions should be based on more general arguments, provided by the functional-analytic properties of the underlaying transport operator. This would lead to the completeness proof for a whole class of operators which have certain common properties.

The functional analytic approach to the problem was considered by Hangelbroek [3] and by Larsen and Habetler [4], where it was essentially shown that the Case eigenfunction expansion formula represents the resolution of the identity of a transport operator, but again only after resorting to a kind of Hölder continuity requirement. There remains some ambiguity about the notion of the eigenfunction, which is also referred to by Baird and Zweifel [5], and the structure of the space of eigenfunctions remains unclear.

Here we propose a completeness proof which is based on the theory of eigenfunction expansions for self adjoint operators in rigged Hilbert spaces, as expounded in the treatise by Berezanskiĭ [6]. A rigged Hilbert space, of the type to be considered, is a triple of separable Hilbert spaces $H_+ \subset H \subset H_-$, where H_+, the positive space, is dense in H, and H_-, the negative space, is isometric to the dual space of H_+ and contains H as a dense subspace. Rigged Hilbert spaces are the most appropriate spaces for spectral decomposition of self adjoint operators [7], this being due to the following properties. Any continuous linear functional on H_+ may be represented by some element of H_- by means of the scalar product in H, instead of the scalar product in H_+ itself. If H_+ consists of finitely differentiable functions, then the elements of H_- are generalized functions of finite order. If the embedding of H_+ into H is quasi-nuclear, i.e. the embedding operator is Hilbert-Schmidt, then any self adjoint operator in H, which admits the so called rigged extension, possesses a complete set of eigenfunctions which are elements of H_-.

The proposed approach is undertaken here for the special case of the one-speed transport equation with isotropic scattering, and c, the number of secondaries per collision, smaller than 1. We were unable to extend it to the case with $c \geqq 1$, even though no difficulties are encountered here in the constructive proofs. It is applicable to the self adjoint eigenvalue problem of the form $Tf = \nu f$, or to the eigenvalue problem $Af = \nu B$, with self adjoint A and positive definite B.

2. The Eigenvalue Problem

Consider the one-speed steady-state transport equation with isotropic scattering in plane-parallel geometry [1, 2]

$$\mu \frac{\partial}{\partial x} \Psi(x, \mu) + \Psi(x, \mu) = \frac{c}{2} \int_{-1}^{1} \Psi(x, \mu) \, d\mu \,, \tag{1}$$

with $0 < c < 1$.

By seeking its solutions in the form

$$\Psi_\nu(x, \mu) = \varphi_\nu(\mu) \exp(-x/\nu) \,,$$

where ν is a separation parameter to be determined, the task of solving Eq. (1) is reduced to the formal eigenvalue problem posed by the equation

$$\mu \varphi_\nu(\mu) = \nu \left[\varphi_\nu(\mu) - \frac{c}{2} \int_{-1}^{1} \varphi_\nu(\mu) \, d\mu \right] . \tag{2}$$

It may be shown that the integral of φ_ν cannot vanish unless this function is identically equal to zero.

The eigenvalue problem (2) will be placed into the Hilbert space $L_2(-1, 1)$, with the usual scalar product to be denoted by $(\; , \;)_0$. By introducing two operators A and B,

$$Af = \mu f(\mu)$$

$$Bf = f(\mu) - \frac{c}{2} \int_{-1}^{1} f(\mu) \, d\mu \,,$$

where f is in $L_2(-1, 1)$, the solutions of Eq. (2) may be sought as eigenfunctions of the eigenvalue problem

$$A\varphi = \nu B \varphi \,. \tag{3}$$

The operator A, multiplication by the independent variable in $L_2(-1,1)$, is continuous and self adjoint, and has a purely continuous spectrum consisting of the closed interval $[-1,1]$.

The operator B is continuous, one-to-one, and, for $0<c<1$, positive definite, since $(Bf,f)_0>(1-c)(f,f)_0$. By writing B as $E-cP$, where E is the identity operator and P the projection

$$Pf=\frac{1}{2}\int_{-1}^{1} f(\mu)\,d\mu\,,$$

the positive square roots $B^{\frac{1}{2}}$ and $B^{-\frac{1}{2}}$ may be evaluated by means of the Neumann series, and one gets

$$B^{\pm\frac{1}{2}}=E+[(1-c)^{\pm\frac{1}{2}}-1]\,P\,.$$

This enables us to rewrite Eq. (3) in the form of an equivalent self adjoint eigenvalue problem

$$T\Phi=\nu\Phi\,, \tag{4}$$

where

$$T=B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\,,$$

and $\Phi=B^{\frac{1}{2}}\varphi$. Explicitly this equation reads as

$$\mu\Phi(\mu)+[(1-c)^{-\frac{1}{2}}-1]\int_{-1}^{1}(\mu+\mu')\,\Phi(\mu')\,d\mu'=\nu\,\Phi(\mu)\,. \tag{5}$$

The continuous and self adjoint operator T is, as evident from Eq. (5), equal to the sum of the operator A and a completely continuous integral operator. The operator T has the same continuous spectrum as the operator A, since this part of the spectrum is conserved under completely continuous perturbations according to the Weyl-von Neumann theorem [8]. The integral operator in Rq. (5) contributes two real eigenvalues to the spectrum of T, and it may be shown that these eigenvalues satisfy the equation

$$1-\frac{1}{2}c\nu\log\frac{\nu+1}{\nu-1}=0\,. \tag{6}$$

To summarize, the spectrum of T consists of a continuous part, being the closed interval $[-1,1]$, and of two eigenvalues, to be denoted by ν_0 and $-\nu_0$, which are the roots of Eq. (6).

The eigenfunctions Φ_{ν_0} and $\Phi_{-\nu_0}$ of T, which correspond to eigenvalues ν_0 and $-\nu_0$, are orthogonal, and the orthogonality relation may be written as $(B\varphi_{\nu_0},\varphi_{-\nu_0})_0=0$, where $\varphi_{\pm\nu_0}=B^{-\frac{1}{2}}\Phi_{\pm\nu_0}$ are the eigenfunctions of the eigenvalue problem (3). By taking into account Eq. (3), this relation may be written in the familiar form of

$$\int_{-1}^{1}\mu\,\varphi_{\nu_0}(\mu)\,\varphi_{-\nu_0}(\mu)\,d\mu=0\,.$$

The operator T does not have a complete set of eigenfunction within $L_2(-1,1)$. However, such a set may be found in the more general setup of a rigged Hilbert space.

3. The Rigged Hilbert Space

The self adjoint eigenvalue problem (4) will now be considered in the rigged Hilbert space

$$W_2^1([-1,1]) \subset L_2(-1,1) \subset W_2^{-1}([-1,1]) . \tag{7}$$

The Sobolev space $W_2^1([-1,1])$, the positive space of the rigged Hilbert space (7), is a Hilbert space obtained by completion of the space $C^1([-1,1])$ of functions with continuous first derivative on the closed interval $[-1,1]$, by means of the scalar product

$$(u, v)_+ = (u, v)_0 + (u', v')_0 ,$$

where u and v are in $C^1([-1,1])$. The space $W_2^{-1}([-1,1])$, the negative space of the rigged Hilbert space (7), is isometric to the dual space of $W_2^1([-1,1])$. It is obtained by completion of $L_2(-1,1)$ by means of the sclar product [9]

$$(f, g)_- = \int_{-1}^{1} \int_{-1}^{1} G(\mu, \mu') \, f(\mu) \, g(\mu') \, d\mu \, d\mu' ,$$

where f and g are in $L_2(-1,1)$, and G is the Green function of the boundary value problem

$$-u''(\mu) + u(\mu) = 0 ,$$

$$u'(-1) = u'(1) = 0 .$$

The space $W_2^1([-1,1])$ is dense in $L_2(-1,1)$, and its embedding is quasi-nuclear [10]. The same holds for $L_2(-1,1)$ with respect to $W_2^{-1}([-1,1])$. According to the Sobolev embedding theorem [11] $W_2^1([-1,1])$ is a subspace of $C([-1,1])$, the space of continuous functions on the closed interval $[-1,1]$. The elements of $W_2^1([-1,1])$ are characterized as follows [12]: a function is in $W_2^1([-1,1])$ if and only if it is in $L_2(-1,1)$, and has a derivative, in the generalized sense of Sobolev, which is in $L_2(-1,1)$. The elements of $W_2^{-1}([-1,1])$ are generalized functions of the first order. Any continuous linear functional on $W_2^1([-1,1])$ can be uniquely represented by some element α of $W_2^{-1}([-1,1])$ by means of the scalar product $(u,\alpha)_0$, where u is in $W_2^1([-1,1])$. The space $W_2^{-1}([-1,1)]$ contains the Dirac delta-function $\delta(\mu - v)$, which is continuous with respect to v on $[-1,1]$, and whose operation on elements of $W_2^1([-1,1])$ is given by $(u, \delta(\mu - v))_0 = u(v)$, $v \in [-1,1]$ [13]. The scalar product $(\delta(\mu - v), \delta(\mu - v'))_-$ is equal to $G(v, v')$, where G is the above mentioned Green's function. The space spanned by delta-functions $\delta(\mu - v)$, with v in a dense subset of the interval $(-1,1)$, is dense in $W_2^{-1}([-1,1])$. The space $W_2^{-1}([-1,1])$ contains the set of all real-valued measures of bounded variation defined on the Borel subsets of the closed interval $[-1,1]$ [14].

The choice of the rigged Hilbert space (7) was motivated by the desire to keep it as tight as possible, so that the negative space is only as big as necessary. Since the functions in the positive space are only continuous on $[-1,1]$, the negative space does not contain generalized functions of higher order, such as the derivatives of the delta-function.

4. Eigenfunctions and Completeness

The self adjoint operator T admits a rigged extension [15] in the rigged Hilbert space (7): it continuously maps the space $W_2^1([-1,1])$ into itself. There exists the rigged extension $\tilde{T}$ of T, which continuously maps $W_2^{-1}([-1,1])$ into itself, so that

$$(Tu, \Phi)_0 = (u, \tilde{T}\Phi)_0\,,$$

for all u in $W_2^1([-1,1])$ and all Φ in $W_2^{-1}([-1,1])$. This extension is defined by

$$\tilde{T}\Phi = \mu\Phi + \tfrac{1}{2}[(1-c)^{-\frac{1}{2}} - 1]\,[\mu(1,\Phi)_0 + (\mu,\Phi)_0]\,, \tag{8}$$

with Φ in $W_2^{-1}([-1,1])$.

Analogously, the operators A, B, and $B^{\pm\frac{1}{2}}$ have rigged extensions which are given by

$$\begin{aligned} \tilde{A}\Phi &= \mu\Phi\,, \\ \tilde{B}\Phi &= \Phi - \tfrac{c}{2}(1,\Phi)_0\,, \\ \widetilde{B^{\pm\frac{1}{2}}}\Phi &= \Phi + \tfrac{1}{2}[(1-c)^{-\frac{1}{2}} - 1]\,(1,\Phi)_0\,. \end{aligned} \tag{9}$$

The operator $\tilde{T}$ is then equal to $\widetilde{B^{-\frac{1}{2}}}\tilde{A}\widetilde{B^{-\frac{1}{2}}}$, and the inverse of $\widetilde{B^{\pm\frac{1}{2}}}$ is $\widetilde{B^{\mp\frac{1}{2}}}$.

To the self adjoint operator T in the rigged Hilbert space (7) then apply the completeness and eigenfunction expansion theorems [16], which can be summarized as follows.

There exists a non-negative finite measure ϱ, the spectral measure of the operator T, defined on the Borel subsets of the real line, with support on the spectrum of T, and ϱ-almost everywhere there exists the operator-valued function $P(\nu)$ from $W_2^1([-1,1])$ into $W_2^{-1}([-1,1])$, whose values are positive Hilbert-Schmidt operators. The operator $P(\nu)$ is the derivative, in terms of the Hilbert operator norm, of the resolution of the identity E_ν of T with respect to the measure ϱ. Operators $P(\nu)$ project $W_2^1([-1,1])$ into $W_2^{-1}([-1,1])$, and this projection is orthogonal: if $(P(\nu)\,v, u)_0 = 0$ for all v in $W_2^1([-1,1])$, then $P(\nu)\,u = 0$.

The range of $P(\nu)$ is the generalized eigenspace of the operator T corresponding to the eigenvalue ν. Its elements $\Phi_\nu = P(\nu)\,v$, $v \in W_2^1([-1,1])$, are such that

$$(\Phi_\nu, (T - \nu E)\,u)_0 = 0 \tag{10}$$

for all u in $W_2^1([-1,1])$. Equivalently, they are eigenfunctions of the rigged extension $\tilde{T}$,

$$\tilde{T}\Phi_\nu = \nu\Phi_\nu\,. \tag{11}$$

The set of eigenvalues of $\tilde{T}$ is the spectrum of T. The eigenvalues are non-degenerate, as can be verified from Eq. (8), so that the range of $P(\nu)$ is one-dimensional.

Equation (11) has two continuous solutions Φ_{ν_0} and $\Phi_{-\nu_0}$ which correspond to eigenvalues ν_0 and $-\nu_0$, respectively. For ν on $[-1,1]$ the solutions Φ_ν are generalized functions from $W_2^{-1}([-1,1])$.

The set of eigenfunctions Φ_ν of Eq. (11) is complete in the sense that any function u from $W_2^1([-1, 1])$ can be expanded in terms of this set as

$$u = \int_{-\infty}^{+\infty} (u, \Phi_\nu)_0 \, \Phi_\nu \, d\varrho \,, \tag{12}$$

with $(\Phi_\nu, \Phi_\nu)_- = 1$, where the expansion coefficients satisfy the closure relation

$$(u, u)_0 = \int_{-\infty}^{+\infty} (u, \Phi_\nu)_0^2 \, d\varrho \,. \tag{13}$$

Analogous results may be formulated for the original eigenvalue problem, Eq. (3), where we obtain that the functions

$$\varphi_\nu = \widetilde{B^{-\frac{1}{2}}} \Phi_\nu \,,$$

which also belong to $W_2^{-1}([-1, 1])$, are solutions, in the sense of Eq. (10), of the eigenvalue problem

$$\tilde{A} \varphi_\nu = \nu \tilde{B} \varphi_\nu \,.$$

If Eq. (12) is written for functions of the form $B^{\frac{1}{2}} u$, $u \in W_2^1([-1, 1])$, and then multiplied by $\widetilde{B^{-\frac{1}{2}}}$, we obtain the corresponding eigenfunction expansion and closure relation as

$$u = \int_{-\infty}^{+\infty} (Bu, \varphi_\nu)_0 \, \varphi_\nu \, d\varrho \,, \tag{14}$$

and

$$(Bu, u)_0 = \int_{-\infty}^{-\infty} (Bu, \varphi_\nu)_0^2 \, d\varrho \,.$$

A comparison of Eq. (14) with the explicit Case's eigenfunction expansion formula [1, 2] and taking into account that $(Bu, \varphi_\nu)_0 = (1/\nu)(Au, \varphi_\nu)_0$, shows that $\varrho(\pm \nu_0) = \nu_0 / N(\nu_0)$, $d\varrho = \nu \, d\nu / N(\nu)$, for ν in $(-1, 1)$, and $d\varrho = 0$ everywhere else.

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