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Quantized Fields in Interaction with External Fields

I. Exact Solutions and Perturbative Expansions

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Abstract. We consider a massive, charged, scalar quantized field interacting with an external classical field. Guided by renormalized perturbation theory we show that whenever the integral equations defining the Feynman or retarded or advanced interaction kernel possess non perturbative solutions, there exists an S-operator which satisfies, up to a phase, the axioms of Bogoliubov, and is given for small external fields by a power series which converges on coherent states. Furthermore this construction is shown to be equivalent to the one based on the Yang-Källen-Feldman equation. This is a consequence of the relations between chronological and retarded Green's functions which are described in detail.

Introduction

Numerous papers have been devoted to the study of the interactions of particles with external fields, within the framework of quantum field theory.

The formal aspects were well developped twenty years ago, in particular through the work of Feynman [1], Matthews and Salam [2] and Schwinger [3]. A good summary can be found e.g. in Thirring's book on Quantum Electrodynamics [4].

Mathematically rigorous non perturbative treatments were given in some particular cases by several authors. Capri [5] has explained lucidly "the reduction to c-number problem" (cf. also Wightman [6]). Bongaarts [7] has treated the case of spin 1/2 particles in a stationary external field. Seiler [8], using the results of a paper by Schroer, Seiler and Swieca [9], proved the existence of a fixed time evolution operator in the following cases: scalar and pure electric external fields for spin 0 and spin 1/2 fields. Wightman [6] has given necessary and sufficient conditions for the existence of a unitary S-operator for arbitrary spins, in the case of external field coupled with quantized fields which fulfil a first order system of partial differential equations.

The perturbative aspects of quantum field theory described for instance in Bogoliubov and Shirkov's classical book [10] have recently been further developped by Epstein and Glaser [11], Steinmann [12] and Zimmermann [13].

These authors have proved the existence of an S-operator as an operator valued formal power series, for arbitrary Wick polynomial Lagrangeans.

The aim of this paper is to exhibit the connection between Bogoliubov's version of perturbation theory and non pertubative methods developped by the above mentioned authors. For the sake of simplicity we shall consider the problem of a charged scalar boson field interacting with a classical external field.

Possible extensions to more general cases will be mentioned at each step. The main problem is to construct a solution of the Bogoliubov axioms as an operator in the Fock space of the in-field.

In Section I, we first consider S as a formal power series with respect to the external field. We show (Theorem I.1.1, Section 1.2 and Proposition I.3.3) that S is, up to a phase, equal to an explicit function $S_0(I)$ of the two-point interaction kernel "I". This is, in the present framework, "the reduction to c-number problem". It can be seen that "I" is defined, in perturbation theory by an integral equation [cf. Eq. (II.0.1)].

It seems therefore reasonable, in order to formulate a non perturbative treatment, to find exact solutions of this equation, which allow to construct $S_0(I)$. The actual existence of such suitable solutions will be studied in a number of cases in Part II [16] of this work.

In fact we shall first define a class of two point functions, which we call NP kernels, and assuming the existence of NP solutions of Eq. (II. 0. 1), we shall study some of their properties:

In particular we explicitly establish the connection between methods using the hamiltonian formalism [8] the Yang-Källen-Feldman equation [5, 6, 14], and the perturbative theory [1, 2, 3, 4] (cf. Theorem II.4.1, and Section II.5). As a corollary, the two-points function "I" can be computed by solving the Cauchy problem for the Klein-Gordon equation in an external field (Theorem II.4.1), Remark II.2.11).

In the case of scalar boson fields, we also show the redundancy already remarked by Wightman [6], of some of the general necessary and sufficient conditions given by him [6] for the existence of the S-operator (cf. Proposition II.5.2).

In the last section, assuming the existence of an $N-P$ interaction kernel "I", we show that $S_0(I)$ defines in Fock space, a covariant, unitary and causal up to a phase, S-operator in the sense of Bogoliubov [10]. We also prove some results about analyticity of $S_0(I)$ with respect to "I". We construct an interpolating field, which is solution of the Yang-Källen-Feldman equations for which the in and out field are related by the S-operator $S_0(I)$.

This construction explicitly made for a charged scalar field, can be extended to fields with arbitrary spins if one is careful enough. At each step, we indicate how this can be done.

It seems that some difficulties appear in the case of fermion fields although it is possible to define an S-operator (cf. Labonte [15]).

In a forthcoming paper [16], we will show the existence of $N-P$ solutions, in the cases of spin 0, 1/2 and 1 fields, for special choices of the external field.

I. Perturbative Theory

I.1. The Bogoliubov S-Operator

The S-operator, which describes the interaction of a quantum field with an external field, is easily constructed in the framework of Bogoliubov's perturbation theory [10], by applying the Epstein-Glaser formalism [11].

In the simple case of a charged scalar boson, interacting with a scalar field and with an electromagnetic vector potential the corresponding (quadratic) lagrangean density is:

$$\mathscr{L}_I(x;\underline{g}) = v(x) :\varphi^*\varphi: (x) + iA_\mu(x) :\varphi^* \overleftrightarrow{\partial}^\mu \varphi: (x)$$
$$\underline{g} = (v, A_0, A_1, A_2, A_3) \in \mathscr{S}(\mathbb{R}^4)^{\times 5} \tag{I.1.1}$$

and φ is the free field describing the scalar bosons (see Appendix 1). Unitarity holds if v and A_μ are real valued.

By applying Wick's theorems [11, 17] (see Appendix 2) one can show the following theorem (whose proof is given in Appendix 3).

Theorem I.1.1. *Given the lagrangean density I.1.1, there exists in the sense of formal power series a solution of the causal Bogoliubov axioms such that:*

$$S(\underline{g}) = (\Omega, S(\underline{g})\,\Omega) :\exp i \int \varphi^*(x)\, I(x,y)\, \varphi(y)\, dx\, dy: . \tag{I.1.2}$$

Here, "I" is the distribution kernel valued formal power series defined by:

$$I = A + A\Delta_F I = A + I\Delta_F A \tag{I.1.3}$$

with:

$$A(x,y) = \delta(x-y)\,(v(x) + A_\mu A^\mu(x)) + i[A_\mu(x) + A_\mu(y)]\,\partial^\mu \delta(x-y) \tag{I.1.4}$$

and Δ_F the two-point Green function:

$$\Delta_F(x) = (2\pi)^{-4} \int e^{ipx}(p^2 - m^2 + i0)^{-1}\, d^4p. \tag{I.1.5}$$

Remark I.1.2. Let ψ be a free (fermion or boson) field, whose twopoint Green function is given by:

$$S_F(x) = \int e^{ipx} P(p)\,(p^2 - m^2 + i0)^{-1}\, d^4p \tag{I.1.6}$$

where P is a matrix valued polynomial. Let $\mathscr{L}_\psi(x;\mathcal{O})$ be the following (quadratic) lagrangean density:

$$\mathscr{L}_\psi(x;\mathcal{O}) = :\overline{\psi}_\alpha(x)\,\psi_\beta(x):\, \mathcal{O}_{\alpha\beta}(x) \tag{I.1.7}$$

where $\mathcal{O}_{\alpha\beta}(x) \in \mathscr{S}(\mathbb{R}^4)$.

The previous theorem holds if we replace φ by ψ, Δ_F by S_F, $\mathscr{L}_I$ by $\mathscr{L}_\psi$ and $A(x,y)$ by $\mathcal{O}(x)\,\delta(x-y)$.

Remark I.1.3. One can express "I" in term of Feynman graphs (see Fig. 1).

Remark I.1.4. One can see that:

$$\int d^4x d^4y :\varphi^*(x)\,\varphi(y):\, A(x,y) = \int d^4x : \mathscr{L}^{\text{can}}(x,\underline{g}): \qquad \text{a) (I.1.8)}$$

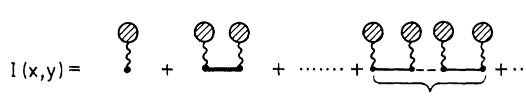

Fig. 1

where $\mathscr{L}^{\text{can}}$ is the canonical lagrangean for this problem:

$$:\mathscr{L}^{\text{can}}(x;g): = [v(x) + A_\mu A^\mu(x)] :\varphi^*\varphi: (x) + iA_\mu(x) :\varphi^* \overleftrightarrow{\partial}^\mu \varphi: (x) . \qquad \text{b)}$$

We note that $\mathscr{L}^{\mathrm{can}}$ is different from $\mathscr{L}_I$ because, by definition [11, 17], the latter has to be of the first order in $(v, A_\mu) = \underline{g}$; $\mathscr{L}^{\mathrm{can}}$ can be recovered upon making the following choice of renormalization:

$$(\Omega | T\partial^\mu \varphi(x)\, \partial^\nu \varphi(y)\, \Omega) = -i[\partial^\mu \partial^\nu \Delta_F(x-y) + g^{\mu\nu}\delta(x-y)]\,. \qquad \text{(I.1.9)}$$

I.2. The Two Point Green Functions

The two point Green function can be defined in the sense of formal power series by the following equation [3]

$$G_F = \Delta_F + \Delta_F A G_F = \Delta_F + G_F A \Delta_F\,. \qquad \text{(I.2.1)}$$

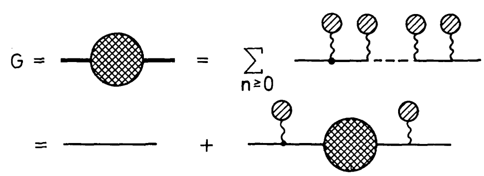

Fig. 2

Its expression in terms of Feynman graphs is given in Fig. 2. It is easy to see that G_F is equal to

$$G_F = \Delta_F + \Delta_F I \Delta_F \qquad \text{(I.2.2)}$$

or equivalently:

$$I = A + A G_F A\,. \qquad \text{(I.2.3)}$$

Equation (I.2.1) shows us that G_F satisfies the following Klein Gordon equation

$$\begin{aligned} &([\partial_{\mu,x} - iA_\mu(x)]\,[\partial^\mu_x - iA^\mu(x)] + m^2 - v(x))\, G_F(x,y) = \delta(x-y) \\ &([\partial_{\mu,y} + iA_\mu(y))\,[\partial^\mu_y + iA^\mu(y)] + m^2 - v(y))\, G_F(x,y) = \delta(x-y) \end{aligned} \qquad \text{(I.2.4)}$$

with the conditions [3]:

if $x \notin (\mathrm{supp}\,\underline{g} + \bar{V}^+)$, $G_F(x,y)$ propagates only positive frequencies
if $x \notin (\mathrm{supp}\,\underline{g} + \bar{V}^-)$, $G_F(x,y)$ propagates only negative frequencies.

I.3. The Physical Meaning of I

Let $\mathfrak{H}_\varepsilon$ be the one particle Hilbert space with energy of the sign of $\varepsilon = \pm 1$ (cf. Appendix 1). Let us define

$$\tilde{I}(p,g) = 2\pi^{-4} \int e^{i(p\cdot x - q\cdot y)} I(x,y)\, d^4x\, d^4y\,. \qquad \text{(I.3.1)}$$

Let $I_{\varepsilon\varepsilon'}$ be the following operator from $\mathfrak{H}_{\varepsilon'}$ to $\mathfrak{H}_\varepsilon$ (cf. Appendix 1):

$$(I_{\varepsilon\varepsilon'} f)(p) = 2\pi \int_{q'^0 = (\vec{q}'^2 + m^2)^{1/2}} \tilde{I}(\varepsilon p, \varepsilon' q')\, f(q')\, \tfrac{1}{2}(\vec{q}'^2 + m^2)^{-1/2}\, d^3\vec{q}' \qquad \text{(I.3.2)}$$

with $\varepsilon=\pm 1$, $\varepsilon'=\pm 1$ and $f\in\mathfrak{H}_{\varepsilon'}$. Then "$I$" defines the following operator on $\mathfrak{H}_+\oplus\mathfrak{H}_-$

$$I_S=\begin{pmatrix} I_{++} & I_{+-} \\ I_{-+} & I_{--}\end{pmatrix}. \tag{I.3.3}$$

Proposition I.3.1. *To any order in the external fields I_{++}, I_{--} are bounded operators, I_{+-} and I_{-+} are in the Hilbert-Schmidt class* [18–20].

Proof. To order "n", "I" reads:

$$\tilde{I}_n(p,q)=\overbrace{A\Delta_F\ldots A\Delta_F A}^{n\text{-times } A}(p,q). \tag{I.3.4}$$

From Appendix 5, Lemma A.5.3 and Corollary A.5.7 we can deduce that $\tilde{I}_n\in\mathcal{N}(m,\alpha,1/4)\,\forall m\geqq 0$ and $\forall\alpha\geqq 0$. Using Lemma A.5.4,

$$\mathrm{Tr}(I_{+-}I^*_{+-})|_{\text{order}\,``n"}=\sum_{p=0}^{n}\mathrm{Tr}((I_{+-})_p\,(I^*_{+-})_{n-p}). \tag{I.3.5}$$

Proposition I.3.1 then follows.

Remark I.3.2. In the case of fields with higher spins described in Remark I.1.2, the results of Appendix 5 (in particular Lemma A.5.5) allow us to modify Proposition I.3.1 as follows:

– I_{+-} and I_{-+} are in the Hilbert Schmidt class.

– I_{++} (resp. I_{--}) is densely defined on a domain D_+ (resp. D_-) such that $I_{++}D_+\subset D_+$ (resp. $I_{--}D_-\subset D_-$).

To see this we have to remark that (see Remark I.1.2) $\forall m\geqq 0\,\forall\alpha\geqq 0\;\tilde{\mathcal{O}}(p,q)=\tilde{\mathcal{O}}(p-q)\in\mathcal{N}(m,\alpha,0)$ if $\mathcal{O}(x)\in\mathcal{S}(\mathbb{R}^4)$ and we have to use Lemmas A.5.5 and A.5.6.

Proposition I.3.3. *In the sense of formal power series, the operator I_S is the building block of the following physical quantities*

a) $$|(\Omega,S\Omega)|^2=\det(\mathbb{1}_{++}-I_{+-}(I_{+-})^*). \tag{I.3.6}$$

b) *The probability $P_{e^+e^-}$ of creation of one pair is given by*

$$P_{e^+e^-}=\det(\mathbb{1}_{++}-I_{+-}I^*_{+-})\,\mathrm{Tr}(I_{+-}I^*_{+-}). \tag{I.3.7}$$

c) *The scattering amplitude for the process particle→particle is given by*

$$(\Omega,S\Omega)(\mathbb{1}_{++}+iI_{++})(p,q). \tag{I.3.8}$$

d) *The complete S operator is given by:*

$$S=e^{i\omega}S_0(I)=e^{i\omega}\det(\mathbb{1}_{++}-I_{+-}I^*_{+-})^{1/2}:e^{i(a^+I_{+-}b^{+}+a^+I_{++}a^-+b^-I_{--}b^{+}+b^-I_{-+}a^-)}: \tag{I.3.9}$$

where ω is a phase depending on I.

The results of this proposition are well-known [3, 4, 6, 8]. They can be proved by using the unitary of the S-operator and the definitions of the out-field and the out-vacuum

$$\Omega_{\text{out}} = S^{-1}\Omega = (\Omega, S\Omega)^{-1} e^{-ia^+ I_+^* - b^+}\Omega, \tag{I.3.10}$$

$$\varphi_{\text{out}}(x) = S^{-1}\varphi(x) S. \tag{I.3.11}$$

Remark I.3.4. Using Nelson's notations [21] for second quantized operators one has:

$$:\exp i(a^+ I_{++} a^- + b^- I_{--} b^+): = \Gamma((\mathbb{1}_{++} + iI_{++}) \otimes (\mathbb{1}_{--} + iI^t_{--})) \tag{I.3.12}$$

with

$$\Gamma(A \otimes B)\, \Phi_{r,s} = (A^{\otimes r} \otimes B^{\otimes s})\, \Phi_{r,s} \tag{I.3.13}$$

if

$$\Phi_{r,s} \in \mathfrak{H}_+^{\vee r} \otimes \mathfrak{H}'^{\vee s}_- \quad \text{(cf. Appendix 1)}$$

Remark I.3.5. In a non perturbative framework we shall use the results of Proposition I.3.3 to reduce the problem to the proof of the existence of a kernel I such that I_{++}, I_{--} are bounded operators, and I_{+-}, I_{-+} are in the Hilbert-Schmidt class.

I.4. The Interpolating Field

In perturbation theory the interpolating field is given by the formula [11]

$$\psi(x) = S^{-1}(g, J) \frac{\delta S}{\delta J(x)} (g, J)\bigg|_{J=0}. \tag{I.4.1}$$

Here $S(g, J)$ is the Bogoliubov S-operator constructed from the lagrangean density

$$\mathscr{L}'(x; g, J) = \mathscr{L}_I(x; g) + J(x)\,\varphi(x). \tag{I.4.2}$$

Proposition I.4.1. *The interpolating field is given by the formula:*

$$\psi(x) = |\langle S\rangle|^2 :e^{-i\int\varphi^*\bar{I}\varphi}::\int(\delta(x-y) + \Delta_F I(x, y))\,\varphi(y)\, d^4y\, e^{i\int\varphi^* I\varphi}:. \tag{I.4.3}$$

It satisfies the Yang-Källen-Feldman equations [5, 14, 22];

$$\begin{aligned} \psi(x) &= \varphi(x) + \int \Delta_r(x-z)\, A(z, y)\, \psi(y)\, d^4y\, d^4z && \text{a)} \\ \psi(x) &= \varphi_{\text{out}}(x) + \int \Delta_a(x-z)\, A(z, y)\, \psi(y)\, d^4y\, d^4z. && \text{b)} \end{aligned} \tag{I.4.4}$$

The proof of Eq. (I.4.3) is given in Appendix 3. Let us give the proof of (I.4.4a)

$$\Delta_F = \Delta_r - \Delta_+ \Rightarrow (1 - \Delta_r A)(1 + \Delta_F I) = (1 - \Delta_+ I) \tag{I.4.5 [3]}$$

because of (I.1.3) for I.

Thus:

$$(1 - \Delta_r A)\,\psi = :e^{-i\int\varphi^*\bar{I}\varphi}: :(1 - \Delta_+ I)\,\varphi e^{i\int\varphi^* I\varphi}: |\langle S\rangle|^2 \tag{I.4.6}$$

where

$$\bar{I}(x, y) = I(y, x)^*. \tag{I.4.7}$$

Wick's theorem yields:

$$:(1 - \Delta_+ I)\,\varphi e^{i\int\varphi^* I\varphi}: = :e^{i\int\varphi^* I\varphi}: \varphi. \tag{I.4.8}$$

The unitarity of S yields the required result. The proof of (I.4.4b) is similar.

Remark I.4.2. Capri [5] and Wightman [6] have used these equations to construct the interpolating field and the out-field for this problem. Following another approach, Seiler, Schroer and Swieca [8, 9] (see also [6]) solve these equations for c-numbers function, and prove the existence of the classical time evolution and S-operator, which is in our formalism (in the sense of formal power series)

$$S_{\mathrm{cl}} = (1 - \Delta_a A)(1 - \Delta_r A)^{-1} . \tag{I.4.9}$$

In Capri's notations [5, 6] $S_{\mathrm{cl}} = T_a T_r^{-1}$ with $T_{r_a} = (1 - \Delta_{r_a} A)$

Thus,

$$\varphi_{\mathrm{out}} = S_{\mathrm{cl}} \varphi . \tag{I.4.10}$$

II. Algebraic Properties of the Two-Point Green Functions

II.0. Introduction

In this chapter we shall investigate the algebraic properties of the solutions of the defining equations for I (I.1.3).

We shall similarly define three other kernels $\bar{I}$, J_r, J_a,

$$\begin{aligned} &I = A + A\Delta_F I = A + I\Delta_F A \qquad && \bar{I} = A + A\Delta_{\bar{F}}\bar{I} = A + \bar{I}\Delta_{\bar{F}} A \quad \text{a)} \\ &J_r = A + A\Delta_r J_r = A + J_r \Delta_r A \qquad && J_a = A + A\Delta_a J_a = A + J_a \Delta_a A \,. \quad \text{b)} \end{aligned} \tag{II.0.1}$$

Keeping in mind how the quantum theory relies on the c-number theory and in view of the results of perturbation theory (Remark I.3.5) we shall need solutions of II.0.1 with specific properties described in Definitions II.2.1 and II.2.2. These solutions will be called N-P kernels (N-P = non perturbative).

We shall also define W-N-P kernels (W = weak) (cf. Definition II.2.3) and will establish later [16] that for any external field in $\mathscr{C}_0^\infty(\mathbb{R}^4)$ [23] II.0.1 has one and only one W-N-P solution. N-P solution however can be shown to exist for some restricted classes of external fields [8, 9, 16].

This section is devoted to the proof of some exact relations between N-P (or W-N-P) solution of II.0.1 based on elementary properties of the free two-point functions.

Some of these relations can be found for instance in J. Schwinger's work [3], some others in Wightman's [6] and Seiler's [8] more recent papers (cf. Remark I.4.2) and are related to general properties of the Bogoliubov transformations [24].

The main result of this section is the equivalence between the equations defining I and J_r, in so far as the existence and unicity of a solution are concerned (Theorem II.4.1). As a corollary (in some sense) it is necessary and sufficient, in order to solve II.0.1a) to solve the Cauchy problem for the Klein-Gordon equation in an external field [6].

Another result is that the necessary and sufficient conditions given by Wightman ([6], Formula 4.12) for the uniqueness of the out-vacuum, is automatically verified for bosons of spin zero and real external fields (Proposition II.5.2) as already remarked in [6].

Everything so far mentioned can be extended without modification for fields with integer spins. For half-integer spin field there are some changes to be made which are stated at each step.

II.1. Preliminaries

We first need some definitions.

Let K, L be kernels in $\mathscr{S}'(\mathbb{R}^4 \times \mathbb{R}^4)$. We define the Fourier transform:

$$\tilde{K}(p,q) = (2\pi)^{-4} \int e^{i(px-qy)} K(x,y)\, d^4x\, d^4y\,, \tag{II.1.1}$$

the product (if it exists):

$$KL(x,y) = \int K(x,z)\, L(z,x)\, d^4z\,, \tag{II.1.2}$$

the adjoint

$$K^\dagger(x,y) = K(y,x)^*\,, \tag{II.1.3}$$

the unit kernel

$$1(x,y) = \delta(x-y)\,, \tag{II.1.4}$$

the kernel associated with the convolution through a tempered distribution:

$$T(x,y) = T(x-y)\,. \tag{II.1.5}$$

Then:

$$\begin{array}{llll} \tilde{\tilde{K}} = K & \tilde{K} + \tilde{L} = \widetilde{K+L} & \tilde{K}\tilde{L} = \widetilde{KL} & \text{a)} \\ K^{\dagger\dagger} = K & K^\dagger + L^\dagger = (K+L)^\dagger & (KL)^\dagger = L^\dagger K^\dagger & \text{b)} \\ (\widetilde{K^\dagger}) = (\tilde{K})^\dagger & K \cdot 1 = 1 \cdot K = K & & \text{c)} \\ \tilde{1} = 1 = 1^\dagger\,. & & & \text{d)} \end{array} \tag{II.1.6}$$

The operators $\hat{K}_{\varepsilon\varepsilon'}(\varepsilon = \pm 1, \varepsilon' = \pm 1)$ are defined (if these expressions have a meaning) by:

$$(\hat{K}_{\varepsilon\varepsilon'} f)(\vec{p}) = 2\pi \int_{q^0 = \omega_q,\, p^0 = \omega_p} \tilde{K}(\varepsilon p, \varepsilon' q)\, f(\vec{q}) \frac{d^3\vec{q}}{2\omega_q} \tag{II.1.7}$$

with $\omega_q = (1 + \vec{q}^2)^{1/2}$ and

$$\hat{K} = \begin{bmatrix} K_{++} & K_{+-} \\ K_{-+} & K_{--} \end{bmatrix} \tag{II.1.8}$$

Recall that

$$\Delta_+(\xi) = -i(2\pi)^{-3} \int e^{i(\omega_p \xi^0 - \vec{p}\vec{\xi})} d^3\vec{p}/2\omega_p = \Delta_-(-\xi)\,. \tag{II.1.9}$$

Then we have (in kernel notations)

$$\begin{array}{lll} \Delta = \Delta_+ - \Delta_- = \Delta_r - \Delta_a & \operatorname{supp} \Delta_{r_a} \subset \{(x-y) \in \bar{V}^\pm\} & \text{a)} \\ \Delta_F = \Delta_r - \Delta_+ = \Delta_a - \Delta_- & & \text{b)} \\ \Delta_{\bar{F}} = \Delta_r + \Delta_- = \Delta_a + \Delta_+ & & \text{c)} \end{array} \tag{II.1.10}$$

$$\Delta_r^\dagger = \Delta_a \qquad \Delta_\pm^\dagger = -\Delta_\pm \qquad \Delta_F^\dagger = \Delta_{\bar{F}}\,. \tag{II.1.11}$$

We will also define [see Eq. (II.0.1)]

$$\begin{array}{ll} I_s = \hat{I} \quad \bar{I}_s = \hat{\bar{I}} \quad J_R = \hat{J}_r \Gamma \quad J_A = \hat{J}_a \Gamma & \text{a)} \\ \Gamma = \begin{pmatrix} \mathbb{1}_{++} & 0 \\ 0 & -\mathbb{1}_{--} \end{pmatrix}. & \text{b)} \end{array} \tag{II.1.12}$$

where

II.2. Non Perturbative Solutions (N-P and W-N-P Kernels)

Definition II.2.1. A kernel $K \in \mathscr{S}'(\mathbb{R}^4 \times \mathbb{R}^4)$ will be called regular if it defines a linear continuous map from $\mathcal{O}_M(\mathbb{R}^4)$ to $\mathscr{S}(\mathbb{R}^4)$ [23, 25].

Definition II.2.2. A regular kernel will be called W-N-P if

i) $\hat{K}$ is a bounded operator in $\mathfrak{H}_+ \oplus \mathfrak{H}_-$,

ii) $\hat{K}_{+-}$ and $\hat{K}_{-+}$ are compact [18–20].

Definition II.2.3. A W-N-P kernel will be called N-P if $\hat{K}_{+-}$ and $\hat{K}_{-+}$ are in the Hilbert-Schmidt class [18–20].

Proposition II.2.4. *The kernel A defined in I.1.4 is N-P.*

Proof. cf. Appendix 5 Lemma A.5.3 and A.5.4.

Proposition II.2.5. *Let D be the dense domain in $\mathfrak{H}_+ \oplus \mathfrak{H}_-$ whose elements are functions of $\mathscr{S}(\mathbb{R}^3)$. Then if K is a regular kernel, $\hat{K}$ is defined on D and*

$$\hat{K} D \subset D\,. \tag{II.2.1}$$

Proof. Let $f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix}$ be an element of D. Then if

$$\hat{f} = \delta\,\frac{(p^0 - \omega_p)}{2\omega_p}\, f_+(\vec{p}) + \delta\,\frac{(p^2 + \omega_p)}{2\omega_p}\, f_-(\vec{p})\,. \tag{II.2.2}$$

One can see that $\hat{f} \in \mathcal{O}_c'(\mathbb{R}^4)$ and its Fourier transform $\tilde{\hat{f}}$ is in $\mathcal{O}_M(\mathbb{R}^4)$ [25]. Therefore, the regularity of K gives:

$$K\tilde{\hat{f}} \in \mathscr{S}(\mathbb{R}^4)\,. \tag{II.2.3}$$

The definition of $\hat{K}$ [cf. Eq. (II.1.7)] tells us that $\hat{K} f$ is the mass-shell restriction of the Fourier transform of $K\tilde{\hat{f}}$ and thus $\hat{K} f \in D$.

Proposition II.2.6. a) *Let $K_1, \ldots, K_n$ be regular kernels. Let $\Delta_1, \ldots, \Delta_{n-1}$ be a family of tempered distributions chosen in the set $\{\Delta_+, \Delta_-, \Delta_r, \Delta_a, \Delta_F, \Delta_{\bar{F}}\}$. Then the kernel $K_1 \Delta_1 \ldots K_{n-1} \Delta_{n-1} K_n$ is regular.*

b) *$K_1 \Delta_1$ (resp. $\Delta_1 K_1$) maps linearly and continuously $\mathscr{S}(\mathbb{R}^4)$ into $\mathscr{S}(\mathbb{R}^4)$ (resp. $\mathcal{O}_M(\mathbb{R}^4)$ into $\mathcal{O}_M(\mathbb{R}^4)$).*

Proposition II.2.7. *If one of the Eq.* (II.0.1) *has a regular solution, this solution is unique.*

Proof. For instance, in view of Proposition II.2.6 and Eq. (II.0.1), $(1 + I\Delta_F)$ is the inverse of $1 - A\Delta_F$.

Proposition II.2.8. *I (resp. J_r) is a regular kernel (or W-N-P, or N-P kernel) if and only if $\bar{I}$ (resp. J_a) is.*

Proof. Indeed:

$$\bar{I}(A) = I(A^\dagger)^\dagger \qquad J_r(A) = J_a(A^\dagger)^\dagger\,. \tag{II.2.4}$$

Definition II.2.9. The external field will be called "physical" if it has only real components.

Proposition II.2.10. a) *The external field is physical if and only if*

$$A = A^{\dagger}\,. \tag{II.2.5}$$

b) *If the external field is real and if Eq.* (II.0.1) *have a regular solution, then these solutions obey:*

$$J_r = J_a^{\dagger} \qquad I^{\dagger} = \bar{I}\,. \tag{II.2.6}$$

Proof. See the Proposition II.2.8 and the definition of A [Eq. (I.1.4)].

Remark II.2.11. Let G_r be defined by:

$$G_r = \Delta_r + \Delta_r J_r \Delta_r\,. \tag{II.2.7}$$

Then, one can easily convince oneself that G_r is the solution of the integral equation:

$$G_r = \Delta_r + \Delta_r A G_r = \Delta_r + G_r A \Delta_r\,. \tag{II.2.8}$$

Or equivalently G_r is the unique solution of the Klein-Gordon [cf. Eq. (I.2.4)] with external field, whose support is in $(x, y); (x-y) \in \bar{V}^+\}$ (Cauchy problem [6]). One can see that J_r can be reconstructed from G_r by the formula:

$$J_r = A + A G_r A\,. \tag{II.2.9}$$

II.3. Unitarity

Proposition II.3.1. *Let* $I, \bar{I}, J_r, J_a$ *be regular solutions of* II.0.1; *then*

a)

$$\begin{aligned} (\mathbb{1} + iI_s)^{-1} &= (\mathbb{1} - i\bar{I}_s) \quad &\text{a)} \\ (\mathbb{1} + iJ_R)^{-1} &= (\mathbb{1} - iJ_A) \quad &\text{b)} \end{aligned} \tag{II.3.1}$$

on the domain D.

b) *If the external field is physical, then:*

$(\mathbb{1} + iI_s)$ *can be extended as a unitary operator on* $\mathfrak{H}_+ \oplus \mathfrak{H}_- = \mathfrak{H}$,

$(\mathbb{1} + iJ_R)$ *can be extended as a pseudo unitary operator on* $\mathfrak{H}$ *with respect to the metric* Γ.

Remark II.3.2. The details of relation II.3.1 a) and b) are given in Table 1.

Remark II.3.3. This proposition can be extended without change to the case of fields with integer spins. In the case of half-integers spin field, a) is also true. But in b) we have the following modification

$(\mathbb{1} + iI_s)$ is pseudo unitary with respect to Γ,

$(\mathbb{1} + iJ_R)$ is unitary.

Proof of the Proposition II.3.1. Following Schwinger [3], and applying Propositions II.2.4 and II.2.6:

$$A = (1 - A\Delta_F)\, I = \bar{I}(1 - \Delta_{\bar{F}} A)\,. \tag{II.3.2}$$

Therefore, multiplying to the left by $1 + I\Delta_F$, and to the right by $1 + \Delta_{\bar{F}}\bar{I}$, we obtain:

$$0 = I - \bar{I} + I(\Delta_+ + \Delta_-)\,\bar{I}\,. \tag{II.3.3}$$

Table 1

1. $(\mathbb{1}+iI_S)(\mathbb{1}-i\bar{I}_S)=\mathbb{1}$	2. $(\mathbb{1}-i\bar{I}_S)(\mathbb{1}+iI_S)=\mathbb{1}$
$(\mathbb{1}_+ + iI_{++})(\mathbb{1}_+ - i\bar{I}_{++})=\mathbb{1}_+ - I_{+-}\bar{I}_{-+}$	$(\mathbb{1}_+ - i\bar{I}_{++})(\mathbb{1}_+ + iI_{++})=\mathbb{1}_+ - \bar{I}_{+-}I_{-+}$
$(\mathbb{1}_- + iI_{--})(\mathbb{1}_- - i\bar{I}_{--})=\mathbb{1}_- - I_{-+}\bar{I}_{+-}$	$(\mathbb{1}_- - i\bar{I}_{--})(\mathbb{1}_- + iI_{--})=\mathbb{1}_- - \bar{I}_{-+}I_{+-}$
$(\mathbb{1}_+ + iI_{++})\bar{I}_{+-}=I_{+-}(\mathbb{1}_- - i\bar{I}_{--})$	$\mathbb{1}_- - i\bar{I}_{--})(I_{-+})=\bar{I}_{-+}(\mathbb{1}_+ + iI_{++})$
$(\mathbb{1}_- + iI_{--})\bar{I}_{-+}=I_{-+}(\mathbb{1}_+ - i\bar{I}_{++})$	$(\mathbb{1}_+ - i\bar{I}_{++})I_{+-}=\bar{I}_{+-}(\mathbb{1}_- + iI_{--})$
3. $(\mathbb{1}+iJ_R)(\mathbb{1}-iJ_A)=\mathbb{1}$	4) $(\mathbb{1}-iJ_A)(\mathbb{1}+iJ_R)=\mathbb{1}$
$(\mathbb{1}_+ + iJ_{r++})(\mathbb{1}_+ - iJ_{a++})=\mathbb{1}_+ + J_{r+-}J_{a-+}$	$(\mathbb{1}_+ - iJ_{a++})(\mathbb{1}_+ + iJ_{r++})=\mathbb{1}_+ + J_{a+-}J_{r-+}$
$(\mathbb{1}_- - iJ_{r--})(\mathbb{1}_- + iJ_{a--})=\mathbb{1}_- + J_{r-+}J_{a+-}$	$(\mathbb{1}_- + iJ_{a--})(\mathbb{1}_- - iJ_{r--})=\mathbb{1}_- + J_{a-+}J_{a+-}$
$(\mathbb{1}_+ + iJ_{r++})J_{a+-}=J_{r+-}(\mathbb{1}_- + iJ_{a--})$	$(\mathbb{1}_- + iJ_{a--})J_{r-+}=J_{a-+}(\mathbb{1}_+ + iJ_{r++})$
$(\mathbb{1}_- - iJ_{r--})J_{a-+}=J_{r--}(\mathbb{1}_+ - iJ_{a++})$	$(\mathbb{1}_+ - iJ_{a++})J_{r+-}=J_{a+-}(\mathbb{1}_- - iJ_{r--})$

Adding the unit kernel to both sides we immediatly obtain II.3.1 upon mass-shell restriction.

Now, if the external field is real one easily finds:

$$\bar{I}_s \subset I_s^* \qquad J_A \subset \Gamma J_R^* \Gamma^{-1} \tag{II.3.4}$$

where Γ is defined by II.1.12.

Proposition II.3.4. *Let us assume that the external field is physical:*
a) *if I is regular, then:*

$$0 \leqq \mathbb{1}_{++} - I_{+-}I_{+-}^* \leqq \mathbb{1}_{++} \tag{II.3.5}$$

and I_S is bounded.
b) *If I is N-P then*

$$0 \leqq \det(\mathbb{1}_{++} - I_{+-}I_{+-}^*) \leqq 1\,. \tag{II.3.6}$$

c) *If J_{r_a} is regular*

$$\mathbb{1}_{++} + J_{r+-}J_{a-+} = \mathbb{1}_{++} + J_{r+-}J_{r+-}^* \geqq \mathbb{1}_{++}\,. \tag{II.3.7}$$

Therefore this operator has a bounded inverse.

Remark II.3.5. The result b) shows that Proposition I.3.3 is likely to hold in a non-perturbative sense since:

$$0 \leqq |(\Omega, S\Omega)|^2 = \det(\mathbb{1}_{++} - I_{+-}I_{+-}^*) \leqq 1\,. \tag{II.3.8}$$

Proof. We have only to remark that, since the external field is physical

$$I_{+-}^* = \bar{I}_{-+} \qquad J_{a-+} = J_{r+-}^*$$
$$I_{++}^* = \bar{I}_{++} \qquad J_{a++} = J_{r++}^*\,.$$

By looking at Table I, Proposition II.3.4 is proved.

II.4 Relationship between I and J_r:

The main theorem is the following:

Theorem II.4.1. 1) *The equations (A) $J_{r_a} = A + A\Delta_{r_a}J_{r_a} = A + J_{r_a}\Delta_{r_a}A$ have a unique regular (resp. W-N-P, N-P) solution such that $\mathbb{1}_+ + J_{a+-}J_{r-+}$ is invertible,* *if and only if the equations* (B) $I = A + A\Delta_F I = A + I\Delta_F A, \bar{I} = A + A\Delta_{\bar{F}}\bar{I} = A + \bar{I}\Delta_{\bar{F}} A$ *have a unique regular (resp. W-N-P, N-P) solution such that* $\mathbb{1}_+ - I_{+-}\bar{I}_{-+}$ *is invertible.*

2) *If the external field is physical, equations* (A) *have a unique regular (resp. W-N-P, N-P) solution if and only if equations* (B) *have a unique regular (resp. W-P-N, N-P) solution such that* $\mathbb{1}_+ - I_{+-}I^*_{+-}$ *is invertible.*

Proof. a) Let us suppose that I and J_r are regular solution of (A) and (B) respectively. Then we find

$$\begin{aligned} J_r &= (1 + J_r\Delta_r)\,A = (1 + J_r\Delta_r)\,(1 - A\Delta_F)\,I \\ &= (1 + J_r\Delta_r)\,(1 - A\Delta_r + A\Delta_+)\,I = (1 + J_r\Delta_+)\,I \qquad \text{(II.4.1)} \\ J_r &= I + J_r\Delta_+ I\,. \end{aligned}$$

By interchanging the different kernels $J_a, J_r, I, \bar{I}$ we obtain similar relations, whose details, when restricted on the mass-shell, are given in Table 2.

b) Now let J_{r_a} be a regular solution of (A); If $\mathbb{1}_+ + J_{r+-}J_{a-+}$ is invertible, then, in view of the results of Table I, $\mathbb{1}_- - iJ_{r--}$ and $\mathbb{1}_+ + iJ_{r++}$ are invertible on the domain D.

Therefore we can define an operator I_S such that the relations of Table 2 hold.

We now define $I(x, y)$ by:

$$\tilde{I}(p, q) = \tilde{J}_r(p, q) + 2i\pi \int \frac{d^3r_1 d^3r_2}{2\omega_{r_1} 2\omega_{r_2}}\, \tilde{J}_r(p_1 - r_1)\,(\mathbb{1} - iJ_{r-})^{-1}\,(r_1, r_2)\,\tilde{J}_r(-r_2, q)\,. \qquad \text{(II.4.2)}$$

By construction and using the Proposition II.2.5 I is regular and satisfies:

$$\hat{I} = I_S\,; \qquad (1 + J_r\Delta_+)\,I = J_r = I(1 + \Delta_+ J_r)\,. \qquad \text{(II.4.3)}$$

Thus since

$$(1 - A\Delta_r)\,(1 + J_r\Delta_+) = (1 - A\Delta_F)\,. \qquad \text{(II.4.4)}$$

I is a regular solution of (B).

Then, we can remark that

$$(\mathbb{1}_+ - I_{+-}\bar{I}_{-+})^{-1} = \mathbb{1}_+ + J_{r+-}J_{a-+}\,. \qquad \text{(II.4.5)}$$

Now, if J_r is W-N-P (resp. N-P), I is also W-N-P (resp. N-P).

c) The converse can be shown by the same procedure.

d) The second part of the theorem is then trivial, by using Proposition II.3.4.

Table 2

1. $(\mathbb{1}_- - iJ_{r--})^{-1} = \mathbb{1}_- + iI_{--}$	$J_{r++} - I_{++} = -iJ_{r+-}I_{-+} = -iI_{+-}J_{r-+}$
$I_{+-} = J_{r+-}(\mathbb{1}_- + iI_{--})$	$J_{r-+} = (\mathbb{1}_- - iJ_{r--})\,I_{-+}$
2. $(\mathbb{1}_+ + iJ_{r++})^{-1} = \mathbb{1}_+ - iI_{++}$	$J_{r--} - I_{--} = iJ_{r-+}\bar{I}_{+-} = i\bar{I}_{-+}J_{r+-}$
$\bar{I}_{+-} = (\mathbb{1}_+ - i\bar{I}_{++})\,J_{r+-}$	$J_{r-+} = \bar{I}_{-+}(\mathbb{1}_+ + iJ_{r++})$
3. $(\mathbb{1}_+ - iJ_{a++})^{-1} = \mathbb{1}_+ + iI_{++}$	$I_{a--} - I_{--} = -iJ_{a-+}I_{+-} = -iI_{-+}J_{a+-}$
$I_{+-} = (\mathbb{1}_+ + iI_{++})\,J_{a+-}$	$J_{a-+} = I_{-+}(\mathbb{1}_+ - iJ_{a++})$
4. $(\mathbb{1}_- + iJ_{a--})^{-1} = (\mathbb{1}_- - i\bar{I}_{--})$	$J_{a++} - \bar{I}_{++} = iJ_{a+-}\bar{I}_{-+} = i\bar{I}_{+-}J_{a-+}$
$\bar{I}_{+-} = J_{a+-}(\mathbb{1}_- - i\bar{I}_{--})$	$J_{a-+} = (\mathbb{1}_- + iJ_{a--})\,\bar{I}_{-+}$

Remark II.4.2. The results 2) of Theorem II.4.1 can be restated as follows: If J_r is an N-P solution of (A) and if the external field is physical, then

$$0 \neq \det(\mathbb{1}_+ - I_{+-} I^*_{+-}) = \det(\mathbb{1}_+ + J_{r+-} J^*_{r+-})^{-1}. \qquad \text{(II.4.6)}$$

Remark II.4.3. Theorem II.4.1 can be extended without change to the case or arbitrary integer spin fields. In the case of half integer spin fields $(I, \bar{I})$, have to be exchanged with (J_r, J_a).

II.5. The Classical S-Matrix [5, 6, 8]

We suppose here that II.0.1 b) has a regular solution.

Proposition II.5.1. *Upon mass-shell restriction, the kernel* $S_{\text{cl}} = (1 - \Delta_a A) \cdot (1 - \Delta_r A)^{-1}$ *defines a unique operator* $\hat{S}_{\text{cl}}$ *on* $D \subset \mathfrak{H}_+ \oplus \mathfrak{H}_-$ *(cf. Proposition II.2.5.) such that*

$$\hat{S}_{\text{cl}} = \mathbb{1} + iJ_R. \qquad \text{(II.5.1)}$$

Proof. Using II.0.1, we easily find:

$$S_{\text{cl}} = 1 + (\Delta_+ - \Delta_-)\, J_r. \qquad \text{(II.5.2)}$$

This relation is sufficient to prove the proposition.

Proposition II.5.2. *If the external field is physical, then,* $\mathbb{1}_+ + iJ_{r++} = (\hat{S}_{\text{cl}})_{++}$ *has a dense image.*

Proof. Indeed, in this case, $\mathbb{1}_+ + iJ_{r,++}$ has a bounded inverse (cf. Proposition II.3.4).

Remark II.5.3. The result of the previous proposition is a necessary and sufficient condition for the unicity of the out-vacuum as shown by Wightman (cf. [6], Eq. (4.12)).

This result can be extended to arbitrary integer spin fields. In the case of half integer spins, it might turn out to be wrong for strong external field [6, 15].

Corollary II.5.4. (Seiler [8]).

1) $\hat{S}_{\text{cl}}$ *is pseudo unitary with respect to* Γ.

2) *We have*

$$(I_{-+})^* = \bar{I}_{+-} = (\hat{S}_{\text{cl}})^{-1}_{++} (\hat{S}_{\text{cl}})_{+-}. \qquad \text{(II.5.3)}$$

Proof. We have only to use Proposition II.3.1, Table 1 and the relation

$$\hat{S}_{\text{cl}} = \mathbb{1} + iJ_R.$$

Remark II.5.5. For integer spin fields, this corollary always holds. For half integer spin fields $\hat{S}_{\text{cl}}$ is unitary [8] (cf. Remark II.3.3).

III. Construction of the Bogoliubov *S*-Operator in Fock Space

III.1. Definition of S(g)

Following perturbation theory it seems natural to define the *S*-operator by:

$$S_0(I) = \det(\mathbb{1}_+ - I_{+-} I^*_{+-})^{1/2} :\exp i \int \varphi^*(x)\, I(x, y)\, \varphi(y)\, dx\, dy:. \qquad \text{(III.1.1)}$$

If this expression is well defined, it is expected to be, up to a phase, a re-summation of the S-operator of the perturbation theory.

We shall, in fact show in this section that if I is an N-P kernel, $S_0(I)$ is well defined on the dense domain of Fock space, generated by coherent states.

Then, if I is the N-P solution of the integral Eq. II.0.1, we shall use an integral representation for $S_0(I)$ (Section III.3) and show that $S_0(I)$ can be continued as a unitary operator. Next we shall show that $S_0(I)$ satisfies the axioms of Bogoliubov [10, 11] up to a phase (Section III.4).

Finally we shall study the interpolating field, defined formally as in perturbation theory, and show that it is an operator valued tempered distribution, satisfying the Yang-Källen-Feldman equations [14, 22].

It is unfortunate that this last part can only be generalized to fields with integer spins in view of the intensive use of coherent states in the computation of $S_0(I)$.

However, we should like to remark that the existence of $S_0(I)$ as a unitary covariant and causal operator [10, 11], up to a phase can be shown, also by using the representation theory of quasi free states, and the characterisation of unitarily implementable Bogoliubov transformations (cf. for instance Berezin [24], Powers and Størmer [26], Manuceau and Verbeure [27], Van Daele and Verbeure [28]).

This method has the advantage that it allows to treat both cases: boson fields as well as fermion fields.

But we have prefered to follow a more explicit method which will be used in Part II [16] of this work to prove the convergence, on coherent states, of the perturbation expansion for $S_0(I)$ in the case of weak external fields.

III.2. Coherent States

Definition III.2.1 (see Appendix 1). Let f be in $\mathfrak{H}_+$, G be in $\mathfrak{H}'_-$. The coherent state with wave functions f and G is defined as the vector of $\mathfrak{F}$

$$\Phi_{f,G} = e^{a^+(f) + G(b^+)}\Omega\,. \tag{III.2.1}$$

Let us note the formula

$$(\Phi_{f,G} | \Phi_{f',G'}) = e^{(f'|f) + (G',G)}\,. \tag{III.2.2}$$

Proposition [29] III.2.2. *The vector space $\mathscr{D}_c$, spanned by the coherent states is a dense domain of $\mathfrak{F}$. Any finite family of coherent states is linearly independent.*

Proposition III.2.3. *The application $\Phi:(f, G) \to \Phi_{f,G}$ from $\mathfrak{H}_+ \times \mathfrak{H}'_-$ to $\mathfrak{F}$ is an analytic function in $\mathfrak{H}_+ \times \mathfrak{H}'_-$.*

Proof. Let us recall [30] that a function $x \to f(x)$ from a complex Banach space E to a denumerable normed space F is analytic in an open set U of E if and only if (see [30], §3.3)

a) f is continuous,

b) f is locally bounded,

c) there exists a total set H on the dual space of F such that $\forall a \in U, \forall h \in E$, the function

$$t \to u \circ f(a + th)$$

is holomorphic in a neighbourhood of zero in $\mathbb{C}$, $\forall u \in H$. The result is then trivial if we use Eq. (III.2.2).

Proposition III.2.4. *Let P be a polynomial in two variables X and y. Then we have:*

$$:P(\varphi^*, \varphi): \Phi_{f,G} = P(b^+ + f, a^+ + G)\,\Phi_{f,G} \qquad \text{(III.2.3)}$$

where $a^+ + G$ *(resp.* $b^+ + f$*) is defined as the operator valued function defined on* $\mathfrak{H}_+$ *(resp. on* $\mathfrak{H}'_-$*)* $h \to a^+(h) + \langle G, h\rangle \mathbb{1}$ *(resp.* $H \to b^+(H) + \langle H, f\rangle \mathbb{1}$*).*

Proof. Using recursively the canonical commutation relations one finds:

$$a^-(F)^m\, b^-(g)^n\, \Phi_{f,G} = \langle F, f\rangle^m \langle G, g\rangle^n\, \Phi_{f,G}\,. \qquad \text{(III.2.4)}$$

Equation (III.2.3) follows by linear extension.

III.3 Definition of $S_0(I)$

Propositions III.2.4 and III.2.2 allow us to define $:e^{i\varphi^* I \varphi}:$ as follows

$$:e^{i\varphi^* I\varphi}: \Phi_{(f,G)} = e^{ia^+ I_{+-} b^+}\, e^{i\langle G, I_{-+} f\rangle}\, \Phi_{(\mathbb{1}_+ + iI_{++})f,\, G(\mathbb{1}_- + iI_{--})}\,. \qquad \text{(III.3.1)}$$

Therefore, it is sufficient to show the existence of $e^{ia^+ I_{+-} b^+}$ as an operator densely defined on $\mathscr{D}_c$. The main result is the following:

Theorem III.3.1. a) *The vector* $e^{ia^+ K b^+}\Phi_{f,G}$ *belongs to* $\mathfrak{F}$ *if and only if K is a H.S. operator with*

$$\|K\|_{\text{op}} < 1 \qquad \text{(II.3.2)}$$

b) *if* $K, K' \in \hat{\mathfrak{S}}_2 = \{K; k \in \mathscr{L}(\mathfrak{H}_-, \mathfrak{H}_+),\ \|K\|_{\text{op}} < 1,\ \|K\|_{\text{H.S.}} < +\infty\}$ *then*

$$(e^{ia^+K'b^+}\Phi_{f',G'}|e^{ia^+Kb^+}\Phi_{f,G}) = \frac{\exp\left\langle (f'^*, G), \begin{bmatrix} \mathbb{1}_+ & -iK \\ -iK'^* & \mathbb{1}_- \end{bmatrix}^{-1} \begin{pmatrix} f \\ G'^* \end{pmatrix}\right\rangle}{\det(\mathbb{1}_+ - KK'^*)}\,. \qquad \text{(III.3.3)}$$

c) *The mapping* $F: (K, f, G) \to e^{ia^+Kb^+}\Phi_{f,G}$ *defined in* $\hat{\mathfrak{S}}_2 \times \mathfrak{H}_+ \times \mathfrak{H}'_-$ *with values in* $\mathfrak{F}$ *is analytic.*

Let K be a H.S. operator. Then there exist [19, 20] two orthonormal basis $(\varphi_i)_{i\in\mathbb{N}}$ in $\mathfrak{H}_+$ and $(\Gamma_i)_{i\in\mathbb{N}}$ in $\mathfrak{H}'_-$ such that:

$$\begin{aligned}
&\alpha) \quad -iK = \sum_{i'\in\mathbb{N}} \lambda_i \varphi_i \otimes \Gamma_i \quad \text{converges in H.S. norm}\,.\\
&\beta) \quad \lambda_i \geqq 0 \quad \forall i \in \mathbb{N}\,,\\
&\gamma) \quad \sum_{i\in\mathbb{N}} \lambda_i^2 = \operatorname{tr}(KK^*) = \|K\|^2_{\text{H.S.}} < +\infty \qquad \text{(III.3.4)}\\
&\delta) \quad \sup_{i\in\mathbb{N}} \lambda_i = \|K\|_{\text{op}}\,.
\end{aligned}$$

Now, one defines a_i^+ and b_i^+ as follows:

$$a_i^+ = a^+(\varphi_i) \qquad b_i^+ = b^+(\Gamma_i)\,. \qquad \text{(III.3.5)}$$

Then, one has also the following proposition:

Proposition III.3.2. *The following formula holds:*

$$e^{ia^+Kb^+} = s\text{-}\lim_{N\to\infty} \prod_{i=1}^{N} \frac{1}{\lambda_i} \int e^{-\frac{|z|^2}{\lambda_i}} e^{i(za_i^+ + \bar{z}b_i^+)} \frac{dz\, d\bar{z}}{2i\pi} \tag{III.3.6}$$

on the dense domain $\mathscr{D}_c$ of coherent states.

Proof of Theorem III.3.1. i) Let us remark that if K is not a H.S. operator then $e^{ia^+Kb^+}\Phi_{f,G}$ does not belong to $\mathfrak{F}$ as can be seen by computing the one particle – one antiparticle component.

ii) One will use the following notations:

$$\mathfrak{F}_i = \text{Fock space spanned by } (a_i^+, b_i^+) \qquad \text{a)}$$

$$\Omega_i = \text{vacuum of } \mathfrak{F}_i. \qquad \text{b)}$$

If $f \in \mathfrak{H}_+$ and $G \in \mathfrak{H}'_-$ one puts:

$$f_i = (\varphi_i | f) \qquad G_i = (\Gamma_i | G) \qquad \text{c)}$$

$$\Phi_i = \exp(-\lambda_i a_i^+ b_i^+ + f_i a_i^+ + G_i b_i^+)\,\Omega_i. \qquad \text{d)} \tag{III.3.7}$$

Hence:

$$\Omega = \bigotimes_{i\in\mathbb{N}} \Omega_i \qquad \mathfrak{F} = \bigotimes_{i\in\mathbb{N}}^{\Omega} \mathfrak{F}_i, \qquad \text{e)}$$

$$e^{ia^+Kb^+}\Phi_{f,G} = \bigotimes_{i\in\mathbb{N}} \Phi_i. \qquad \text{f)}$$

iii) One can easily see the following results:

– $\Phi_i \in \mathfrak{F}_i$ if and only if $\lambda_i < 1$ and

–
$$\|\Phi_i\|^2 = (1-\lambda_i^2)^{-1} \exp\left\langle (\bar{f}_i, G_i), \begin{bmatrix} 1 & \lambda_i \\ \lambda_i & 1 \end{bmatrix}^{-1} \begin{pmatrix} f_i \\ \bar{G}_i \end{pmatrix} \right\rangle \tag{III.3.8}$$

as can be seen from Lemma A.4.2 in Appendix 4.

– Hence, the vector:

$$e^{ia^+Kb^+}\Phi_{f,G} = \bigotimes_{i\in\mathbb{N}} \Phi_i$$

belongs to $\mathfrak{F}$ only if $\|K\|_{\text{op}} < 1$, in view of the formulae III.3.4 δ) and III.3.8.

– The infinite product $\prod_{i\in\mathbb{N}} \|\Phi_i\|^2$ converges in $\mathbb{R}_+$ since the sequences $(\lambda_i)_{i\in\mathbb{N}}$, $(f_i)_{i\in\mathbb{N}}$, $(G_i)_{i\in\mathbb{N}}$ are in $l^2(\mathbb{N})$.

– Therefore, the vector $\bigotimes_{i\in\mathbb{N}} \Phi_i$ belongs to $\mathfrak{F}$. Indeed, let $\Phi_{(N)}$ be defined as:

$$\Phi_{(N)} = \left(\bigotimes_{i=1}^{N} \Phi_i\right) \otimes \left(\bigotimes_{i=N+1}^{\infty} \Omega_i\right) \tag{III.3.9}$$

Then, if $M < N$, one has:

$$\|\Phi_{(N)} - \Phi_{(M)}\|^2 \leqq \left(\prod_{i\in\mathbb{N}} \|\Phi_i\|^2\right)\left(1 - \prod_{M+1}^{N} \|\Phi_i\|^2\right). \tag{III.3.10}$$

Thus $\Phi_{(N)}$ is a Cauchy sequence in $\mathfrak{F}$, and is convergent.

iv) Let $\hat{\mathfrak{S}}_2^F$ be the dense subset of $\hat{\mathfrak{S}}_2$ whose elements are finite-rank operators. By Lemma A.4.2 (Appendix 4), it can be seen that III.3.3 holds if K and K' are in $\hat{\mathfrak{S}}_2^F$. Putting $E=\mathfrak{S}_2\times\mathfrak{H}_+\times\mathfrak{H}'_-$, $E^F=\hat{\mathfrak{S}}_2^F\times\mathfrak{H}_+\times\mathfrak{H}'_-$, and, $x=(K,f,G)\in E$,

$$\begin{aligned}F(x)&=e^{ia^+Kb^+}\Phi_{f,G}\\ H(x',x)&=(F(x')|F(x))\,.\end{aligned}\tag{III.3.11}$$

One deduces that, $(x',x)\to H(x',x)$ is continuous on $E^F\times E^F$, as can be seen from Formula III.3.3. Therefore, $H(x',x)$ can be uniquely continued as a continuous function on $E\times E$. On the other hand, one also has:

$$\|F(x)-F(x')\|^2=H(x,x)+H(x',x')-H(x',x)-H(x,x')\,.\tag{III.3.12}$$

Thus, $x\to F(x)$ is a continuous function on E. Now, F is clearly locally bounded, and we can see, from Equation III.3.3 that:

$$x\to(\Phi_{f,G}|F(x))\tag{III.3.13}$$

is analytic on E. Since $\mathscr{D}_c$ is total in $\mathfrak{F}$, all the necessary and sufficient conditions hold for the analycity of F (cf. [30], § 3.3, or the proof of Proposition III.2.3).

Proof of Proposition III.3.2. By the previous proof [see Eq. (III.3.7)] 10/11).

$$e^{ia^+Kb^+}=\operatorname*{s-lim}_{N\to\infty}\prod_{i=1}^{N}e^{-\lambda_i a_i^+ b_i^+}\tag{III.3.14}$$

on a dense domain of coherent states.

It is then sufficient to prove (III.3.4) when

$$-iK=\lambda_i\varphi_i\otimes\Gamma_i\,.$$

But it is easy to verify that, if $\lambda_i>0$: (use Lemma A.4.2 Appendix 4) and Theorem III.3.1)

$$\forall f'\in\mathfrak{H}_+\qquad\forall G'\in\mathfrak{H}'_-$$

$$(\Phi_{f',G'}|e^{-\lambda_i a_i^+ b_i^+}\Phi_{f,G})=\frac{1}{\lambda_i}\int e^{-\frac{|z|^2}{\lambda_i}}(\Phi_{f',G'}|e^{i(a_i^+z+b_i^+\bar z)}\Phi_{f,G})\frac{d\bar z\,dz}{2i\pi}\,.\tag{III.3.15}$$

However, Schwarz's inequality implies:

$$|(\Psi|e^{i(a_i^+z+b_i^+z)}\Phi_{f,G})|\leqq\|\Psi\|\,e^{1/2(\|f+iz_i\varphi_i\|^2+\|G+iz_i\Gamma_i\|^2)}\tag{III.3.16}$$

so that the integral converges in the strong sense if and only if $\lambda_i<1$.

III.4. Bogoliubov's Axioms for $S_0(I)$ [10, 11]

Theorem III.4.1. *Let I be an N-P solution of the equation*

$$I=A+I\Delta_F A=A+A\Delta_F I\tag{III.4.1}$$

for a physical external field A, such that $\det(\mathbb{1}_+-I_{+-}I^*_{+-})\neq0$. *Then:*

a) *$S_0(I)$ densely defined on $\mathscr{D}_c$ by*

$$S_0(I)\,\Phi_{(f,G)}=\det(\mathbb{1}_+-I_{+-}I^*_{+-})^{1/2}\cdot e^{i\langle G,I_{-+}f\rangle}\cdot e^{ia^+I_{+-}b^+}\Phi_{(\mathbb{1}_++iI_{++})f,G(\mathbb{1}_-+iI_{--})}\,.\tag{III.4.2}$$

b) $S_0(I)$ *can be continued as a unitary operator.*
c) $S_0(I)$ *is causal up-to-a-phase.*
d) $S_0(I)$ *is relativistically covariant.*

Proof. a) Since $\det(\mathbb{1}_+ - I_{+-}I^*_{+-}) > 0, I_{+-}$ satisfies all the conditions of Theorem III.3.1. Therefore a) is proved, since I is an N-P kernel (see Section II).

b) By Eq. (III.4.2) we find

$$(S_0(I)\,\Phi_{f_0'G_0'}|S_0(I)\,\Phi_{f_0,G_0}) = (e^{ia^+I_{+-}b^+}\Phi_{f',G'}|e^{ia^+I_{+-}b^+}\Phi_{f,G})\,\eta\det(\mathbb{1}_+ - I_{+-}I^*_{+-}) \tag{III.4.3}$$

with

$$f' = (\mathbb{1}_+ + iI_{++})f_0' \qquad G' = G_0'(\mathbb{1}_- + iI_{--})$$
$$f = (\mathbb{1}_+ + iI_{++})f_0 \qquad G = G_0(\mathbb{1}_- + iI_{--}) \tag{III.4.4}$$

and

$$\eta = \exp i(\langle G_0, I_{-+}f_0\rangle - \overline{\langle G_0', I_{-+}f_0'\rangle}). \tag{III.4.5}$$

By Theorem III.3.1, we find

$$(S_0(I)\,\Phi_{f_0'G_0'}|S_0(I)\,\Phi_{f_0G_0}) = \exp\left\langle (f_0'^*, G_0), B\begin{pmatrix} f_0 \\ G_0'^* \end{pmatrix}\right\rangle \tag{III.4.6}$$

with

$$B = \begin{bmatrix} 0 & -i(I_{-+})^* \\ iI_{-+} & 0 \end{bmatrix} + \begin{bmatrix} \mathbb{1}_+ - iI^*_{++} & 0 \\ 0 & \mathbb{1}_- + iI_{--} \end{bmatrix}$$
$$\cdot\begin{bmatrix} (\mathbb{1}_+ - I_{+-}I^*_{+-})^{-1} & i(\mathbb{1}_+ - I_{+-}I^*_{+-})^{-1}I_{+-} \\ i(\mathbb{1}_- - I^*_{+-}I_{+-})^{-1}I^*_{+-} & (\mathbb{1}_- - I^*_{+-}I_{+-})^{-1} \end{bmatrix}\begin{bmatrix} \mathbb{1}_+ + iI_{++} & 0 \\ 0 & \mathbb{1}_- - iI^*_{--} \end{bmatrix}. \tag{III.4.7}$$

With the results of Table I, we find

$$B = \mathbb{1}. \tag{III.4.8}$$

Thus, $\forall f_0', f_0 \in \mathfrak{H}_+, \forall G_0', G_0 \in \mathfrak{H}'_-$:

$$(S_0(I)\,\Phi_{f_0'G_0'}|S_0(I)\,\Phi_{f_0'G_0}) = (\Phi_{f_0'G_0'}|\Phi_{f_0'G_0}). \tag{III.4.9}$$

On the other hand, it can be seen without difficulty that

$$S_0(\bar{I}) \subset S_0(I)^*. \tag{III.4.10}$$

Thus $S_0(I)$ is unitary.

c) In order to show the causality property we have to compute $S_0(I_1)\,S_0(I_2)$. We shall use the integral representation given by Proposition III.3.2 and the boundedness of $S_0(I_1)$ for real external fields:

$$S_0(I_1)\,\operatorname*{s-lim}_{N\to\infty} A_N = \operatorname*{s-lim}_{N\to\infty} S_0(I_1)\,A_N \tag{III.4.11}$$

if A_N is a sequence of operators.

On the other hand, the strong convergence of integral III.3.6 allows us to push $S_0(I_1)$ under the integral sign. By computation of several Gaussian integrals we find, $\forall f_0 \in \mathfrak{H}_+, \forall G_0 \in \mathfrak{H}'_-$:

$$S_0(I_1)\,S_0(I_2)\,\Phi_{f_0G_0} = \eta S_0(I_3)\,\Phi_{f_0,G_0} \tag{III.4.12}$$

with

$$\eta = \frac{\det(\mathbb{1}_+ - I_{1+-}I^*_{1+-})^{1/2}\det(\mathbb{1}_+ - I_{2+-}I^*_{2+-})^{1/2}}{\det(\mathbb{1}_+ - I_{3+-}I^*_{3+-})^{1/2}\det(\mathbb{1}_+ + I_{1+-}I_{2-+})} \tag{III.4.13}$$

and

$$\begin{aligned} \mathbb{1}_+ + iI_{3++} &= (\mathbb{1}_+ + iI_{1++})(\mathbb{1}_+ + I_{1+-}I_{2-+})^{-1}(\mathbb{1}_+ + I_{2++}) \quad \text{a)} \\ \mathbb{1}_- + iI_{3--} &= (\mathbb{1}_- + iI_{2--})(\mathbb{1}_- + I_{2-+}I_{1+-})^{-1}(\mathbb{1}_- + I_{1--}) \quad \text{b)} \end{aligned} \tag{III.4.14}$$

$$\begin{aligned} I_{3-+} &= I_{2-+} + (\mathbb{1}_- + iI_{2--})(\mathbb{1}_- + I_{1-+}I_{2+-})^{-1} I_{1-+}(\mathbb{1}_+ + iI_{2++}) \quad \text{a)} \\ I_{3+-} &= I_{1+-} + (\mathbb{1}_+ + iI_{1++})(\mathbb{1}_+ + I_{2+-}I_{1-+})^{-1} I_{2+-}(\mathbb{1}_- + iI_{1--}). \quad \text{b)} \end{aligned} \tag{III.4.15}$$

With the help of the results of Table 1, it can be seen that $\mathbb{1} + I_{3,S}$ is also unitary and that η is a complex number with modulus 1.

Now, if we construct $J_{3,R}$ from $I_{3,S}$ by the formulae given in Table II, we find as a consequence of Theorem II.4.1.

$$(\mathbb{1} + iJ_{3,R}) = (\mathbb{1} + iJ_{1,R})(\mathbb{1} + iJ_{2,R}). \tag{III.4.16}$$

This formula suggests that I_3 is associated with a problem in which the corresponding retarded kernel $J_{r,3}$ is

$$J_{3,r} = J_{1,r} + J_{2,r} + J_{1,r}\Delta J_{2,r} \tag{III.4.17}$$

where $J_{i,r}(i=1,2)$ is the solution of

$$J_{i,r} = A_i + A_i\Delta_r J_{i,r} = A_i + J_{i,r}\Delta_r A_i. \tag{III.4.18}$$

Let us now assume that ([10] p. 167, [11])

$$\operatorname{supp}(v_1, A_{1\mu}) \cap (\bar{V}^- + \operatorname{supp}(v_2, A_{2\mu})) = \emptyset. \tag{III.4.19}$$

Then we easily find

$$A_1\Delta_a A_2 = J_{1,r}\Delta_a J_{2,r} = J_{1,r}\Delta_a A_2 = A_1\Delta_a J_{2,r} = 0. \tag{III.4.20}$$

Since we have the same formula by simultaneously interchanging 1 and 2, Δ_a and Δ_r,

$$J_{1,r}\Delta J_{2,r} = J_{1,r}\Delta_r J_{2,r} \tag{III.4.21}$$

and

$$J_{3,r} = (A_1 + A_2)(1 + \Delta_r J_{3,r}) = (1 + J_{3,r}\Delta_r)(A_1 + A_2). \tag{III.4.22}$$

Therefore, I_3 is a kernel associated with the external field

$$((v_1 + v_2), (A_{1,\mu} + A_{2,\mu})) \tag{III.4.23}$$

which proves causality up-to-a phase.

d) The covariance of $S_0(I)$ is easy to show. Recall that

$$^{(a,\Lambda)}(v, A_\mu)(x) = (v, \Lambda_\mu{}^\nu A_\nu)(\Lambda^{-1}(x-a)). \tag{III.4.24}$$

Then:

$$^{(a,\Lambda)}I(x,y) = I(\Lambda^{-1}(x-a), \Lambda^{-1}(y-a)). \tag{III.4.25}$$

If $U_\varepsilon(a,\Lambda)$ is the representation of the Poincaré group in $\mathfrak{H}_\varepsilon$, it is clear from II.1.7, that in momentum space:

$$(^{a,\Lambda}I)_{\varepsilon\varepsilon'} = U_\varepsilon(a,\Lambda)\, I_{\varepsilon\varepsilon'}\, U_{\varepsilon'}(a,\Lambda)^{-1}. \tag{III.4.26}$$

Therefore, $\det(\mathbb{1}_+ - I_{+-} I^*_{+-})$ is invariant, and

$$:e^{i\int \varphi^*(a,\Lambda) I \varphi}: = U(a,\Lambda) :e^{\int i\varphi^* I \varphi}: U(a,\Lambda)^{-1} . \tag{III.4.27}$$

Theorem III.4.2. i) *The mapping $I \to S_0(I)$ is strongly continuous when we norm the space of N-P kernel with*

$$\|I\|_{\text{N-P}} = \max\{\|I_{++}\|_{\text{op}}, \|I_{--}\|_{\text{op}}, \|I_{+-}\|_{\text{H.S.}} \|I_{-+}\|_{\text{H.S.}}\} .$$

ii) *The mapping $(I, f, G) \to \det(\mathbb{1}_+ - I_{+-} I^*_{+-})^{-1/2} S_0(I) \Phi_{f,G}$ is analytic in the domain*

$f \in \mathfrak{H}_+$, $G \in \mathfrak{H}'_-$, $I_{+-} \in \hat{\mathfrak{S}}_2$, $I_{-+} \in \hat{\mathfrak{S}}_2$, $I_{++} \in \mathscr{L}(\mathfrak{H}_+)$, $I_{--} \in \mathscr{L}(\mathfrak{H}_-)$.

Proof. a) it is well known that [20]

$$I_{+-} \to \det(\mathbb{1}_+ - I_{+-} I^*_{+-})^{1/2}$$

is continuous with respect to the H.S. norm.

b) From the analyticity [30] of

$$\begin{gathered} (I_{++}, f) \to I_{++} f \\ \mathscr{L}(\mathfrak{H}_+) \times \mathfrak{H}_+ \quad \mathfrak{H}_+ \end{gathered}$$

and from Theorem III.3.1, ii) follows.

c) i) is then a simple consequence of ii).

III.5. The Interpolating Fields

Theorem III.5.1. *Let ψ be the field formally defined by:*

$$\psi(x) = :e^{i\int \varphi^* \bar{I} \varphi}: :\int d^4y (1 + \Delta_F I)(x, y) \varphi(y) e^{i\int \varphi^* I \varphi}:. \tag{III.5.1}$$

a) *$\psi(f) = \int \psi(x) f(x) d^4x$ is densely defined on $\mathscr{D}_c$ $\forall f \in \mathscr{S}(\mathbb{R}^4)$.*

b) *$\psi(x)$ is an operator valued distribution,* [32], *which is a solution of the Yang-Källen-Feldman equations, on the dense domain $\mathscr{D}_c$.*

Proof. a) First of all, $f \to \int d^4x f(x)(1 + \Delta_F])(x, y)$ is a continuous linear map from $\mathscr{S}(\mathbb{R}^4)$ to $\mathscr{S}(\mathbb{R}^4)$, and therefore $(1 + \Delta_F I)\varphi$ is a good operator valued distribution.

On the other hand $:e^{i\int \varphi^* \bar{I} \varphi}:$ can be continued as a bounded operator, as follows from Theorem III.4.1. Therefore, we have to show that if $f \in \mathfrak{H}_+$, $G \in \mathfrak{H}'_-$, then

$$:\varphi(h) e^{i\int \varphi^* I \varphi}: \Phi_{f,G} \in \mathfrak{F} \qquad \forall h \in \mathscr{S}(\mathbb{R}^4) . \tag{III.5.2}$$

By using properties of the coherent states (Proposition III.2.4) it is sufficient to show that

$$e^{ia^+ I_{+-} b^+} \Phi_{f,G} \tag{III.5.3}$$

is in the domain of b^+ (or a^+) for any wave function f (or G). In order to see this, recall that the domain of b^+ (or a^+) is the set of vectors Ψ such that

$$\sum_{s,r} s \|\Psi\|^2_{r,s} < +\infty \qquad \left(\text{or} \sum_{s,r} r \|\Psi\|^2_{r,s} < +\infty\right) . \tag{III.5.4}$$

Now, let us put for $B > 0$

$$\|\Psi\|_B^2 = \sum_{r,s} B^{r+s} \|\Psi\|_{r,s}^2 . \tag{III.5.5}$$

We see that

$$\|e^{ia^+ I_{+-} b^+} \Phi_{f,G}\|_B^2 = \|e^{iBa^+ I_{+-} b^+} \Phi_{\sqrt{B}f,\ \sqrt{B}G}\|^2 \tag{III.5.6}$$

which, by Theorem III.3.1 is finite if and only if

$$\|I_{+-}\|_{\mathrm{op}} < 1/B . \tag{III.5.7}$$

In view of the condition $\|I_{+-}\|_{\mathrm{op}} < 1$, there exists $B_0 > 1$ such that $\|I_{+-}\|_{\mathrm{op}} < \frac{1}{B_0}$. Therefore

$$B \to \|e^{ia^+ I_{+-} b^+} \Phi_{f,G}\|_B^2 \tag{III.5.8}$$

is analytic in an open domain containing the closed set $B \leqq 1$. Therefore

$$\frac{d}{dB} \|e^{ia^+ I_{+-} b^+} \Phi_{f,G}\|_B|_{B=1} = \sum_{r,s} (r+s) \|\ldots\|_{r,s}^2 < +\infty . \tag{III.5.9}$$

Remark III.5.2. By the same method we also prove that $e^{ia^+ I_{+-} b^+} \Phi_{f,G}$ is in the domain of $(a^+)^m (b^+)^n \ \forall m, \forall n$.

b) Now it is easy to compute the matrix elements of $\psi(x)$ between coherent states. We have only to use Theorem III.3.1 and Table I. We find that

$$\begin{aligned} (1 - \Delta_r A)(\Phi_{f',G'} | \psi(x) \Phi_{f,G}) &= (\Phi_{f',G'} | \varphi(x) \Phi_{f,G}) \\ (1 - \Delta_a A)(\Phi_{f',G'} | \psi(x) \Phi_{f,G}) &= (\Phi_{f'G'} | \varphi_{\mathrm{out}}(x) \Phi_{fG}) \end{aligned} \tag{III.5.10}$$

which are the Yang-Källen-Feldman equations [14, 22].

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Appendix I

Fock Space and Two Point Functions

1. Let $\mathfrak{H}$ be a Hilbert space. We denote by $\mathfrak{H}'$ its dual. By Riez's theorem [18] we can find an antiisomorphism

$$J : x \to x^*$$

from $\mathfrak{H}$ to $\mathfrak{H}'$, defined by

$$\forall y \in \mathfrak{H} \qquad \langle x^*, y \rangle = (x | y) . \tag{A.1.1}$$

$\mathfrak{H}'$ is a Hilbert space with inner product:

$$(x' | y')_{\mathfrak{H}'} = (J^{-1} y | J^{-1} x)_{\mathfrak{H}} \tag{A.1.2}$$

We shall denote also $J^{-1} x'$ by x'^*, which is not confusing, by Eq. (A.1.2).

2. In order to construct the Fock space for charged bosons, we need two Hilbert spaces $\mathfrak{H}_+$ and $\mathfrak{H}_-$. We shall define $\mathfrak{F}$ by

$$\mathfrak{F} = \bigoplus_{r,s} (\mathfrak{H}_+)^{\vee r} \otimes (\mathfrak{H}'_-)^{\vee s} . \qquad (A.1.3)$$

Here $\vee$ means "symmetrical tensor product".

If $\Phi \in \mathfrak{F}$ we can write

$$\Phi = (\Phi_{ns})_{r,s \in \mathbb{N}^2} \qquad \Phi_{r,s} \in \mathfrak{H}_+^{\vee r} \otimes \mathfrak{H}_-^{\prime \vee s} . \qquad (A.1.4)$$

We put:

$$\Omega = (\Omega_{r,s})_{(r,s) \in \mathbb{N}^2} \qquad \Omega_{0,0} = 1 \qquad \Omega_{r,s} = 0 \quad \text{if} \quad (r,s) \neq 0 . \qquad (A.1.5)$$

3. Creation and annihilation operators for particles or antiparticles are defined as follows

$$\forall F \in \mathfrak{H}'_+ , \quad \forall f \in \mathfrak{H}_+ , \quad \forall G \in \mathfrak{H}'_- , \quad \forall g \in \mathfrak{H}_-$$

$$a^-(F)\,\Omega = 0 , \quad b^-(g)\,\Omega = 0 , \qquad \text{a)}$$

$$[a^\pm, b^\pm] = 0 . \qquad (A.1.6)$$

$$[a^-(F), a^+(f)] \subset \langle F, f \rangle \mathbb{1}_{\mathfrak{F}} , \quad [b^-(g), b^+(G)] \subset \langle G, g \rangle \mathbb{1}_{\mathfrak{F}} . \qquad \text{b)}$$

$$a^+(f) = a^-(f^*)^* \quad b^+(G) = b^-(G^*)^* \qquad \text{c)}$$

4. The field is defined if we can find four linear continuous maps defined on the space of test function $\mathscr{S}(\mathbb{R}^4)$ $u_\pm, u'_\pm$ such that [6, 33, 34]

$$\operatorname{Ran}(u_\pm) \subset \mathfrak{H}_\pm \qquad \operatorname{Ran}(u'_\pm) \subset \mathfrak{H}'_\pm . \qquad (A.1.7)$$

Then $\forall h \in \mathscr{S}(\mathbb{R}^4)$

$$\left.\begin{array}{ll} \varphi(h) = a^-(u'_+ h) + b^+(u'_-(h)) & \text{a)} \\ \varphi^*(h) = a^+(u_+(h)) + b^-(u_-(h)) & \text{b)} \end{array}\right. \qquad (A.1.8)$$

with the condition (if $h \to h^*$ is the natural involution in $\mathscr{S}(\mathbb{R}^4)$).

$$u_\pm(h)^* = u'_\pm(h) . \qquad (A.1.9)$$

For more complete details about covariance see for instance [6, 33, 34].

5. For a scalar field we choose $\mathfrak{H}_+ = \mathfrak{H}_- = \mathfrak{H}'_+ = \mathfrak{H}'_- = \mathscr{L}^2(H_m, d\Omega_m(p))$ where

$$H_m = \{p \in \mathbb{R}^4 ; p^0 = \sqrt{\vec{p}^2 + m^2} = \omega_p\} \qquad d\Omega_m(p) = \frac{d^3\vec{p}}{2\omega_p} . \qquad (A.1.10)$$

We have to put:

$$\forall h \in \mathscr{S}(\mathbb{R}^4)$$

$$u_+(h) = u'_-(h) = p \to (2\pi)^{-3/2} \int e^{-ip\cdot x} h(x)\, d^4x|_{p^0 = \omega_p} \qquad (A.1.11)$$

$$u_-(h) = u'_+(h) = p \to (2\pi)^{-3/2} \int e^{ip\cdot x} h(x)\, d^4x|_{p^0 = \omega_p} .$$

Then

$$[\varphi(h), \varphi^*(k)] \subset i\left(\int h(x)\, k(y)\, \Delta(x-y)\, dx dy\right) \mathbb{1}_{\mathfrak{F}} \qquad (A.1.12)$$

with $\Delta = \Delta_+ - \Delta_-$ and

$$\Delta_\pm(\xi) = \frac{-i}{(2\pi)^{3/2}} \int_{Hm} e^{\pm ip\cdot\xi} d\Omega_m(p) . \qquad (A.1.13)$$

We then define Δ_r and Δ_a by

$$\begin{aligned} &\Delta = \Delta_r - \Delta_a && \text{a)} \\ &\operatorname{supp} \Delta_r \subset \bar{V}^+ , \quad \operatorname{supp} \Delta_a \subset \bar{V}^- && \text{b)} \\ &(\Box + m^2)\, \Delta_{r_a} = \delta . && \text{c)} \end{aligned} \tag{A.1.14}$$

Appendix 2

Wick's Theorems

1. Weak Wick's Theorem [17]

Let φ be a free scalar field. Then

$$\prod_{i=1}^{n} \frac{:\varphi^{\alpha_i}:}{\alpha_i!}(x_i) = \sum_{0 \leqq \beta_i \leqq \alpha_i} \left(\Omega \left| \prod_{i=1}^{n} \frac{:\varphi^{\beta_i}:}{\beta_i}(x_i)\, \Omega \right. \right) : \prod_{i=1}^{n} \frac{\varphi^{\alpha_i - \beta_i}}{(\alpha_i - \beta_i)!}(x_i) : . \tag{A.2.1}$$

Let now $\lambda = (\lambda_i(x))_{i \in \mathbb{N}}$ be a family of test functions, a finite number of which only are non zero. We define

$$\mathscr{L}^{(\alpha)}(x; \lambda) = \sum_{\beta \geqq 0} \lambda_{\alpha+\beta}(x) \frac{:\varphi^{\beta}:}{\beta!}(x) . \tag{A.2.2}$$

With $\mathscr{L}^{(0)} = \mathscr{L}$ Eq. (A.2.1) can be written in the formal power series sense with respect to λ:

$$e^{i \int d_x^4 \mathscr{L}(x; \lambda)} = :e^{\sum_{\alpha \leqq 0} \int d_x^4 \mathscr{L}^{(\alpha)}_{((x); \lambda)} \frac{\delta}{\delta \lambda'_\alpha(x)}} : (\Omega)\, e^{i \int d_x^4 \mathscr{L}(x; \lambda')} \Omega) . \tag{A.2.3}$$

Formulae (A.2.1, A.2.3) can be generalized without difficulty, by replacing α by a multiindex $(\alpha_1, \ldots, \alpha_N)$ if $\varphi = (\varphi_1, \ldots, \varphi_N)$. If some components describe a fermion field, the corresponding "coupling" constants $\lambda_{(\cdot)}$ have to take values in a formal Grassman algebra to give the correct signs in (A.2.1).

2. Strong Wick's Theorem

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an n-index. Let $\Gamma_{(\alpha)}$ be the set of graphs with vertices $1, 2, \ldots, n$ such that α_i lines meet at vertex "i".

If l is a line of the graph G, we denote by $u(l)$ and $v(l)$ its extremity, with this convention:

$$u(l) < v(l) .$$

Then we have

$$\left(\Omega \left| \prod_{i=1}^{n} \frac{:\varphi^{\alpha_i}:}{\alpha_i!}(x_i)\, \Omega \right. \right) = \sum_{G \in \Gamma_\alpha} \prod_{l \in G} \Delta_+(x_{u(l)} - x_{v(l)}) . \tag{A.2.4}$$

The right-hand side is defined in $\mathscr{S}'(\mathbb{R}^4)$ as the boundary value of a function analytic in the tube:

$$\operatorname{Im}(x_i - x_{i+1}) \in V^+ \qquad i = 1, \ldots, n-1 . \tag{A.2.5}$$

If G is a tree-graph, the product $\prod_{l \in G} \ldots$ is a simple tensor product of distribution [11, 25].

Formula (A.2.4) can be easily generalised for field with several components by introducing several kinds of lines and other "propagators".

3. Wick's Theorem for Chronological Product [11]

One can formally replace products of fields by chronological products in (A.2.1) or (A.2.4). Epstein and Glaser [11] have shown that it is possible to define a T-product such that (A.2.3) holds, and such that (A.2.4) holds for tree graphs (see also Bogoliubov and Chirkov [10]).

Appendix 3

Proof of Theorem I.1.1

1. We want to compute $S(g, J)$ for the following lagrangean density:

$$\mathscr{L}_I(x, \underline{g}\underline{J}) = v :\varphi^* \varphi: + iA_\mu :\varphi^* \overleftrightarrow{\partial}^\mu \varphi: + J\varphi + \bar{J}\varphi^* + J_\mu \partial^\mu \varphi + \bar{J}_\mu \partial^\mu \varphi^* \tag{A.3.1}$$

with

$$\underline{g} = (v, A_\mu) \qquad \underline{J} = (J, \bar{J}, J_\mu, \bar{J}_\mu)\,. \tag{A.3.2}$$

It is more convenient to write the field in the Petiau-Duffin-Kemmer [35] form:

$$\hat{\psi} = \begin{bmatrix} \varphi \\ \partial^\mu \varphi \end{bmatrix} \qquad \hat{\psi}^* = [\varphi^*, \partial^\mu \varphi^*] \qquad j = [J, J_\mu] \qquad \bar{j} = \begin{bmatrix} \bar{J} \\ \bar{J}_\mu \end{bmatrix} \tag{A.3.3}$$

and

$$\boldsymbol{A} = \begin{bmatrix} v & iA_\mu \\ -iA_\mu & 0 \end{bmatrix}.$$

Then, (A.3.1) becomes:

$$\mathscr{L}_I(x, \underline{g}, \underline{J}) = j \cdot \hat{\psi} + \hat{\psi}^* \bar{j} + :\hat{\psi}^* \boldsymbol{A} \hat{\psi}:\,. \tag{A.3.4}$$

2. We can apply the weak Wick theorem [Appendix 2 (A.2.3)]:

$$\begin{aligned} S(\underline{g}, \underline{J}) &= T(\exp i \textstyle\int \mathscr{L}(x, \underline{g}, \underline{J})\, d^4x) \\ &= :e^{i \int \mathscr{L}_I(x, \underline{g}, \underline{J}) d^4_x + \int (j + \hat{\psi}^* \boldsymbol{A})(x) \frac{\delta}{\delta j'(x)} + \frac{\delta}{\delta j'(x)} (\boldsymbol{A}\hat{\psi} + \bar{j})(x) d^4_x} : (\Omega, S(\underline{g}, \underline{J})\, \Omega)|_{J'=0} \end{aligned} \tag{A.3.5}$$

(use also Taylor's formula).

On the other hand, by using the strong Wick theorem for tree graphs (App. 2 § 2 and 3) we can find:

$$\begin{aligned} &\frac{\delta}{\delta j(x)} (\Omega | S(\underline{g}, \underline{J})\, \Omega) = i \textstyle\int \boldsymbol{G}_F(x, y)\, \bar{j}(y)\, dy^4 \quad (\Omega | S(\underline{g}, \underline{J})\, \Omega) && \text{a)} \\ &\frac{\delta}{\delta J(g)} (\Omega | S(\underline{g}, \underline{J})\, \Omega) = i \textstyle\int j(x) \boldsymbol{G}_F(x, y)\;\, dx^4 \quad (\Omega | S(\underline{g}, \underline{J})\, \Omega) && \text{b)} \end{aligned} \tag{A.3.6}$$

where $\boldsymbol{G}_F$ is defined by:

$$\begin{aligned} \boldsymbol{G}_F &= S_F + S_F \boldsymbol{A} \boldsymbol{G}_F = S_F + \boldsymbol{G}_F \boldsymbol{A} S_F && \text{a)} \\ S_F(x-y) &= (\Omega | T(\hat{\psi}(x)\, \hat{\psi}^*(y))\, \Omega)\,. && \text{b)} \end{aligned} \tag{A.3.7}$$

Therefore (see Schwinger [3]) the unique solution of (A.3.6) is

$$(\Omega | S(\underline{g}, \underline{J})\, \Omega) = (\Omega | S(\underline{g}, 0)\, \Omega)\, e^{i \int j \boldsymbol{G_F} \bar{j}} . \tag{A.3.8}$$

Omitting integrations, we can write, by Taylor's formula

$$\begin{aligned} S(\underline{g}, \underline{J}) &= e^{i j \boldsymbol{G_F} \bar{j}} :e^{i \int \hat{\psi}^* \boldsymbol{I} \hat{\psi} + j(\mathbb{1} - S_F \boldsymbol{A})^{-1} \hat{\psi} + \hat{\psi}^*(1 - \boldsymbol{A} S_F)^{-1} \bar{j}}: \\ &\quad \ldots (\Omega | S(g, 0)\, \Omega) \\ \boldsymbol{I} &= \boldsymbol{A} + \boldsymbol{A} \boldsymbol{G_F} \boldsymbol{A} . \end{aligned} \tag{A.3.9}$$

3. Now, we have to introduce the distribution kernel $I(x, y)$ defined by

$$\int \varphi^* I \varphi = \int \hat{\psi}^* \boldsymbol{I} \hat{\psi} . \tag{A.3.10}$$

In order to compute I we have to compute S_F. Indeed power counting tells us that [11]

$$(\Omega | T(\partial_\mu \varphi(x)\, \partial_\nu \varphi^*(y))\, \Omega)$$

is defined up to a counter-term of the form

$$a g_{\mu\nu} \delta(x - y) \qquad a \in \mathbb{R} . \tag{A.3.11}$$

By choosing

$$(\Omega | T(\partial_\mu \varphi(x)\, \partial_\nu \varphi^*(y))\, \Omega) = \frac{1}{i} (\partial_x \partial_\nu \Delta_F + g_{\mu\nu} \delta)(x - y) \tag{A.3.12}$$

we find

$$I = A + A \Delta_F I = A + I \Delta_F A \tag{A.3.13}$$

with

$$A(x, y) = [v(x) + A_\mu A^\mu(x)]\, \delta(x - y) + i[A_\mu(x) + A_\mu(y)]\, \partial^\mu \delta(x - y) . \tag{A.3.14}$$

And therefore

$$S(\underline{g}, 0) = S(\underline{g}) = :e^{i \int \varphi^* I \varphi}: (\Omega | S(\underline{g})\, \Omega) . \tag{A.3.15}$$

Remark. This choice of renormalisation is associated with the minimal coupling for the two points Green's function.

4. In order to compute the interpolating field we have only to derive (A.3.9)

$$S(\underline{g}, \underline{J})^{-1} \frac{\delta S}{\delta J(x)} (\underline{g}, \underline{J}) \bigg|_{\underline{J} = 0} = \psi(x) . \tag{A.3.16}$$

We must remark that:

$$\frac{\delta}{\delta J(x)} j(1 - S_F \boldsymbol{A})^{-1} \hat{\psi} = (1 - \Delta_F A)^{-1} \varphi = (1 + \Delta_F I)\, \varphi \tag{A.3.17}$$

because

$$(1 + \Delta_F I)(1 - \Delta_F A) = (1 - \Delta_F A)(1 + \Delta_F I) = 1 . \tag{A.3.18}$$

Therefore Proposition I.4.1 is proved.

Remark. The same calculation can be performed for any free field and any quadratic lagrangean. It gives us a "preferred" choice of renormalisation for fields with high spins.

Appendix 4

Gaussian Integrals

Let $E = \mathbb{C}^n$ with its canonical Hilbert structure. If $x \in E$, let $x^* \in E'$ the linear form on E defined by (cf. Appendix 1. § 2)

$$\langle x^*, y\rangle = (x | y) \qquad \forall y \in E. \tag{A.4.1}$$

Then:

Lemma A.4.1. *The following formula holds, provided*

$$A + A^* > 0 \qquad \text{a)}$$

$$e^{-\langle\alpha, A\beta\rangle} \det(A) = \int e^{-\langle z^*, A^{-1}z\rangle + i\langle\alpha, z\rangle + i\langle z^*, \beta\rangle} d\varrho_n(z, z^*). \qquad \text{b)} \tag{A.4.2}$$

Here $\alpha \in E'$ *and* $\beta \in E$.

$$d\varrho_n(z, z^*) = d^n(\operatorname{Re} z)\, d^n(\operatorname{Im}(z))\, \pi^{-n}. \qquad \text{c)}$$

Remark. If $A + A^* > 0$ then A is regular.

Now let $(\varphi_i)_{i\in\mathbb{N}}$ and $(\Gamma_i)_{i\in\mathbb{N}}$ be two orthogonal basis in $\mathfrak{H}_+$ and $\mathfrak{H}_-$ respectively. Let a_i^+, b_i^+ be defined by

$$a_i^+ = a^+(\varphi_i) = \langle a^+, \varphi_i\rangle, \qquad b_i^+ = b^+(\Gamma_i) = \langle\Gamma_i, b^+\rangle. \tag{A.4.3}$$

Then we have the following result.

Lemma A.4.2. *Let* λ, λ' *be in* $\mathscr{L}(\mathbb{C}^n)$, α, α' *be in* $\mathbb{C}^n$, β, β' *in* $\mathbb{C}^{n'}$. *We put:*

$$\Psi(\lambda, \alpha, \beta) = \exp[\Sigma\lambda_{ij} a_i^+ b_j^+ + \Sigma(\alpha_i a_i^+ + \beta_i b_i^+)]\,\Omega. \tag{A.4.4}$$

Then we have

a) $\qquad \Psi(\lambda, \alpha, \beta) \in \mathfrak{F}$ *if and only if* $0 \leqq \lambda\lambda^* < \mathbb{1}$.

b) $$(\Psi(\lambda', \alpha', \beta') | \Psi(\lambda, \alpha\beta)) = \frac{\exp\left\langle (\alpha'^*, \beta)\begin{bmatrix} \mathbb{1}_n & -\lambda \\ -\lambda'^* & \mathbb{1}_n\end{bmatrix}\begin{pmatrix}\alpha \\ \beta'^*\end{pmatrix}\right\rangle}{\det_n(\mathbb{1} - \lambda\lambda'^*)}. \tag{A.4.5}$$

Proof. To establish formula (IV.4.5) we can use Bargman's representation [36] of Fock space with a finite number of degree of freedom.

Then, $\Psi(\lambda, \alpha, \beta)$ is representaed by the analytic entire function

$$\hat{\Psi}_{\lambda,\alpha,\beta}(\xi, \eta) = \exp(\langle\xi, \lambda\eta\rangle + \langle\xi, \alpha\rangle + \langle\beta, \eta\rangle) \tag{A.4.6}$$

with $\xi \in E', \eta \in E$.

The scalar product is given by

$$(\hat{\Psi}', \hat{\Psi}) = \int e^{-\langle\xi,\xi^*\rangle - \langle\eta^*,\eta\rangle}\, \hat{\Psi}'(\xi, \eta)^*\, \Psi(\xi, \eta)\, d\varrho_n(\xi, \xi^*)\, d\varrho_n(\eta, \eta^*). \tag{A.4.7}$$

It is then sufficient to use Lemma A.4.1.

Remark A.4.3. 1. In Lemma A.4.3 (α'^*, β) is the linear form on $E \times E$ defined by

$$\left\langle (\alpha'^*, \beta), \begin{pmatrix}\alpha \\ \beta'^*\end{pmatrix}\right\rangle = \langle\alpha'^*, \alpha\rangle + \langle\beta, \beta'^*\rangle. \tag{A.4.8}$$

2. We also have the formula

$$\begin{pmatrix} \mathbb{1} & -\lambda \\ -\lambda'^* & \mathbb{1} \end{pmatrix}^{-1} = \begin{pmatrix} (\mathbb{1}-\lambda\lambda'^*)^{-1} & (\mathbb{1}-\lambda\lambda'^*)^{-1}\lambda \\ (\mathbb{1}-\lambda'^*\lambda)^{-1}\lambda'^* & (\mathbb{1}-\lambda'^*\lambda)^{-1} \end{pmatrix}. \tag{A.4.9}$$

Appendix 5

A Class of Banach Spaces of Kernels

In this Appendix we use the notations of Section II.1.

Definition A.5.1. Let $\alpha \in \mathbb{R}_+, M \in \mathbb{R}, m \in \mathbb{N}$. Let E be a finite dimensional Banach space. Then $\mathcal{N}_E(m, \alpha, M)$ is the Banach space of kernels from $\mathbb{R}^4 \times \mathbb{R}^4$ to E, normed by:

$$\|K\|_{\alpha,m,M} = \sum_{\substack{|\mu|,|\nu| \\ \leqq m}} \sup_{\mathbb{R}^4 \times \mathbb{R}^4} \left\{ \frac{[1+(p^0-q^0)^2]^\alpha [1+(\vec{p}-\vec{q})^2]^\alpha}{(1+|p|^2)^M (1+|q|^2)^M} \|\partial_p^\mu \partial_q^\nu K(p,q)\|_E \ . \right\} \tag{A.5.1}$$

Here for $p \in \mathbb{R}^4$ we denote $p = (p^0, \vec{p})$; $p^0 \in \mathbb{R}, \vec{p} \in \mathbb{R}^3$ and

$$|p| = (p^{0\,2} + \vec{p}^2)^{1/2} . \tag{A.5.2}$$

Remark A.5.2. The space $\mathcal{N}_E(m, \alpha, M)$ is increasing with M and decreasing with α and m.

Let $S_{\text{ex}}(x) = P(i\partial)\,\Delta_{\text{ex}}(x)$ where P is a polynomial of of degree σ over $\mathbb{R}^4$ with values in $\mathcal{L}(E)$ [cf. Eq. (II.1.10–11)].

Let $(A_\mu)_{\mu=0,1,2,3}$ and v be functions in $\mathcal{S}(\mathbb{R}^4)$. We will put:

$$\begin{aligned} \tilde{A}(p,q) &= \tilde{v}(p-q) + \widetilde{A_\mu A^\mu}(p-q) + i\tilde{A}_\mu(p-q)(p^\mu + q^\mu) \quad &\text{a)} \\ \tilde{\mathcal{A}}(p,q) &= \gamma_\mu \tilde{A}^\mu(p-q) + \tilde{v}(p-q) \quad &\text{b)} \end{aligned} \tag{A.5.3}$$

where the tilde denote Fourier transform, and γ_μ denote the Dirac matrices.

Lemma A.5.3. *The mapping* $(v, A_\mu) \to \tilde{A}$ *(resp.* $(v, A_\mu) \to \tilde{\mathcal{A}}$*) from* $\mathcal{S}(\mathbb{R}^4)^{\times 5}$ *to* $\mathcal{N}(m, \alpha, 1/4)$ *(resp. to* $\mathcal{N}_{\mathbb{C}^4}(m, \alpha, 0)$ *) is continuous.*

Proof. Since $\tilde{v}$ and $\tilde{A}_\mu$ belongs to $\mathcal{S}(\mathbb{R}^4)$, [25] we can find for given α and m, a constant C such that for $|\mu| \leqq m$ and $|\nu| \leqq m$:

$$\left| \frac{\partial^\mu}{(\partial p)^\mu} \frac{\partial^\nu}{(\partial q)^\nu} \tilde{A}(p,q) \right| \leqq C \frac{\operatorname{Inf}((1+|q|^2)^{1/2}(1+|p|^2)^{1/2})}{(1+(p^0-q^0)^2]^\alpha [1+(\vec{p}-\vec{q})^2]^\alpha} \tag{A.5.4}$$

noticing that in (A.5.4) a) one can use decompositions of $p_\mu + q_\mu$ as $2p_\mu + (q_\mu - p_\mu)$ or as $2q_\mu + (p_\mu - q_\mu)$, so that we get at most a linear growth in $|p|$ or in $|q|$ due to the fast decrease in $p - q$.

Using the inequality

$$\operatorname{In} f(|a|, |b|) \leqq \sqrt{|a|\,|b|} \tag{A.5.5}$$

the proof of Lemma A.5.3 is complete.

Lemma A.5.4. *If* $\alpha > 3/2$ *and* $K \in \mathcal{N}_E(m, \alpha, 1/4)$, *then* $\hat{K}_{++}$ *and* $\hat{K}_{--}$ *are bounded operators, and* $\hat{K}_{+-}, \hat{K}_{-+}$ *are in the Hilbert-Schmidt class. Moreover* *the mapping*

$$K \to (\hat{K}_{++}, \hat{K}_{--}, \hat{K}_{+-}, \hat{K}_{-+}) \tag{A.5.6}$$

from $\mathcal{N}_E(m, \alpha, 1/4)$ *is linear and continuous.*

Proof. a) Let us first examine $\hat{K}_{+-}$:

$$\begin{aligned} \|\hat{K}_{+-}\|_{\mathrm{H.S.}} &= C^{te} \int |K(p, -q)|^2 \, d\Omega_m(p) \, d\Omega_m(q) \\ &\leqq C^{te} \|K\|^2_{m,\alpha,1/4} \int \frac{d^3\vec{p} d^3\vec{q}}{[1+(\omega_p+\omega_q)^2]^\alpha [1+(\vec{p}-\vec{q})^2]^\alpha} \\ &\leqq C^{te} \|K\|^2_{m,\alpha,1/4} \left(\int \frac{d^3\vec{p}}{(1+\vec{p}^2)^\alpha} \right) \left(\int \frac{d^3\vec{q}}{[1+(\vec{p}-\vec{q})^2]^\alpha} \right). \end{aligned} \tag{A.5.7}$$

Both integrals are equal and converge for $\alpha > 3/2$.

b) By a theorem from Dunford and Schwarz [37], one has the following: Let (S, Σ, μ) be a positive measured space, and R a measurable function on $S \times S$ such that

$$\operatorname*{ess\,sup}_t \int |R(s,t)| \, d\mu(s) \leqq M \qquad \operatorname*{ess\,sup}_s \int |R(s,t)| \, d\mu(t) \leqq M . \tag{A.5.8}$$

Then the operator T defined by

$$\begin{aligned} g &= Tf \\ g(s) &= \int R(s,t) f(t) \, d\mu(t) \end{aligned} \tag{A.5.9}$$

is a bounded operator in $\mathcal{L}^2(S, \Sigma, \mu)$, and we have

$$\|T\| \leqq M . \tag{A.5.10}$$

This result generalizes $\mathcal{L}^2$-space inequalities for convolutions. Here we choose:

$$S = \mathbb{R}^3 \qquad d\mu(\vec{p}) = \frac{d^3\vec{p}}{2\omega_p} \qquad R = \hat{K}_{++}$$

$$\int_{q^p = \omega_p} |\hat{K}_{++}(p,q)| \frac{d^3 q}{2\omega_q} \leqq C^{te} \|K\|_{m,\alpha,1/4} (1+|p|^2)^{1/4} \int (1+(\vec{p}-\vec{q})^2)^{-\alpha} (2\omega_q)^{-1/2} \, d^3\vec{q} \tag{A.5.11}$$

if $\alpha > 3/2$, Eq. (A.5.20) gives

$$\operatorname{ess\,sup} \int |\hat{K}_{++}(p,q)| \, d\Omega_m(q) \leqq C^{te} \|K\|_{m,\alpha,1/4} \operatorname{ess\,sup} \frac{(1+|p|^2)^{1/4}}{(1+\vec{p}^2)^{1/4}} \leqq C^{te} \|K\|_{m,\alpha,1/4} \tag{A.5.12}$$

since on mass-shell

$$1 + |p|^2 = 2(1+\vec{p}^2) . \tag{A.5.13}$$

The proof is complete.

Lemma A.5.5. *If* $\alpha - M > 3/2$ *and* $K \in \mathcal{N}(m, \alpha, M + 1/4)$ *then* $\hat{K}_{+-}$ *and* $\hat{K}_{-+}$ *are in the Hilbert-Schmidt class, and there exists a dense domain* D_+ *in* $\mathfrak{H}_+$ *(resp.* D_- *in* $\mathfrak{H}_-$*) such that* $\hat{K}_{++}$ *(resp.* $\hat{K}_{--}$*) is defined on* D_+ *(resp.* D_-*) with*

$$\hat{K}_{++} D_+ \subset D_+ \qquad (\textit{resp. } \hat{K}_{--} D_- \subset D_-) . \tag{A.5.14}$$

Proof. The proof is the same as before. The domain D can be chosen as the space of $\mathscr{C}^m$ functions of fast decrease on the mass-shell.

Lemma A.5.6. *Let $m \geqq 1$ be an integer, α, β be real positive numbers, M, N real numbers such that $M+N+\frac{\sigma-1}{2} \geqq 0$. Then, there exist a constant C and indices γ and P such that*

$$\|KS_{\text{ex}}L\|_{m,\gamma,P} \leqq C\|K\|_{m,\alpha,M}\ \|L\|_{m,\beta,N}\,. \tag{A.5.15}$$

If there exist $A \geqq$, $B \geqq 0$ with

$$A+B=M+N+\frac{\sigma-1}{2}, \qquad \sup(\alpha-A, \beta-B) > 3/2 \qquad \inf(\alpha-A, \beta-B) \geqq 0\,. \tag{A.5.16}$$

Then

$$\gamma \leqq \operatorname{Inf}(\alpha-A, \beta-B); \qquad P \geqq \operatorname{Sup}(M+A, N+B)\,. \tag{A.5.17}$$

Proof. a) We need some auxiliary results.

i) The following inequalities hold: $\forall\, x \in \mathbb{R}^n, \forall\, y \in \mathbb{R}^n$

$$\begin{aligned} 1+\|x\|^2+\|y\|^2 &\leqq (1+\|x\|^2)(1+\|y\|^2) && \text{a)}\\ (1+\|x+y\|^2) &\leqq 2(1+\|x\|^2)(1+\|y\|^2) && \text{b)}\\ \text{if } \|x\| \geqq 1 \quad \frac{1}{\|x\|} &\leqq \frac{\sqrt{2}}{(1+\|x\|^2)^{1/2}}, && \text{c)} \end{aligned} \tag{A.5.18}$$

ii) If $x, y, z)$ are in $\mathbb{R}^n$, and $\theta \geqq 0$ the following inequality holds [38]

$$\begin{aligned} &(1+\|x-y\|^2)^{-\theta}(1+\|y-z\|^2)^{-\theta}\\ &\quad \leqq C_\theta(1+\|x-z\|^2)^{-\theta}\{(1+\|x-y\|^2)^{-\theta}+(1+\|z-y\|^2)^{-\theta}\}\,. \end{aligned} \tag{A.5.19}$$

Indeed, $\forall\, \theta$, $\exists K_\theta$ such that

$$(|a|+|b|)^\theta \leqq K_\theta(|a|^\theta+|b|^\theta) \qquad \forall\, a\, \forall\, b \in \mathbb{R}\,.$$

And by

$$(1+\|x-y\|^2) \leqq 1+2(\|x-y\|^2+\|y-z\|^2) \leqq 2[(1+\|x-y\|^2)+(1+\|y-z\|^2)]$$

we obtain easily (A.5.19).

iii) If x, z, and y are in $\mathbb{R}^n$, $\alpha \geqq 0$, $\beta \geqq 0$ then if $\sup(\alpha, \beta) > \frac{n}{2}$

$$\int \frac{d^n y}{(1+\|x-y\|^2)^\alpha(1+\|y-z\|^2)^\beta} \leqq \frac{C}{(1+\|x-z\|^2)\operatorname{Inf}(\alpha, \beta)} \tag{A.5.20}$$

as can be seen by application of (A.5.19) and of the Holder inequality.

b) We can without loss of generality, choose the particle mass m to be unity

i) Up to a multiplicative constant, we have in momentum space

$$\widetilde{KS_+L}(p,q) = c^{te} \int_{r^0=\omega_r} \frac{\tilde{K}(p,r)\,P(r)\,\tilde{L}(r,q)\,d^3\vec{r}}{(1+\vec{r}^2)^{1/2}} \qquad \omega_r = (1+\vec{r}^2)^{1/2}\,.$$

This expression can be majorised by

$$\left\|\frac{\partial^\mu}{\partial p^\mu}\frac{\partial^\nu}{\partial q^\nu}KS_+L\right\| \leqq c^{te}\|K\|_{m,\alpha,M}\|N\|_{m,\beta,N}(1+|p|^2)^M(1+|q|^2)^N$$
$$\ldots\int\frac{(1+|r|^2)^{M+N+\frac{\sigma-1}{2}}d^3\vec{r}}{(1+(p^0-\omega_r)^2)^\alpha(1+(q^0-\omega_r)^2)^\alpha(1+(\vec{p}-\vec{r})^2)^\alpha(1+(\vec{r}-\vec{q})^2)^\alpha} \tag{A.5.21}$$

Because $r^0=\omega_r\Rightarrow 1+|r|^2=2(1+\vec{r}^2)$.

Let us apply A.5.19, and remark that (A.5.18 b)

$$\frac{1+\vec{r}^2}{1+(\vec{p}-\vec{r})^2}\leqq 2(1+\vec{p}^2). \tag{A.5.22}$$

Then, if $A\geqq 0$, $B\geqq 0$ satisfy (A.5.16), and γ and P are defined by (A.5.17) we find, with (A.5.20) the result for $S_{ex}=S_+$.

ii) Let us prove the lemma for $S_{ex}=S_r$.

Up to a constant, we have

$$\tilde{S}_r(r)=\frac{P(r)}{(r^0-i0)^2-\omega_r^2}=\frac{P(r)}{\omega_r^1}\left[\frac{1}{r^0-i0-\omega_r}-\frac{1}{r^0-i0+\omega_r}\right]. \tag{A.5.23}$$

Thus, KS_rL can be split into two parts which can both be majorised in similar ways.

For instance

$$\int\frac{\tilde{K}(p,r)P(r)\tilde{L}(r,q)d^4r}{2\omega_r(r^0-i0-\omega_r)}=\int_{|r^0-\omega|\geqq 1}\cdots+\int_{|r^0-\omega|\leqq 1}. \tag{A.5.24}$$

We can immediatly majorise the second term of (A.5.24)

$$\int_{|r^0-\omega_r|\leqq 1}\cdots=\int_{|r^0-\omega_r|\leqq 1}\frac{d^4r}{2\omega_r}\frac{\widetilde{KPL}(p,q,r)-\widetilde{KPL}(p,q,r)|_{r^0=\omega_r}}{r^0-\omega_r-i0}$$
$$+\int\frac{d^3\vec{r}}{2\omega_r}\widetilde{KPL}(p,q,r)_{r^0=\omega_r}\int_{\omega-1}^{\omega+1}\frac{dr_0}{r^0-\omega_r+i0}. \tag{A.5.25}$$

But

$$\int_{-1}^{+1}\frac{dx}{x-i0}=-i\pi \tag{A.5.26}$$

and the proof i) for $S_{ex}=S_r$ is sufficient to obtain the good majorisation for the second term of (A.5.25).

The first term of A.5.25 is majorised in the same way because

$$\left\|\frac{\widetilde{KPL}(p,q,r)-\widetilde{KPL}|_{r^0=\omega_r}}{r^0-\omega_r-i0}\right\|\leqq\sup_{|r^0-\omega_r|\leqq 1}\left\|\frac{\partial}{\partial r_0'}\widetilde{KPL}(p,q,r')\right\|. \tag{A.5.27}$$

It is at this point that we need the derivability of $\tilde{K}$ and $\tilde{L}(m\geqq 1)$. If we remark that

$$(1+|r|^2)\leqq C^{te}(1+\vec{r}^2)\quad\text{if}\quad |r^0-\omega_r|\leqq 1 \tag{A.5.28}$$

and $\int_{|r^0-\omega_r|\leqq 1}dr^0=2$ we find the same majorisation as in i).

iii) Now we have to majorise $\int\limits_{|r^0-\omega_r|\geqq 1} \cdots$ in (A.5.24)

$$\left\|\frac{\partial^\mu}{\partial p^\mu}\frac{\partial^\nu}{\partial q^\nu}\int\limits_{|r^0-\omega_r|\geqq 1}\right\| \leqq C^{te}\|K\|_{m,\alpha,M}\,\|L\|_{m,\beta,N}(1+|p|^2)^M(1+|q|^2)^N$$

$$\cdots\int\frac{(1+|r|^2)^{M+N+\frac{\sigma-1}{2}}(1+|p-r|^2)^{-A}(1+|q-r|^2)^{-B}}{(1+(p^0-r^0)^2)^{\alpha-A}[1+(\vec{p}-\vec{r})^2]^{\alpha-A}[1+(q^2-r^0)^2]^{\beta-B}[1+(\vec{q}-\vec{r})^2]^{\beta-B}}. \tag{A.5.29}$$

In this expression we have used for $|r^0-\omega_r|\geqq 1$ [by (A.5.18) c) and b)]

$$\begin{aligned}\frac{(1+|r|^2)^{1/2}}{(1+\vec{r}^2)\,|r^0-\omega_r-io|} &\leqq \frac{\sqrt{2}(1+|r|^2)^{1/2}}{(1+\vec{r}^2)(1+(r^0-\omega_r)^2)^{1/2}}\\ &\leqq \frac{C^{te}(1+|r|^2)^{1/2}}{(1+r^0)^{1/2}(1+(\vec{r}-|r_0|)^2)^{1/2}}\leqq C^{te}.\end{aligned} \tag{A.5.30}$$

Using again (A.5.22, A.5.20) we find the expected result.

iv) If S_{ex} is any other two point function, it can be expressed in term of S_r and S_+. Therefore the lemma is shown.

Corollary A.5.7. *Let $m\geqq 1$ be an integer, α,β be real positive numbers with $\alpha\leqq\beta$. Then:*

$$\begin{aligned}\|K\Delta_{\text{ex}}L\|_{m,\alpha,1/4} &\leqq C^{te}\|K\|_{m,\alpha,1/4}\,\|L\|_{m,\beta,1/4} \quad \text{a)}\\ \|KS_{\text{ex}}L\|_{m,\alpha,0} &\leqq C^{te}\|K\|_{m,\alpha,0}\,\|L\|_{m,\beta,0} \quad \text{b)}\end{aligned} \tag{A.5.31}$$

where

$$S_{\text{ex}}=(-i\gamma_\mu\partial^\mu+m)\,\Delta_{\text{ex}}\,.$$

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