

Analyticity Properties of the Correlation Functions for the Anisotropic Heisenberg Model

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Abstract. It is shown for the Heisenberg model that the correlation functions are analytic in h and T if $\operatorname{Re}(h) \neq 0$ and T is positive.

Introduction

The analyticity properties of the Ising model, when there is no phase transition, were established by Lee and Yang [6, 11] and by Lebowitz and Penrose [5]. The theorem of Lee-Yang about the zeros of the partition function of the system plays a prominent part in these papers. The generalization of this famous theorem to the case of the Heisenberg model was made by Asano [1] and Suzuki-Fisher [10]. With the help of this generalization we obtain analogous results as those obtained by Lebowitz and Penrose for the Ising model: the correlation functions are analytic in h and T if $\operatorname{Re}(h) \neq 0$ and T is positive. The proof follows closely that of Lebowitz and Penrose. We use essentially the theorem of Lee-Yang and the technique introduced by Asano [1]. Our proof is only valid if the total magnetization commutes with the Hamiltonian, and does not extend to the general case considered by Suzuki and Fisher [10].

Notation and Definition of the Model

The model is defined on the lattice \mathbb{Z}^{ν} . With each point of the lattice we associate a spin $-1/2$, which we describe by a Hilbert space \mathcal{H}_i isomorphic to \mathbb{C}^2 , and by the Pauli matrices σ_i^x , σ_i^y , σ_i^z . We consider first a system restricted to a finite subset A of \mathbb{Z}^{ν} . The corresponding Hilbert space is $\mathcal{H}_A = \bigotimes_{i \in A} \mathcal{H}_i$ and we choose the Hamiltonian as follows:

$$H_A = - \sum_{\substack{i \neq j \\ i, j \in A}} H(i, j) + h \sum_{i \in A} (\sigma_i^z + 1) \quad (1)$$

with

$$H(i, j) = K(i - j) (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) + J(i - j) \sigma_i^z \sigma_j^z. \quad (2)$$

In this formula $H(i, j)$ describes an interaction between two spins. The interaction will be a ferromagnetic one:

$$J(x) = J(-x) \geq 0, \quad K(x) = g(x) J(x) \quad (3a)$$

with

$$-1 \leq g(x) = g(-x) \leq +1. \quad (3b)$$

The function g allows us to introduce some anisotropy in the coupling between the spins. It is, however, very important for the rest of the paper that (4) holds:

$$\left[\sum_{i \in A} \sigma_i^z, H \right] = 0. \quad (4)$$

The constant h may be interpreted as a magnetic field. We impose also two conditions on the decrease of the interactions K and J for large distances:

$$\sup_{\substack{|s| > r > 0 \\ s \in \mathbb{Z}^v}} r^v J(s) = u(r) \rightarrow 0, \quad r \rightarrow \infty, \quad (5a)$$

$$\sum_{x \neq 0} J(x) < \infty, \quad \sum_{x \neq 0} |K(x)| < \infty. \quad (5b)$$

Such conditions ensure the existence of the thermodynamic limit of the correlation functions [4, 7]. When we take the thermodynamic limit, this means that we choose a sequence of finite subsets of \mathbb{Z}^v , $(A_n)_{n \in \mathbb{N}}$ such that $A_n \subset A_{n+1}$ for all n and for every finite subset A of \mathbb{Z}^v there is a number $N(A)$ with the property that A is contained in A_p for all $p > N(A)$.

We denote the partition function by

$$P(h, T, A) = \text{Tr}_{\mathcal{H}_A} \exp(-\beta H_A), \quad \beta = (kT)^{-1}, \quad (6)$$

and the correlation functions by

$$\langle \sigma_{x_1}^{i_1} \dots \sigma_{x_m}^{i_m} \rangle (h, T, A) = \text{Tr}_{\mathcal{H}_A} (\sigma_{x_1}^{i_1} \dots \sigma_{x_m}^{i_m} \exp(-\beta H_A)) / P(h, T, A), \quad (7)$$

where x_1, \dots, x_m are m sites of A and $i_j = x, y$ or z , $1 \leq j \leq m$.

Remark. All finite subsets A which we shall consider have the property: If x and y are two points of A , then there exists a set of points $\{x_0, \dots, x_n\}$ with $x_i \in A$, $x_0 = x$, $x_n = y$ and where the spins at x_i and x_{i+1} interact with one another.

Results and Proofs

If A is a finite set, H_A is a matrix and we may without difficulty consider complex values of h . Asano showed under the hypothesis 3 a) and 3 b) that $P(h, T, A) \neq 0$ if $\text{Re}(h) \neq 0$. On the other hand it is easy to see that $P(0, T, A) \neq 0$. We may thus define $\langle \sigma_{x_1}^{i_1} \dots \sigma_{x_m}^{i_m} \rangle (h, T, A)$ if $\text{Re}(h) \neq 0$ or h real. Our first result is

Theorem I. *For the model defined above [in particular if (3a), (3b), (5a), and (5b) hold].*

1) *If T is a positive fixed number*

$$\lim_{A \rightarrow \infty} \langle \sigma_{x_1}^{i_1} \dots \sigma_{x_m}^{i_m} \rangle (h, T, A) = \langle \sigma_{x_1}^{i_1} \dots \sigma_{x_m}^{i_m} \rangle (h, T) \quad (8)$$

converges locally uniformly in h , both when $\text{Re}(h) > 0$ and when $\text{Re}(h) < 0$.

2) *The function $\langle \sigma_{x_1}^{i_1} \dots \sigma_{x_m}^{i_m} \rangle (h, T)$ is analytic in h and T in the region $\text{Re}(h) > 0$ (and $\text{Re}(h) < 0$) and T in a complex neighbourhood of the positive real axis.*

Remark. If we introduce the variable $z = e^{-\beta h}$, then the domains of analyticity for the new variable become $|z| < 1$ and $|z| > 1$.

Proof. We shall use the following result, which we shall prove later.

Lemma 1. *Let A be a finite subset of \mathbb{Z}^v . Then the inequality below is uniform in A*

$$|\langle \sigma_{x_1}^{i_1} \dots \sigma_{x_m}^{i_m} \rangle (z, T, A)| \leq \left(\frac{1+r}{1-r} \right)^{2m}, \quad \text{if } |z| \leq r < 1, r \text{ fixed.} \quad (9)$$

This lemma means that the family of analytic functions $\langle \sigma_{x_1}^{i_1} \dots \sigma_{x_m}^{i_m} \rangle (z, T, A_p)$ with $p \in \mathbb{N}$ is a normal family [2] on the unit open disc $E = \{z \mid |z| < 1\}$. On the other hand it has been proved by Ginibre [4] that for every finite interval I of the positive real axis there exists a complex neighbourhood U of $z=0$ such that

$$\lim_{p \rightarrow \infty} \langle \sigma_{x_1}^{i_1} \dots \sigma_{x_m}^{i_m} \rangle (z, T, A_p)$$

converges and defines an analytic function of z and T if $z \in U$ and $T \in I$. (The Hamiltonian in [4] is slightly different, but the proof of the above statement with our Hamiltonian is the same except for minor changes. The difference between the two Hamiltonians is a boundary term for the finite systems.) We apply then the theorem of Vitali [3] and we obtain the first result, the second one follows directly from the theorem of [5, p. 104]. Details of the proof may be found in the paper of Lebowitz and Penrose [5]. We obtain the same results in the case where $|z| > 1$ using the symmetry of the model (see e.g. next section).

Let us suppose that there exists an arc γ of the circumference of the unit circle $\{z \mid |z| = 1\}$, on which, if p is sufficiently large, $P(z, T, A_p)$ is non-zero. Without loss of generality we consider the case where γ is given by the inequalities $-\varphi < \arg(z) < +\varphi$ with $0 < \varphi < \pi$. Under such assumptions the free energy is analytic in z if $z \in \gamma$ in the thermodynamic limit. We extend this result to the correlation functions.

We consider the situation just described and we denote by $\delta\Gamma$ the circle which passes through the points $e^{i\varphi}$ and $e^{-i\varphi}$ and which is orthogonal to the unit circle. The open set which contains the point 1 and which has the boundary $\delta\Gamma$ is denoted by Γ . Then we obtain:

Theorem II. *The first conclusion of Theorem I is true if we replace the unit disc E by Γ . The function $\langle \sigma_{x_1}^{i_1} \dots \sigma_{x_m}^{i_m} \rangle (z, T)$ coincides with that of Theorem I if $|z| \neq 1$.*

Proof. If we use the notation of Theorem I we can prove

Lemma 2. *The family $\langle \sigma_{x_1}^{i_1} \dots \sigma_{x_m}^{i_m} \rangle (z, T, A_p)$, $p \in \mathbb{N}$, is normal in Γ when p is large enough, i.e. when $P(z, T, A_p) \neq 0$ if $z \in \gamma$.*

With this result the proof follows as before.

Proof of the Lemmas 1 and 2

Introduction

We discuss here the technique used to prove Lemmas 1 and 2. In these proofs the position of the zeros of some polynomials, and also their symmetry properties, take a prominent part. In this section we define these polynomials and give some of their properties. We prove the lemmas in the next section.

We consider polynomials of N complex variables. It is convenient to use the notation of Ruelle [8]. We introduce a finite set Ω of N elements denoted by letters x, y, \dots . We associate a complex variable z_x to each x of Ω . If X is a subset of Ω , then z_X is the set of variables $\{z_x | x \in X\}$ and $z^X = \prod_{x \in X} z_x$. We put $z^\emptyset \equiv 1$. Let $\mathcal{E}(\Omega)$ be the set of all complex-valued functions f defined on the subsets of Ω .

$$f : X \subset \Omega \mapsto f(X) \in \mathbb{C}.$$

Let $\mathcal{P}(\Omega)$ be the set of all complex polynomials with N complex variables, linear in each z_x . Clearly there is a one-to-one correspondence between $\mathcal{E}(\Omega)$ and $\mathcal{P}(\Omega)$:

$$f \in \mathcal{E}(\Omega) \mapsto P_f(z_\Omega) = \sum_{X \subset \Omega} f(X) z^X \in \mathcal{P}(\Omega).$$

On the set $\mathcal{E}(\Omega)$ we define the transformation $D(x, y)$ when $x \neq y$:

$$(D(x, y) f)(X) = \begin{cases} 0 & \text{if } x \in X \text{ and } y \notin X \\ 0 & \text{if } x \notin X \text{ and } y \in X \\ f(X) & \text{otherwise.} \end{cases} \tag{10}$$

The corresponding transformation on $\mathcal{P}(\Omega)$ is

$$(D(z_x, z_y) P_f)(z_\Omega) = P_{D(x, y) f}(z_\Omega). \tag{11}$$

This last operation is not exactly the contraction of Asano [1, 8, 9]. If we write explicitly only the variables z_x and z_y , then we obtain

$$P_f(z_\Omega) = az_x z_y + bz_x + cz_y + d \xrightarrow{D(z_x, z_y)} az_x z_y + d. \tag{12}$$

The class $L(\Omega)$ of polynomials, which interest us, consists of all polynomials of $\mathcal{P}(\Omega)$, which satisfy the property E

$$E: P_f(z_\Omega) = 0 \text{ and } |z_x| \leq 1, \forall x \in \Omega \text{ implies } |z_x| = 1, \forall x \in \Omega. \tag{13}$$

as also a symmetry property S :

$$S: f(X) = f(\Omega - X)^*, \quad \forall X \subset \Omega, \tag{14}$$

where $*$ represents complex conjugation.

Remarks. The property E is equivalent to:

$$\text{if } P_f(z_\Omega) = 0 \text{ and } |z_y| \leq 1 \forall y \in \Omega - \{x\} \text{ and if } \exists y' \in \Omega - \{x\} \tag{15}$$

such that $|z_{y'}| < 1$, then z_x is such that $|z_x| > 1$.

The property S means:

$$(P_f(z_\Omega))^* = \sum_{X \subset \Omega} f(\Omega - X)^* z^{*\Omega - X} = z^{*\Omega} P_f((z^*)_\Omega^{-1}). \tag{16}$$

Proposition. *The class $L(\Omega)$ is stable under the transformation $D(z_x, z_y)$.*

Proof. The property S is evidently conserved [see (10)]. The property E is also conserved (Proposition 3.3, [8]).

We note also two simple facts:

I) If $P_f(z_\Omega) \in \mathcal{P}(\Omega)$, we associate to P_f a polynomial Q_f in one complex variable z by setting

$$Q_f(z) = P_f(z, \dots, z) \equiv a_0 \prod_{i=1}^N (q_i - z), \quad N = |\Omega|.$$

We notice immediately that

$$Q_{D(x,y)f}(z) = a_0 \prod_{i=1}^N (\hat{q}_i - z).$$

II) If $P_f(z_\Omega) \in L(\Omega)$ and if we associate to each $x \in \Omega$ a complex number ω_x of unit modulus, then we can introduce new variables $\bar{z}_x = \omega_x z_x$ and define $P_f(\bar{z}_\Omega) \equiv P_{\bar{f}}(z_\Omega)$. Then

$$Q_{\bar{f}}(z) = a_0 \left(\prod_{x \in \Omega} \omega_x \right) \prod_{i=1}^N (\bar{q}_i - z) \equiv \bar{a}_0 \prod_{i=1}^N (\bar{q}_i - z)$$

and

$$\bar{f}(X) = f(X) \prod_{y \in X} \omega_y. \quad (17)$$

In particular $P_f(z_\Omega)$ satisfies the property E and the symmetry property S_ω :

$$\bar{f}(X) = \omega \bar{f}(\Omega - X)^*, \quad \omega = \prod_{x \in \Omega} \omega_x. \quad (18)$$

Proof of the Lemmas 1 and 2

Lemma 1. We consider the explicit case $i_1 = x$ and $i_2 = y$; the generalization to other correlation functions is immediate. The fact that Λ is a subset of \mathbb{Z}^v is unimportant. Therefore Λ is here a set with N elements $\{1, \dots, N\}$ and we write 1 respectively 2 instead of x_1 respectively x_2 etc. We must show

$$\sup_{|z| \leq r < 1} |\langle \sigma_1^x \sigma_2^y \rangle(z, T, \Lambda)| \leq \left(\frac{1+r}{1-r} \right)^4. \quad (19)$$

The proof consists of expressing $\langle \sigma_1^x \sigma_2^y \rangle(z, T, \Lambda)$ as a sum of four terms; each term is a quotient of polynomials, which possess the properties described in the last section. Then we use Remarks I and II in order to obtain the desired result.

A. Definition of Four Polynomials

In \mathcal{H}_Λ we introduce the vectors $|\{s_j\}\rangle = |s_1\rangle \otimes \dots \otimes |s_N\rangle$ defined by $\sigma_j^z |s_j\rangle = s_j |s_j\rangle$ with $s_j = \pm 1$. These vectors form a basis in \mathcal{H}_Λ and we index them by the subsets of Λ :

$$|\{s_j\}\rangle = |X\rangle : i \in X \Leftrightarrow s_i = +1. \quad (20)$$

The partition function becomes

$$P(z, T, \Lambda) = \sum_{X \subset \Lambda} \langle X | \exp(-\beta H_\Lambda) | X \rangle, \quad \beta = (kT)^{-1}, \quad z = e^{-\beta h}. \quad (21)$$

The operator $M = \sum_{i \in A} (\sigma_i^z + 1)$ commutes with $H_A \equiv H_0 + hM$. Therefore

$$P(z, T, A) = \sum_{X \subset A} \left\langle X \left| \exp\left(\frac{-\beta h M}{2}\right) \exp(-\beta H_0) \exp\left(\frac{-\beta h M}{2}\right) \right| X \right\rangle \quad (22)$$

is a polynomial in z . Trotter's formula allows us to write

$$\exp(-\beta H_0) = \lim_{n \rightarrow \infty} \left(\prod_{i \neq j} \exp\left(\frac{\beta}{n} H(i, j)\right) \right)^n \equiv \lim_{n \rightarrow \infty} A_n \quad (23)$$

with $H(i, j)$ given by (2). We make now the connection with the previous section: Let A and A' be two copies of the set $\{1, \dots, N\}$. We distinguish the elements or the subsets of A' by $'$. We define then Ω as the disjoint union of A and A' and we write the subsets of Ω by the pairs (X, Y') with $X \subset A$ and $Y' \subset A'$. Let f_n be the function of $\mathcal{E}(\Omega)$ defined by

$$f_n(X, Y') = \langle X | A_n | Y' \rangle, \quad (24)$$

and the corresponding polynomial

$$P_{f_n}(z_\Omega) = P_{f_n}(z_A, z_{A'}) = \sum_{\substack{X \subset A \\ Y' \subset A'}} z^X f_n(X, Y') z^{Y'} \quad (25)$$

we first define $g_n \in \mathcal{E}(\Omega)$:

$$g_n = D(3, 3') \dots D(N, N') f_n; \quad (26)$$

then we construct two polynomials in four variables $z_1, z_{1'}, z_2, z_{2'}$, and one complex parameter w :

$$P_n^1(z_1, z_2, z_{1'}, z_{2'}; w) = D(z_1, z_{1'}) D(z_2, z_{2'}) P_{g_n}(z_A, z_{A'}) \quad (27a)$$

and we put $z_3 = z_{3'} = \dots = z_N = z_{N'} = w$,

$$P_n^2(z_1, z_2, z_{1'}, z_{2'}; w) = P_{g_n}(z_A, z_{A'}) \quad (27b)$$

and we put $z_3 = z_{3'} = \dots = z_N = z_{N'} = w$.

Finally we introduce

$$Q_n^1(z; w) = P_n^1(z, z, z, z; w) = a_0(w) \prod_{i=1}^4 (\hat{q}_i(w) - z), \quad (28a)$$

$$Q_n^2(z; w) = P_n^2(z, z, z, z; w) = a_0(w) \prod_{i=1}^4 (q_i(w) - z), \quad (28b)$$

(cf. Remark I).

B. Relation between the Polynomials (27a), (27b), (28a), and (28b), and $\langle \sigma_1^x \sigma_2^z \rangle(z, T, A)$

We see immediately, comparing (22) and (28b), that

$$P(z, T, A) = \lim_{n \rightarrow \infty} Q_n^1(z; z), \quad (29)$$

because

$$\prod_{i=1}^N D(i, i') f_n(X, Y') = \begin{cases} 0 & \text{if } X \neq Y \\ f_n(X, X) & \text{if } X = Y. \end{cases} \quad (30)$$

On the other hand we compute the expression

$$\begin{aligned} & \text{Tr}_{\mathcal{H}_A} \left[\sigma_1^x \sigma_2^y \exp\left(\frac{-\beta h M}{2}\right) \exp(-\beta H_0) \exp\left(\frac{-\beta h M}{2}\right) \right] \\ &= \lim_{n \rightarrow \infty} \text{Tr}_{\mathcal{H}_A} \left[\sigma_1^x \sigma_2^y \exp\left(\frac{-\beta h M}{2}\right) A_n \exp\left(\frac{-\beta h M}{2}\right) \right] \end{aligned} \quad (31)$$

with the aid of the basis

$$\begin{aligned} |\{s_j\}\rangle &= |s_1\rangle^x \otimes |s_2\rangle^y \otimes |s_3\rangle \otimes \cdots \otimes |s_N\rangle \\ &= |s_1, s_2, X\rangle \quad \text{with} \quad X \subset \tilde{A} = A - \{1, 2\}. \end{aligned} \quad (32)$$

We have used the following vectors

$$\begin{aligned} |s_1\rangle^x &= \frac{1}{\sqrt{2}} (s_1 |1\rangle + |-1\rangle), \quad s_1 = \pm 1 \\ |s_2\rangle^y &= \frac{1}{\sqrt{2}} (-i s_2 |1\rangle + |-1\rangle), \quad s_2 = \pm 1, \end{aligned} \quad (33)$$

which satisfy

$$\sigma_1^x |s_1\rangle^x = s_1 |s_1\rangle^x, \quad \sigma_2^y |s_2\rangle^y = s_2 |s_2\rangle^y. \quad (34)$$

Hence we obtain

$$\begin{aligned} & \text{Tr}_{\mathcal{H}_A} \left[\sigma_1^x \sigma_2^y \exp\left(\frac{-\beta h M}{2}\right) A_n \exp\left(\frac{-\beta h M}{2}\right) \right] \\ &= \sum_{X \subset \tilde{A}} (1, 1, X | \dots | 1, 1, X) - \sum_{X \subset \tilde{A}} (-1, 1, X | \dots | -1, 1, X) \\ &\quad - \sum_{X \subset \tilde{A}} (1, -1, X | \dots | 1, -1, X) + \sum_{X \subset \tilde{A}} (-1, -1, X | \dots | -1, -1, X) \end{aligned} \quad (35)$$

[... represents the expression between the square brackets on the left hand side of (35)].

Using (33) and the definition of P_n^2 we see that

$$4 \sum_{X \subset \tilde{A}} (1, 1, X | \dots | 1, 1, X) = P_n^2(z, -iz, z, -iz; z) \quad (36)$$

and the three other terms in (35) have analogous expressions.

C. Estimation of $\langle \sigma_1^x \sigma_2^y \rangle (z, T, A)$

We use now the result proved by Asano [1]:

$$P_{f_n}(z_\Omega) \in L(\Omega).$$

We thus obtain using Remarks I and II

$$P_n^2(z, -iz, z, -iz; w) = \bar{a}_0(w) \prod_{i=1}^4 (\bar{q}_i(w) - z). \quad (37)$$

Hence the n^{th} approximation of $\langle \sigma_1^x \sigma_2^y \rangle (z, T, A)$ is the sum of four terms of the following type:

$$\frac{1}{4} \frac{\bar{a}_0(z)}{a_0(z)} \prod_{i=1}^4 \frac{\bar{q}_i(z) - z}{\hat{q}_i(z) - z}. \tag{38}$$

If $|w| \leq 1$, then $|\bar{q}_i(w)| \geq 1$, $|\hat{q}_i(w)| \geq 1$ and if $|w| = 1$, then $|\bar{q}_i(w)| = |\hat{q}_i(w)| = 1$.
 Consequently

$$\begin{aligned} & \sup_{|z| \leq r < 1} \frac{1}{4} \left| \frac{\bar{a}_0(z)}{a_0(z)} \right| \left| \prod_{i=1}^4 \frac{\bar{q}_i(z) - z}{\hat{q}_i(z) - z} \right| \\ & \leq \sup_{\substack{|z| \leq r < 1 \\ |w|=1}} \frac{1}{4} \prod_{i=1}^4 \left| \frac{\bar{q}_i(w) - z}{\hat{q}_i(w) - z} \right| \leq \frac{1}{4} \left(\frac{1+r}{1-r} \right)^4. \end{aligned} \tag{39}$$

This last expression does not depend on n and A . Hence the lemma is proved.

Lemma 2. We use the same notation as before. Let P_1 and P_2 be two interior points of the arc γ and k be the closed disc whose boundary is the circle passing through P_1 and P_2 and orthogonal to the unit circle. We introduce two subsets of k : $k_1 = \{z \in k \mid |z| \geq 1\}$ and $k_2 = \{z \in k \mid |z| \leq 1\}$. By the transformation $z \mapsto (z^*)^{-1}$ we have $k_1 \mapsto k_2$. Let us take now any interior compact subset k' of k and we put $d(k', \delta k) \equiv d > 0$, where $d(k', \delta k)$ is the distance between k' and the boundary δk of k . Let us consider $\langle \sigma_1^x \sigma_2^y \rangle (z, T, A_p)$. By assumption it is possible to find an integer $N(p, k)$ such that if $n > N(p, k)$ $Q_n^1(z; z) \neq 0$ for $z \in k \cap \gamma$. On the other hand $P_n^1(z_A, z_A) \neq 0$ if all $|z_i| < 1$ or all $|z_i| > 1$. We may apply the proposition p. 268 of [8] and hence we find

$$P_n^1(z_1, z_2, z_{1'}, z_{2'}; w) \neq 0 \quad \text{if } z_1, z_2, z_{1'}, z_{2'}, \text{ and } w \in k. \tag{40}$$

In particular $Q_n^1(z; w) \neq 0$ if z and $w \in k$ therefore

$$\min_{\substack{z \in k' \\ w \in k}} |\hat{q}_i(w) - z| \geq d(k', \delta k) = d. \tag{41}$$

This estimation is independent of p and n . If $w \in k_1$, we know that $|\bar{q}_i(w)| \leq 1$; hence

$$\max_{\substack{w \in k_1 \\ z \in k'}} |\bar{q}_i(w) - z| \leq \phi(\overline{E \cup k_1}) \equiv \delta, \tag{42}$$

where $\phi(\overline{E \cup k_1})$ is the diameter of $\overline{E \cup k_1}$.

Finally we obtain

$$\sup_{z \in k' \cap k_1} |\langle \sigma_1^x \sigma_2^y \rangle (z, T, A_p)| \leq 4 \left(\frac{\delta}{d} \right)^4. \tag{43}$$

We extend the validity of this last estimate using the symmetry properties. The symmetry S of $P_n^1(z_1, z_2, z_{1'}, z_{2'}; w)$ allows us to write

$$\begin{aligned} & \left(a_0(w) \prod_{i=1}^4 (\hat{q}_i(w) - z) \right)^* \\ & = (w^*)^{2N-4} (z^*)^4 a_0(w^{*-1}) \prod_{i=1}^4 (\hat{q}_i(w^{*-1}) - z^{*-1}) \end{aligned} \tag{44}$$

and the symmetry S_{-1} of $P_n^2(z_1, -iz_2, z_1', -iz_2'; w)$

$$\begin{aligned} & \left(\bar{a}_0(w) \prod_{i=1}^4 (\bar{q}_i(w) - z) \right)^* \\ &= -1(w^*)^{2N-4}(z^*)^4 \bar{a}_0(w^{*-1}) \prod_{i=1}^4 (\bar{q}_i(w^{*-1}) - z^{*-1}). \end{aligned} \quad (45)$$

Hence (43) is valid for all z in k' and the lemma is proved.

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References

1. Asano, T.: J. Phys. Soc. Japan **29**, 350 (1970)
2. Ahlfors, L. V.: Complex analysis. New York: McGraw-Hill 1966
3. Behnke, H., Sommer, F.: Theorie der analytischen Funktionen einer komplexen Veränderlichen. Berlin-Heidelberg-New York: Springer 1965
4. Ginibre, J.: Commun. math. Phys. **10**, 140 (1968)
5. Lebowitz, J. L., Penrose, P.: Commun. math. Phys. **11**, 99 (1968)
6. Lee, T. D., Yang, C. N.: Phys. Rev. **87**, 404 (1952)
7. Ruelle, D.: Statistical mechanics. New York: Benjamin 1969
8. Ruelle, D.: Commun. math. Phys. **31**, 265 (1973)
9. Ruelle, D.: Phys. Rev. Letters **26**, 303 (1971)
10. Suzuki, M., Fisher, M. E.: J. Math. Phys. **12**, 235 (1971)
11. Yang, C. N., Lee, T. D.: Phys. Rev. **87**, 410 (1952)

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