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Conformal Groups and Conformally Equivalent Isometry Groups

L. Defrise-Carter

Girton College, Cambridge, England

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Abstract. It is shown that if an n dimensional Riemannian or pseudo-Riemannian manifold admits a proper conformal scalar, every (local) conformal group is conformally isometric, and that if it admits a proper conformal gradient every (local) conformal group is conformally homothetic. In the Riemannian case there is always a conformal scalar unless the metric is conformally Euclidean. In the case of a Lorentzian 4-manifold it is proved that the only metrics with no conformal scalars (and hence the only ones admitting a (local) conformal group not conformally isometric) are either conformal to the plane wave metric with parallel rays or conformally Minkowskian.

§ 1. Introduction

Yano (1955) has shown that in an n dimensional Riemannian or pseudo-Riemannian manifold (M_n, g) every r dimensional Lie group C_r of local conformal transformations that is *simply transitive* is conformally isometric (§ 3). More recently it has been proved for $n \geqq 4$ (Defrise, 1969; Suguri and Ueno, 1972) that for any manifold M_n with a *positive definite* metric tensor $g_{\mu\nu}$, every (local) conformal group C_r is conformally isometric (except in the case when the metric is conformally flat).

In this paper we present a more general result. We show that if a space (M_n, g) admits a *proper conformal scalar* (in the sense of du Plessis (1969) as described in § 4) every (local) conformal group C_r is conformally isometric; if a space (M_n, g) admits a *proper conformal gradient* then every C_r is conformally homothetic (§ 4).

This suggests the conjecture that the converses of these two theorems are true, i.e. that: a space (M_n, g) with no *proper conformal scalar* admits a conformal group C_r that is not conformally isometric; and that a space with *no proper conformal gradient* admits a proper conformal group (i.e. one that is not conformally homothetic).

These conjectures are easily verified for a positive definite metric by using the theorem (proved by Taub 1951) on the order of the conformal group C_r admitted by conformally flat spaces (§ 4).

These conjectures are also true in the physically interesting case of a Lorentzian manifold. We prove in particular (§ 6) that a Lorentzian 4-manifold with no proper conformal scalar is conformally equivalent to the *plane-wave metric* with *parallel rays* or else conformally *minkowskian.*

The plane-wave metric admits a proper conformal gradient and the conformal group is reduced to a *proper homothety group* H_6 or H_7.

In the case of a conformally flat space there is a *proper* conformal group C_{15}.

The preceding results imply that in any other Lorentzian manifold the conformal group C_r is *conformally isometric* [as was shown independently by Bilialov (1963)].

Some cosmological applications of these results have been discussed recently by Eardley (1974).

It is to be understood throughout this paper that the transformations under discussion are only required to be well behaved locally and that the words "group" or "transformation" will stand for "local group" or "local transformation".

§ 2. Conformal, Homothetic and Isometric Groups

We recall (see Schouten, 1954; Yano, 1955; Eisenhart, 1961) that a transformation $T: M_n \to M_n$ acting on (M_n, g) is *conformal* if the corresponding pull-back maps g into $e^{\sigma_T(x)}g$. If $\sigma_T(x)$ is constant T is called *homothetic* and if σ_T is zero T is called *isometric*. In the sequel we shall consider Lie groups C_r of such transformations and their associated Lie algebras of infinitesimal transformations.

If the vector fields $\underset{a}{\xi}$ $(a=1, \dots r)$ form a basis of the Lie algebra of C_r they satisfy the Lie structure equations:

$$\mathscr{L}_{\underset{a}{\xi}} \underset{b}{\xi}^{\mu} = C^d_{ab} \underset{d}{\xi}^{\mu}, \qquad C^d_{(ab)} = 0 \tag{2.1}$$

where the structure constants C^d_{ab} satisfy the Jacobi identities.

The group C_r is conformal if and only there exist r functions $\underset{a}{\phi}$ on M_n such that:

$$\mathscr{L}_{\underset{a}{\xi}} g_{\mu\nu} = 2 \underset{a}{\phi} g_{\mu\nu} \tag{2.2}$$

$\underset{a}{\phi} = \text{const}$ characterizes homothetic groups (denoted by H_r) and $\underset{a}{\phi} = 0$ characterizes isometry groups (denoted by I_r). In the latter case (2.2) reduces to Killing's equations.

Equations (2.1) and (2.2) imply

$$\mathscr{L}_{\underset{a}{\xi}} \underset{b}{\phi} - \mathscr{L}_{\underset{b}{\xi}} \underset{a}{\phi} = C^d_{ab} \underset{d}{\phi} . \tag{2.3}$$

§ 3. Conformally Equivalent Metrics

Let us consider the effect of replacing the metric g on M_n by a conformally equivalent metric $\bar{g}$ given by:

$$\bar{g}_{\mu\nu}(x) = e^{2\sigma(x)} g_{\mu\nu}(x) . \tag{3.1}$$

If $\underset{a}{\xi}$ are the generators of a conformal group C_r acting on (M_n, g) then they will also generate a conformal group $\bar{C}_r$ acting on $(M_n, \bar{g})$ since in consequence of (2.2) we shall have

$$\mathscr{L}_{\underset{a}{\xi}} \bar{g}_{\mu\nu} = 2 \underset{a}{\bar{\phi}} \bar{g}_{\mu\nu} \tag{3.2}$$

with

$$\underset{a}{\bar{\phi}} = \underset{a}{\phi} + \underset{\underset{a}{\xi}}{\mathscr{L}} \sigma . \tag{3.3}$$

Thus in certain cases, it will be possible, with an appropriate choice of $\sigma(x)$, to find a space $(M_n, \bar{g} = e^{2\sigma} g)$ for which the conformal group is reduced to a group of isometries. If this is the case, the conformal group C_r is said to be *conformally isometric*. It will be said *conformally homothetic* if it is possible to find a space $(M_n, \bar{g})$ for which the conformal group $\bar{C}_r$ is an homothety group (i.e. $\underset{a}{\bar{\phi}}$ constants). Finally C_r is said to be a *proper conformal group* if it is neither conformally homothetic or isotropic.

In order for the group C_r to be conformally isometric it is sufficient that there exists a solution σ of the system of equations:

$$\underset{\underset{a}{\xi}}{\mathscr{L}} \sigma + \underset{a}{\phi} = 0 . \tag{3.4}$$

It can be shown that the Conditions (2.1) and (2.3) ensure the integrability of this system except when there exists a linear relation of the form:

$$\lambda^a(x) \underset{a}{\xi}^\alpha = 0 \quad \text{with} \quad \lambda^a(x) \underset{a}{\phi}(x) \neq 0 . \tag{3.5}$$

Hence in particular, if the rank of the matrix $|\underset{a}{\xi}^\alpha|$ is equal to the dimension r of the group C_r $(r \leq n)$, (3.4) is integrable and C_r is conformally isometric (Yano, 1955). However this restriction on the rank implies that C_r is simply transitive and this excludes all the cases of conformal groups with sub-groups of isotropy. In the next section, we show that a conformal group C_r is conformally isometric under much more general conditions.

§ 4. Conformal Scalars and Conformal Gradients

Let two metrics g, $\bar{g}$ be related by (3.1). If a scalar concomitant of the metric tensor g (in the sense of Schouten, 1954) is such that

$$\bar{F} = e^{2p\sigma} F \qquad (p \text{ constant}) \tag{4.1}$$

it is said to be a *conformal scalar* of weight p (du Plessis, 1969).

A vector concomitant Y_μ is said to be a *conformal vector* of weight p is

$$\bar{Y}_\mu = Y_\mu + 2p\sigma,_\mu . \tag{4.2}$$

It is a *conformal gradient* if $Y_{[\mu,\nu]} = 0$. If F is a conformal scalar of weight p, $(\ln F),_\mu$ is a conformal gradient (of same weight). A concomitant scalar or vector that satisfies (4.1) or (4.2) with p equal to zero is said to be *conformally invariant*. We shall use the term *proper* conformal scalar of vector to denote a conformal scalar or vector that is *not* conformally invariant. It is easy to verify that a conformal scalar or vector must satisfy:

$$\underset{\underset{a}{\xi}}{\mathscr{L}} F = 2p \underset{a}{\phi} F \tag{4.3}$$

$$\underset{\underset{a}{\xi}}{\mathscr{L}} Y_\mu = 2p \underset{a}{\phi}{}_{,\mu} , \tag{4.4}$$

where $\underset{a}{\xi}{}^\mu$ are the transformations of a conformal group C_r.

Let us suppose that (M_n, g) admits a conformal group C_r and a *proper* conformal scalar F. Then we may choose $\bar{g}$ such that $\bar{F} = 1$. It follows that one has

$$\underset{\underset{a}{\xi}}{\mathscr{L}} \bar{F} = 2p \underset{a}{\bar{\phi}} \bar{F} = 0 \tag{4.5}$$

and hence each of the scalar $\underset{a}{\bar{\phi}}$ is zero. Thus the conformal group is reduced to a group of isometries and we have:

Theorem 1. *A conformal group acting on a space (M_n, g) admitting a proper conformal scalar is conformally isometric.*

This theorem is particularly powerfull in the case of positive definite metrics. We have:

Corollary 1. *Any conformal group C_r acting on a non conformally flat space (M_n, g) with a positive definite metric is conformally isometric.*

In order to prove this we recall that the necessary and sufficient condition for an n-dimensional Riemannian or pseudo-Riemannian space (M_n, g) to the conformally flat is that the Weyl tensor $C_{\mu\nu\sigma\varrho}$ be zero when $n \geqq 4$ and that Cotton's tensor $C_{\mu\nu\sigma}$ be zero when $n = 3$. (Any two dimensional space is conformally flat.) These tensors transform under (3.1) according to

$$\bar{C}_{\mu\nu\sigma\varrho} = e^{2\sigma} C_{\mu\nu\sigma\varrho} , \tag{4.6}$$

$$\bar{C}_{\mu\nu\sigma} = C_{\mu\nu\sigma} - C_{\mu\nu\sigma}{}^{\varrho} \sigma_{,\varrho} . \tag{4.7}$$

Equation (4.6) implies that $C_{\mu\nu\sigma\varrho} C^{\mu\nu\sigma\varrho}$ is a conformal scalar of weight -2 and thus that it is a *proper* conformal scalar in any space (M_n, g) with positive definite metric for which $C_{\mu\nu\sigma\varrho}$ is non zero. We thus obtain the corollary in the case $n \geqq 4$. [This result was obtained independently by Suguri and Ueno (1972).] In the special case $n = 3$, the condition that the Weyl tensor is identically zero implies by (4.7) that $C_{\mu\nu\sigma} C^{\mu\nu\sigma}$ is a conformal scalar of weight -3, and hence that it is a *proper* conformal scalar if the metric is positive definite and $C_{\mu\nu\sigma}$ is not zero. Since the case $n = 2$ is trivial, this completes the proof of the corollary.

We have a proposition similar to the Theorem 1 in terms of proper conformal gradients.

If (M_n, g) admits a proper conformal gradient Y_μ Eq. (4.2) show that one can always choose a space $(M, \bar{g} = e^{2\sigma} g)$ such that $\bar{Y}_\mu$ is zero. If (M_n, g) admits a conformal group C_r, one derives from (4.4) that the group $\bar{C}_r$ acting on $(M_n, \bar{g})$ is an homothety group. So we have:

Theorem 2. *A conformal group acting on a space (M_n, g) admitting a proper conformal gradient is conformally homothetic.*

These results suggest the possibility of propositions converse to the Theorems 1 and 2 which we reformulate as:

Theorem 1. *A space (M_n, g) admitting a non conformally isometric conformal group C_r, has no proper conformal scalar.*

Theorem 2. *A space (M_n, g) admitting a proper conformal group C_r, has no proper conformal gradient.*

The converses read as follow:

Conjecture 1. *A space (M_n, g) with no proper conformal scalar admits a non conformally isometric conformal group C_r.*

Conjecture 2. *A space (M_n, g) with no proper conformal gradient, admits a proper conformal group C_r.*

These two conjectures are evidently true in the case of spaces with positive definite metrics, since we have seen in the proof of Corollary 1 that such spaces always admit proper conformal scalars (and hence also gradients) unless they are conformally flat. By the theorem of Taub (1951) a conformally flat Riemannian space (M_n, g) always admits a conformal group C_r of order $r = \frac{1}{2}(n+1)(n+2)$. (For $n \geqq 3$, this is the maximum order. When $n = 2$ there is an infinite dimensional conformal group.) However for all n the dimension of the maximum isometry group is $r = \frac{1}{2}n(n+1)$ and of the maximum homothety group is $r = \frac{1}{2}n(n+1)+1$.

In the next two sections we show that Conjectures 1 and 2 are also true in the case of a Lorentzian 4-manifold.

§ 5. Complex Bivectorial Formalism and Canonical Tetrads

In this section, we briefly summarize the complex bivector formalism (based on the isomorphism between the Lorentz group and the complex rotation group $SO(3, C)$) that is described in detail by Debever (1964) and Cahen, Debever, Defrise (1967).

We consider a Lorentzian manifold (M_4, g) of class C^2, piecewise C^5. At each point of M_4 a normalized null tetrad is introduced such that the metric has the following form:

$$ds^2 = 2(\theta^0\theta^3 - \theta^1\theta^2) = \eta_{ab}\theta^a\theta^b (a, b = 0, 1, 2, 3). \tag{5.1}$$

The forms θ^0, θ^3 are real and the two others are complex conjugate $\theta^1 = (\theta^2)^*$. We denote by $\underset{a}{h}^{\mu}$ the components of the co-base vectors canonically associated to the θ^a.

From the base $\{\theta^a\}$, we obtain the associated base Z^i $(i = 1, 2, 3)$ of the complex euclidean space $E_3(C)$ of self-dual 2-forms:

$$Z^1 = \theta^2 \Lambda \theta^3 \qquad Z^2 = \theta^0 \Lambda \theta^1 \qquad Z^3 = \tfrac{1}{2}(\theta^0 \Lambda \theta^3 - \theta^1 \Lambda \theta^2). \tag{5.2}$$

There is a canonical metric on the space $E_3(C)$ given by

$$\gamma_{ij} Z^i Z^j = 2(Z^1 Z^2 - (Z^3)^2). \tag{5.3}$$

The Riemannian connexion determined on M_4 by the metric tensor $g_{\mu\nu}$ may be expressed in terms of the one-forms $\sigma^i{}_j$ defined by:

$$dZ^i + \sigma^i{}_j \Lambda Z^j = 0 \qquad (i, j = 1, 2, 3). \tag{5.4}$$

To the forms $\sigma^i{}_j$ one associates the three forms σ^i given by

$$\sigma^i = \tfrac{1}{2}\, \varepsilon^{ijk} \gamma_{jl} \sigma^l{}_k \tag{5.5}$$

where ε^{ijk} is the antisymmetric Kronecker symbol.

The curvature 2-forms Σ_k are given by:

$$\Sigma_i = \gamma_{ij}\, d\sigma^j - \tfrac{1}{2}\, \varepsilon_{ijk} \sigma^j \Lambda \sigma^k . \tag{5.6}$$

The complex auto-dual form

$$\overset{+}{C}_{\lambda\mu\nu\sigma} = \frac{1}{2}\left(C_{\lambda\mu\nu\sigma} - \frac{i}{2}\, \eta_{\lambda\mu\tau\varrho} C^{\tau\varrho}{}_{\nu\sigma} \right) \tag{5.7}$$

of the Weyl tensor can be obtained from the expression

$$\overset{+}{C}_{\lambda\mu\nu\sigma} = C_{ij} Z^i_{\lambda\mu} Z^j_{\nu\sigma} \tag{5.8}$$

where C_{ij} is a tensor on $E_3(C)$, which is symmetric and trace free (i.e. $4C_{12} = C_{33}$) and which can be read out from the curvature 2-forms by expressing them in the form:

$$\Sigma_i = \left(C_{ij} - \frac{R}{6}\, \gamma_{ij} \right) Z^j + E_{ij*} (Z^j)^* . \tag{5.9}$$

The scalar R is the scalar curvature and the hermitian tensor E_{ij*} determines the trace free part

$$S_{\alpha\beta} = R_{\alpha\beta} - \frac{R}{4}\, g_{\alpha\beta} \tag{5.10}$$

of the Ricci tensor by

$$\overset{+}{E}_{\alpha\beta\gamma\delta} = E_{ij*} Z^i (Z^j)^* \tag{5.11}$$

where $\overset{+}{\mathrm{E}}_{\alpha\beta\gamma\delta}$ is the auto-dual tensor formed from the tensor

$$E_{\alpha\beta\gamma\delta} = -\tfrac{1}{2} (g_{\alpha[\gamma} S_{\delta]\beta} + S_{\alpha[\gamma} g_{\delta]\beta}) . \tag{5.12}$$

From (5.8), it can be seen that the classification of the Weyl tensor is reduced to the classification of the matrix C_{ij}. For each Petrov type it is possible to find a *canonical tetrad* such that the only *non* zero components of C_{ij} are the following: C_{12}, C_{13}, C_{23} for Type I (with the condition that $C_{13} = C_{23}$); C_{12} and C_{23} for Type II (with the condition that $C_{12} = C_{23}$); C_{23} for Type III (with $C_{23} = 1$); C_{12} for type D and C_{22} (with $C_{22} = 1$) for type N.

In each case the vector $\overset{0}{h}{}^\mu$ is a characteristic vector of the Weyl curvature tensor. For the Petrov types I, II, III, the canonical tetrad is univoquely determined and hence the basis vectors $\overset{a}{h}{}^\mu$ are concomitants (in the sense of Schouten, 1954) of the metric tensor $g_{\mu\nu}$. For the type D and N the canonical tetrad is determined up to a transformation which (in both cases) is a subgroup of order two of the Lorentz group $L_+^\uparrow$.

In the case of type N, we shall need to use the explicit representation

$$\begin{cases} {}'\overset{0}{h}{}^\mu = \overset{0}{h}{}^\mu \\ {}'\overset{1}{h}{}^\mu = \overset{1}{h}{}^\mu + \gamma^* \overset{0}{h}{}^\mu \\ {}'\overset{3}{h}{}^\mu = \overset{3}{h}{}^\mu + \gamma \overset{1}{h}{}^\mu + \gamma^* \overset{2}{h}{}^\mu + \gamma\gamma^* \overset{0}{h}{}^\mu , \end{cases} \tag{5.13}$$

of the subgroup, where $\gamma = \gamma_1 + i\gamma_2$ is a complex parameter.

We shall also need the corresponding formulae for the transformation under the group (5.13) of the rotation coefficients $\overset{1}{\sigma}_a$ and $\overset{3}{\sigma}_a$:

$$\begin{cases} {}'\overset{2}{\sigma}_0 = \overset{2}{\sigma}_0 - \gamma^* \overset{2}{\sigma}_1 - \gamma \overset{2}{\sigma}_2 + \gamma\gamma^* \overset{2}{\sigma}_3 \qquad {}'\overset{2}{\sigma}_2 = \overset{2}{\sigma}_2 - \gamma^* \overset{2}{\sigma}_3 \\ {}'\overset{2}{\sigma}_1 = \overset{2}{\sigma}_1 - \gamma \overset{2}{\sigma}_3 \qquad\qquad\qquad\qquad\quad {}'\overset{2}{\sigma}_3 = \overset{2}{\sigma}_3 \\ {}'\overset{3}{\sigma}_0 = \overset{2}{\sigma}_0 - \gamma^* \overset{3}{\sigma}_1 - \gamma \overset{3}{\sigma}_2 + \gamma\gamma^* \overset{3}{\sigma}_3 + \gamma(\overset{2}{\sigma}_0 - \gamma^* \overset{2}{\sigma}_1 - \gamma \overset{2}{\sigma}_2 + \gamma\gamma^* \overset{2}{\sigma}_3) \\ {}'\overset{3}{\sigma}_1 = \overset{3}{\sigma}_1 - \gamma \overset{3}{\sigma}_3 + \gamma(\overset{2}{\sigma}_1 - \gamma \overset{2}{\sigma}_3) \\ {}'\overset{3}{\sigma}_2 = \overset{3}{\sigma}_2 - \gamma^* \overset{3}{\sigma}_3 + \gamma(\overset{2}{\sigma}_2 - \gamma^* \overset{2}{\sigma}_3) \\ {}'\overset{3}{\sigma}_3 = \overset{3}{\sigma}_3 + \gamma \overset{2}{\sigma}_3 . \end{cases} \tag{5.14}$$

§ 6. Conformal Group in a Lorentzian 4-Manifold

In this section, we show that the only Lorentzian 4-manifolds which have no proper conformal scalar are conformally equivalent to the plane wave metric with parallel rays or conformally flat.

If (M_n, g), $(M_n, \bar{g})$ are conformally equivalent, one derives from (4.6) and (5.7) that

$$\overset{\mp}{C}_{\lambda\mu\nu\sigma} = e^{2\sigma} \overset{+}{C}_{\lambda\mu\nu\sigma} . \tag{6.1}$$

Hence, if they are not zero the scalars

$$\overset{(1)}{C} = \overset{+}{C}_{\lambda\mu\nu\sigma} \overset{+}{C}{}^{\lambda\mu\nu\sigma} , \qquad \overset{(2)}{C} = \overset{+}{C}_{\lambda\mu\nu\sigma} \overset{+}{C}{}^{\nu\sigma\kappa\tau} \overset{+}{C}_{\kappa\tau}{}^{\lambda\mu} \tag{6.2}$$

are proper conformal scalars (of respective weight -2 and -3), for the Petrov types I, II and D one at least of these scalars is non zero.

For the Type III, the canonical tetrad described in the last section is concomitant and hence the rotation coefficients $\overset{i}{\sigma}_a$ are concomitant scalars.

For the Type III, the correspondance between the canonical tetrad of (M_4, g) and $(M_4, \bar{g} = e^{2\sigma} g)$ is the following:

$$\begin{aligned} \overset{\bar{0}}{h}_\mu &= e^{-\sigma} \overset{0}{h}_\mu \\ \overset{\bar{1}}{h}_\mu &= e^{\sigma} \overset{1}{h}_\mu , \\ \overset{\bar{3}}{h}_\mu &= e^{3\sigma} \overset{3}{h}_\mu . \end{aligned} \tag{6.3}$$

The correspondence between the rotation coefficients $\overline{\sigma^i_a}$ of $(M,\bar g)$ and σ^i_a is given by:

$$\left\{\begin{array}{lll} \overset{\bar 1}{\sigma}_0 = e^{3\sigma}\overset{1}{\sigma}_0 & \overset{\bar 2}{\sigma}_0 = e^{-\sigma}(\overset{2}{\sigma}_0 - 2\sigma_{,2}) & \overset{\bar 3}{\sigma}_0 = e^{\sigma}(\overset{3}{\sigma}_0 + 2\sigma_{,0}) \\ \overset{\bar 1}{\sigma}_1 = e^{\sigma}\overset{1}{\sigma}_1 & \overset{\bar 2}{\sigma}_1 = e^{-3\sigma}(\overset{2}{\sigma}_1 - 2\sigma_{,3}) & \overset{\bar 3}{\sigma}_1 = e^{-\sigma}(\overset{3}{\sigma}_1 + 3\sigma_{,1}) \\ \overset{\bar 1}{\sigma}_2 = e^{\sigma}(\overset{1}{\sigma}_2 + 2\sigma_{,0}) & \overset{\bar 2}{\sigma}_2 = e^{-3\sigma}\overset{2}{\sigma}_2 & \overset{\bar 3}{\sigma}_2 = e^{-\sigma}(\overset{3}{\sigma}_2 + 3\sigma_{,2}) \\ \overset{\bar 1}{\sigma}_3 = e^{-\sigma}(\overset{1}{\sigma}_3 + 2\sigma_{,1}) & \overset{\bar 2}{\sigma}_3 = e^{-5\sigma}\overset{2}{\sigma}_3 & \overset{\bar 3}{\sigma}_3 = e^{-3\sigma}(\overset{3}{\sigma}_3 + \sigma_{,3}) . \end{array}\right. \tag{6.4}$$

If one equates to zero all the coefficients σ^i_a which are proper conformal scalars, one finds that the space is reduced to be conformally flat.

Finally, for the Type N, the correspondence between the canonical tetrad of (M_4, g), $(M_4, \bar g)$ and between the rotation coefficients $\overset{\bar 2}{\sigma}_a$, $\overset{\bar 3}{\sigma}_a$ and $\overset{2}{\sigma}_a$, $\overset{3}{\sigma}_a$ is

$$\begin{aligned} \overset{\bar 0}{h}_\mu &= \overset{0}{h}_\mu , \\ \overset{\bar 1}{h}_\mu &= e^{\sigma}\overset{1}{h}_\mu , \\ \overset{\bar 3}{h}_\mu &= e^{2\sigma}\overset{3}{h}_\mu . \end{aligned} \tag{6.5}$$

$$\left\{\begin{array}{ll} \overset{\bar 2}{\sigma}_0 = e^{-\sigma}(\overset{2}{\sigma}_0 - 2\sigma_{,2}) & \overset{\bar 3}{\sigma}_0 = \overset{3}{\sigma}_0 + 2\sigma_{,0} \\ \overset{\bar 2}{\sigma}_1 = e^{-2\sigma}(\overset{2}{\sigma}_1 - 2\sigma_{,3}) & \overset{\bar 3}{\sigma}_1 = e^{-\sigma}(\overset{3}{\sigma}_1 + 2\sigma_{,1}) \\ \overset{\bar 2}{\sigma}_2 = e^{-2\sigma}\overset{2}{\sigma}_2 & \overset{\bar 3}{\sigma}_2 = e^{-\sigma}\overset{3}{\sigma}_2 \\ \overset{\bar 2}{\sigma}_3 = e^{-3\sigma}\overset{2}{\sigma}_3 & \overset{\bar 3}{\sigma}_3 = e^{-2\sigma}\overset{3}{\sigma}_3 . \end{array}\right. \tag{6.6}$$

This leads us to restrict the class of Type N spaces by requiring that the following sequence of conformal scalars be zero.:

$$\overset{2}{\sigma}_3 ; \overset{2}{\sigma}_2 ; \overset{3}{\sigma}_3 ; \overset{2}{\sigma}_1 - (\overset{2}{\sigma}_1)^* ; \overset{3}{\sigma}_2 ; (\overset{2}{\sigma}_0)^* + \overset{3}{\sigma}_1 ; \overset{3}{\sigma}_{0,1} - (\overset{3}{\sigma}_{0,2})^* . \tag{6.7}$$

The order is significant since certain coefficients are invariant for the group (5.12) (and hence are concomitant) only when preceding coefficients have been equated to zero [as can be shown from (5.13)].

At this stage we remark that

$$\overset{3}{\sigma}_0 \overset{0}{h}_\mu + \overset{3}{\sigma}_1 \overset{1}{h}_\mu - \overset{2}{\sigma}_0 \overset{2}{h}_\mu - \overset{2}{\sigma}_1 \overset{3}{h}_\mu \tag{6.8}$$

is a proper conformal gradient.

The Conditions (6.7) ensure that the space be equivalent to the plane (wave metric) with parallel rays (denoted ppw metric). To show this, let us take advantage of the Correspondence (6.6) by choosing in the family of conformally equivalent spaces the class for which one has:

$$\overset{\bar 2}{\sigma}_0 = \overset{\bar 2}{\sigma}_1 = 0 . \tag{6.9}$$

This is possible since the Conditions (6.7) ensure that the system of equations in the function σ:

$$\begin{cases} 2\sigma_{,2} = \overset{2}{\sigma}_0 \\ 2\sigma_{,3} = \overset{2}{\sigma}_1 \end{cases} \tag{6.10}$$

is completely integrable.

The Conditions (6.7) and (6.9) put together caracterize the ppw metric (see Debever, 1965). It can be written in the form:

$$ds^2 = 2\,du\,dv + [(\alpha + i\beta)z^2 + (\alpha - i\beta)z^{*2} + 2\gamma\, zz^*]du^2 - 2dz\,dz^* \tag{6.11}$$

where α, β and γ are three functions of u. The ppw metric admits a homothetic group H_6 formed by an isometry group I_5 and the homothetic transformation:

$$\underset{\xi}{\mathscr{L}} g_{\mu\nu} = 2\underset{0}{\phi} g_{\mu\nu}\,. \tag{6.12}$$

If x^μ are defined by

$$x^1 + ix^2 = z\,, \qquad x^3 = u\,, \qquad x^4 = v\,, \tag{6.13}$$

the components of the operator ξ are:

$$\xi^1 = \underset{0}{\phi} x^1\,, \qquad \xi^2 = \underset{0}{\phi} x^2\,, \qquad \xi^3 = 0\,, \qquad \xi^4 = 2\underset{0}{\phi} x^4\,. \tag{6.14}$$

If α, β and γ satisfy the following conditions

$$\alpha + i\beta = a\,e^{ibu}\,, \qquad \gamma = c \tag{6.15}$$

where a, b, c are three constants, the metric admits an isometry group I_6 and an homothety group H_7.

It is easy to verify that H_6 (or H_7) is a *proper* homothetic group.

The only Lorentzian 4-manifold with *no proper conformal gradient* is conformally minkowskian.

Thus we have:

Theorem 3. *The Lorentzian 4-manifolds with no proper conformal scalar are conformally equivalent to the plane wave metric with parallel rays or conformally minkowskian. In the first case there exists a proper conformal gradient and the family of spaces admits a conformal group conformally equivalent to an homothety group H_6 or H_7. In the second case there is no proper conformal gradient and there is a proper conformal group.*

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L. Defrise-Carter
Girton College
Cambridge CB3 OJG, England