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The Mass Gap for the $P(\phi)_2$ Quantum Field Model with a Strong External Field

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Abstract. We consider the $P(\phi)_2$ hamiltonian whose interaction density is given by

$$\lambda P(\phi(x)) + \mu \phi(x)^k$$

where k is odd and $1 \leqq k < \deg P$. For sufficiently large μ we show that there is a gap in the energy spectrum. In addition we obtain new regions of analyticity in λ and μ for the Schwinger functions and the pressure.

§ 1. Introduction

The $P(\phi)$ quantum field hamiltonian in two space time dimensions is given formally by

$$H = H_0 + \lambda \int :P(\phi(x)):dx - E \geqq 0 \tag{1.1}$$

where H_0 is the free hamiltonian of mass m_0, P is a positive polynomial and E is the vacuum energy. In [2, 3], it was shown that for sufficiently weak coupling (λ/m_0^2 small) the vacuum (ground state) of H is unique and that the mass of H is positive. In this paper we consider the Hamiltonian

$$H_\mu = H_0 + \int :P_\mu(\phi(x)):dx - E_\mu \tag{1.2}$$

where

$$P_\mu(\xi) = \lambda P(\xi) + \mu \xi^k \tag{1.3}$$

and k is odd, $1 \leqq k < n \equiv \deg P$. Our main result is that if μ is sufficiently large then the mass of H_μ is positive. We also show that the infinite volume Schwinger functions are analytic in λ and μ, provided $|\lambda|$, $|\mathrm{Im}\,\mu|$ are small, $\mathrm{Re}\,\lambda > 0$ and $|\mu|$ is large. The results of [2] can also be obtained for large μ.

If $P(\xi) = \lambda \xi^4 + \mu \xi$, $\mu \neq 0$, Simon and Griffiths [10] have established uniqueness of the vacuum and analyticity of the pressure (vacuum energy per unit volume) as a function of μ for $\mathrm{Re}\,\mu > 0$. Their proof follows

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from the Lee-Yang theorem and from an approximation of the Euclidean field model by an Ising model. The cluster expansion [3] combined with these results and an argument of Lebowitz and Penrose [5] shows that if $\mu \neq 0$ the pressure $p(\lambda, \mu)$ is real analytic in λ for all $\lambda > 0$. We also show that for small coupling, the mass is a real analytic function of λ, $0 \leqq \lambda \leqq \lambda_0$ for λ_0 sufficiently small.

The proof of our main results relies on a "high temperature" cluster expansion developed in [2, 3] together with some elementary transformations in function space. The idea behind our proof is to transform H_μ so that the following conditions hold:

(a) The coefficients of the interaction are small relative to the transformed bare mass as $\mu \to \infty$.

(b) The vacuum energy per unit volume is small relative to the transformed bare mass as $\mu \to \infty$.

We shall see that these conditions are essentially sufficient for the convergence of the cluster expansion.

Heuristically the low energy spectrum of H is governed by the behavior of $P(\xi)$ at its minima. In the conventional picture if P has a unique minimum then H has a unique vacuum and the curvature of P at its minimum approximates the mass of H. If $n \geqq 4$, note that as $\mu \to \infty$, $P_\mu(\xi)$ has a unique minimum and the curvature at the minimum tends to infinity. The proof of our main result is based on the estimate

$$P_\mu(\xi + a) - P_\mu(a) \geqq c_0\, a^{n-m} \xi^m \tag{1.4}$$

where $c_0 > 0$, $2 \leqq m \leqq n$ and $a = a(\mu, \lambda)$ is the value of ξ which minimizes $P_\mu(\xi)$. This estimate allows us to control the vacuum energy per unit volume i.e. condition (b). To obtain condition (a), we expand $P_\mu(\xi + a) - P_\mu(a)$ in powers of ξ.

$$P_\mu(\xi + a) - P_\mu(a) = \sum_{i=2}^{n} b_i(\mu)\, a^{n-i} \xi^i \tag{1.5}$$

where b_i are uniformly bounded in μ. We shall see that $b_2(\mu) \geqq \varepsilon > 0$ where ε is independent of μ. Thus as a (equivalently μ) becomes large (1.5) shows that the coefficient of ξ^2 dominates the remaining coefficients. Since the coefficient of the quadratic term in ϕ may be incorporated into m_0^2 in H_0 we see formally that the transformation $\phi(x) \to \phi(x) + a$ enables us to establish (a).

§ 2. The Main Results

Our main theorems are formulated and proved in the framework of Euclidean field theory. See [4, 6, 7, 11]. Let $\Lambda = \Lambda_L$ be a square of area L^2 centered at the origin. We define Δ_Λ to be the Laplacian with periodic boundary conditions on Λ (i.e. the Laplacian on the torus). Let dq_{Λ,m_0^2} be the Gaussian measure on $\mathscr{S}'(R^2)$ of mean zero and covariance $(-\Delta_\Lambda+m_0^2)^{-1}$. In general if C is a continuous scalar product on $\mathscr{S}(R^2)$ we denote the corresponding measure of covariance C by dq_C. Let $P(\xi)$ be a polynomial in ξ which is bounded below. We define

$$\mathscr{E}_\Lambda(P)=\exp\left[-\int_\Lambda :P(q(x)):dx\right] \tag{2.1}$$

and the expectation

$$\langle A\rangle_{\Lambda,P,m_0^2}=\frac{\int A\,\mathscr{E}_\Lambda(P)\,dq_{\Lambda,m_0^2}}{\int \mathscr{E}_\Lambda(P)\,dq_{\Lambda,m_0^2}}. \tag{2.2}$$

The pressure is defined to be

$$p(\lambda,\mu,m_0^2)=\lim_{L\to\infty}\frac{1}{L^2}\log\int \mathscr{E}_{\Lambda_L}(\lambda P+\mu\xi^k)\,dq_{\Lambda,m_0^2}. \tag{2.3}$$

The Wick order : : in (2.2) is defined with respect to the measure dq_{Λ,m_0^2}. (A brief discussion of Wick order appears later.) The existence of the periodic pressure has been established in [13]. We thank B. Simon for this information.

If $A=q(x_1)\dots q(x_m)$, then $\langle A\rangle_{\Lambda,P,m_0^2}$ defines the space cutoff Schwinger (correlation) functions. More generally let A be a localized monomial, i.e.

$$A(q)=\int\prod_{i=1}^{l}:q(x_i)^{n_i}:w(x_1,\dots,x_l)\,d^{2l}x$$

where $w\in L^2(\Pi_{i=1}^l\Delta_i)$ and $\Delta_i\subset\Lambda$ denotes a unit lattice square. For $t\in R^2$ let $T^tA(q)=A(q(\cdot-t))$. Also we define

$$D_{x_0}(\varepsilon)=\{x\in\mathbb{C}\,|\,|x-x_0|\leqq\varepsilon\}.$$

Theorem 2.1. *Let A be as above, and let λ_0 be positive. For $\mu_0=\mu_0(\lambda_0)$ sufficiently large, there exists $\varepsilon=\varepsilon(\mu_0,\lambda_0)>0$ such that*

$$\int\mathscr{E}_\Lambda(\lambda P+\mu\xi^k)\,dq_{\Lambda,m_0^2}\neq 0 \tag{2.4}$$

and

$$\lim_{L\to\infty}\langle A\rangle_{\Lambda_L,\lambda P+\mu\xi^k,m_0^2} \tag{2.5}$$

exists for all $\lambda\in D_{\lambda_0}(\varepsilon)$ and $\mu\in D_{\mu_0}(\varepsilon)$. Moreover there is a positive constant m independent of A_1 and A_2 and a constant $C=C(A_1,A_2)$ such that

$$|\langle A_1T^tA_2\rangle-\langle A_1\rangle\langle A_2\rangle|\leqq Ce^{-|t|m}. \tag{2.6}$$

Here $\langle\ \rangle$ denotes the infinite volume expectation.

Remarks. When A is a product of fields and μ and λ are real (2.5) defines Euclidean invariant Schwinger functions from which a Wightman field theory can be constructed. See [8]. Convergence of (2.5) and

(2.4) imply that the Schwinger functions and the pressure are analytic for $\lambda \in D_{\lambda_0}(\varepsilon)$ and $\mu \in D_{\mu_0}(\varepsilon)$. Finally (2.6) implies the existence of a mass gap.

As in Lebowitz-Penrose [5] we use the following theorem (Malgrange-Zerner, see [14]) to extend the domain of analyticity of (λ, μ, m_0^2).

Theorem 2.2. *Let $K \subset \mathbb{C}$ be the closure of a bounded simply connected domain such that a disc D_{μ_0} is contained in K. Let I be a closed interval on the real λ axis, and let D_{λ_0} be the disc whose diameter is I. Let $F(\lambda, \mu)$ be a function with the following properties:*

(i) *$F(\lambda, \mu)$ is jointly analytic in λ and μ for all $\lambda \in D_{\lambda_0}$ and $\mu \in D_{\mu_0}$.*
(ii) *$F(\lambda, \mu)$ is analytic in μ, $\mu \in K$ for each $\lambda \in I$.*
(iii) *There is a constant M such that*

$$|F(\lambda, \mu)| < M$$

for all $\lambda \in I$, $\mu \in K$. Then $F(\lambda, \mu)$ is jointly analytic for $\lambda \in I$ and $\mu \in K$.

Corollary 2.3. *If $P(\xi) = \lambda\xi^4 + \mu\xi$ the pressure $p(\lambda, \mu)$ is jointly real analytic in λ for positive λ and complex μ with* Re $\mu \neq 0$.

Proof. Following Lebowitz and Penrose [5] we set $F(\lambda, \mu) = e^{-p(\lambda, \mu)}$ and verify (i), (ii), and (iii) using Theorem 2.1 and the Simon-Griffiths generalization of the Lee-Yang theorem [10]. Given $\lambda_0 > 0$ we choose μ_0 sufficiently large and define K to be any rectangle contained in the right half plane and containing $D_{\mu_0}(\varepsilon)$. Now (i) follows from Theorem 2.1 and the remark following it. Condition (ii) is an immediate consequence of the LeeYang theorem. Since

$$|e^{p(\lambda, \mu)}| \leqq \lim_{\Lambda \to \infty} [\int \mathscr{E}_\Lambda(\lambda\xi^4 + \operatorname{Re}\mu\xi)\, dq]^{1/|\Lambda|}$$

is uniformly bounded for $\mu \in K$ and for $\lambda \in D_{\lambda_0}(\varepsilon)$ real (iii) holds. Hence it follows that $p(\lambda, \mu)$ is jointly real analytic in λ near λ_0 and complex analytic in $\mu \in K$ where $D_{\mu_0} \subset K$ and K is a compact subset of $\operatorname{Re}\mu > 0$. The corollary now follows.

Remark. A similar result should hold for the Schwinger functions.

We also include another application of Theorem 2.2, which is not directly related to the main results of this paper. Let $m(\lambda)$ be the physical mass for a $\lambda P(\phi)_2$ theory and let $Z(\lambda)$ be the field strength renormalization.

Corollary 2.4. *The functions $m(\lambda)$ and $Z(\lambda)$ are real analytic functions of λ for $0 < \lambda \leqq \lambda_0$ and for λ_0/m_0^2 sufficiently small.*

Proof. The proof of this result relies on the results of [2] and [3]. Let

$$F(\lambda, p) = \int e^{-ix \cdot p} \langle \Phi(x)\, \Phi(0) \rangle^T_{R^2, \lambda P, m_0^2}\, dx\,.$$

By analytic continuation of the Lehman spectral formula, and results of [2] we have for real λ

$$F(\lambda, p) = \frac{Z(\lambda)}{p^2 + m(\lambda)^2} + \int_{4(m_0-\varepsilon)^2}^{\infty} \frac{d\varrho(m^2, \lambda)}{p^2 + m^2}. \tag{2.7}$$

Here $\varepsilon > 0$ can be chosen arbitrarily small by decreasing λ_0. Let $z = (0, p_1)$. Then from (2.7) we have

$$im(\lambda) = \frac{\oint zF(\lambda, z)\, dz}{\oint F(\lambda, z)\, dz} \tag{2.8 a}$$

and

$$\pi i Z(\lambda) = \oint zF(\lambda, z)\, dz. \tag{2.8 b}$$

The contour is defined to be $\{z \,|\, |z - im_0| = m_0/2\}$. To show $m(\lambda)$ is real analytic in λ it suffices to show that $F(\lambda, z)$ is real analytic in λ for z belonging to the contour. Now we observe that for $\lambda_0 > 0$ sufficiently small,

(i) $F(\lambda, z)$ is jointly analytic in z and λ, for $\operatorname{Re}\lambda > 0$, $|\lambda| \leqq \lambda_0$ and $|z| \leqq \delta$, where $\delta > 0$ is sufficiently small. This is an immediate consequence of the exponential cluster property proved in [3] for complex λ.

(ii) For real λ, $0 \leqq \lambda \leqq \lambda_0$, $F(\lambda, z)$ is analytic in z when $z \in A$. Here $A = A_1 \cup A_2$ is a region which contains the contour of the integrals in (2.8) and A_i are simply connected regions containing $|z| \leqq \delta$.

(iii) For real λ, $0 \leqq \lambda \leqq \lambda_0$ and $z \in A$, $F(\lambda, z)$ is uniformly bounded.

(ii) and (iii) follow from (2.7) and uniform bounds on $F(\lambda, 0)$, provided A is bounded away from $\pm im_0$. Hence by Theorem 2.2, with $\mu = z$ and $K = A_1$ or A_2, it follows that $F(\lambda, z)$ is analytic for $0 < \lambda < \lambda_0$, $z \in A$. This completes the proof.

We conclude the section with a review of Wick order. The Euclidean field with a momentum cutoff r is represented by

$$q_r(x) = q(\delta_r(\cdot - x))$$

where δ_r is the Fourier transform of characteristic function of the interval $[-r, r]$. We define the Wick order of $q^n(x)$ with respect to dq_C by the limit as $r \to \infty$ of

$$:q_r(x)^n: = \sum_{j=0}^{[n/2]} c_{nj}(-\sigma_r(x))^j\, q_r(x)^{n-2j} \tag{2.9}$$

where $c_{nj} = n!\, 2^{-j}/j!(n-2j)!$ and

$$\sigma_r(x) = \int q_r(x)^2\, dq_C\,.$$

In [1] it is shown that (2.9) converges as $r \to \infty$ in $L^p(\mathscr{S}', dq_C)$, $1 \leqq p < \infty$ for a large class of covariances. When $C = (-\Delta + m_0^2)^{-1}$, the above definition coincides with the standard definition of Wick order. An easy computation shows that

$$\sigma_r(x) = \int q_r(x)^2\, dq_{\Lambda, m_0^2} \leqq \text{Const.} \log\left(\frac{r^2}{m_0^2} + 1\right).$$

The proof of our main theorem involves changes in the mass appearing in the covariance of our Gaussian measure. For this reason we give a formula relating Wick order $:\;:_{m_i^2}$ with respect to the two covariances

$$C_i = (-\Delta_\Lambda + m_i^2)^{-1}, \qquad i = 1, 2$$

$$:q(x)^n:_{m_1^2} = \sum_j^{[n/2]} c_{nj}\, \delta C(x,x)^j :q(x)^{n-2j}:_{m_2^2} \tag{2.10}$$

where for large L

$$(C_1 - C_2)(x,x) = \delta C(x,x) \doteq \frac{1}{4\pi^2} \int \frac{m_2^2 - m_1^2}{(k^2 + m_1^2)(k^2 + m_2^2)}$$

$$= \frac{1}{4\pi} \log\left(\frac{m_2^2}{m_1^2}\right).$$

See ([4] § II) for details.

§ 3. Proof of Theorem 2.1

In this section we prove Theorem 2.1 assuming estimates on the vacuum energy per unit volume established in the next section. We apply three elementary transformations to H_μ so that condition (a) of § 1 holds.

(α) Translation. The transformation $q(x) \to q(x) + c$ for c real yields the identity

$$\langle A \rangle_{\Lambda, P, m_0^2} = \langle A^c \rangle_{\Lambda, P^c + c m_0^2 \xi, m_0^2} \tag{3.1}$$

where

$$P^c(\xi) = P(\xi + c) - P(c)$$

and

$$A^c(q) = A(q + c).$$

(β) Scaling. By the change of scale $x \to xs$ we have

$$\langle A \rangle_{\Lambda, P, m_0^2} = \langle A_s \rangle_{s\Lambda, s^{-2}P, s^{-2}m_0^2} \tag{3.2}$$

where

$$A_s(q) = A(q(s\,\cdot)).$$

(γ) Mass shift

$$\langle A \rangle_{\Lambda, P + 1/2\, b^2 \xi^2, m_0^2} = \langle A \rangle_{\Lambda, P^\sim, m_0^2 + b^2} \tag{3.3}$$

where the equation $:P:_{m_0^2} = :P^\sim:_{m_0^2 + b}$ defines $P^\sim$.

The proofs of (α) and (γ) follow easily from the lattice approximation of [4] where the Laplacian has periodic boundary conditions. Formally $e^{\int (q \Delta_\Lambda q) dx}$ is approximated by $\Pi_{\langle x,y \rangle}\, e^{-(q_x - q_y)^2}$. The product ranges over all nearest neighbor lattice sites in $T(\Lambda)$, the torus obtained by identifying opposite sides of Λ. To establish (β) note that

$$\begin{aligned} \int q(sx)\, q(sy)\, dq_{s\Lambda, s^{-2}m_0^2} &= (-\Delta_{s\Lambda} + s^{-2} m_0^2)^{-1}(sx, sy) \\ &= (-\Delta_\Lambda + m_0^2)^{-1}(x,y) \\ &= \int q(x)\, q(y)\, dq_{\Lambda, m_0^2}. \end{aligned} \tag{3.4}$$

Remark. We have used periodic boundary conditions in order to simplify the formulation of (γ). Dirichlet boundary conditions make (α) awkward.

Let a be the value of $\xi \in R$ which minimizes Re $P_\mu(\xi)$. Then

$$P_\mu(\xi+a)-P_\mu(a)=\sum_{i\geqq 2} b_i(\mu)\, a^{n-i}\xi^i+Q(\xi)$$

where the $b_i(\mu)$ are real and bounded and

$$Q(\xi)=i\,\mathrm{Im}(P_\mu(\xi+a)-P_\mu(a))\,.$$

We define

$$\begin{aligned} R(\xi)&=\frac{P_\mu(\xi+a)-P_\mu(a)-b_2a^{n-2}\xi^2+m_0^2a\xi}{a^{n-2}} \\ &=\sum_{i\geqq 3} b_i a^{-(i-2)}\xi^i+a^{2-n}Q(\xi)+m_0^2a^{-(n-3)}\xi\,. \end{aligned} \tag{3.5}$$

Next we apply (α), (β), (γ) (in that order) with $c=a$, $\frac{1}{2}b^2=\sigma b_2$ $s^2=a^{n-2}/\sigma$ where σ is a constant independent of a to be determined. We see that

$$\begin{aligned} \langle A\rangle_{\Lambda,P_\mu,m_0^2}&=\langle A^a\rangle_{\Lambda,P_\mu^a(\xi)+m_0^2a\xi,m_0^2} \\ &=\langle A'\rangle_{\Lambda',\sigma R(\xi)+b_2\sigma\xi^2,m_0^2\sigma a^{2-n}} \\ &=\langle A'\rangle_{\Lambda',\sigma R_1,m_1^2} \end{aligned} \tag{3.6}$$

where

$$\Lambda'=s\Lambda\,,\qquad A'=(A^a)_s$$

$$m_1^2=m_0^2a^{2-n}\sigma+2b_2\sigma$$

and

$$:R_1:_{m_1^2}=:R:_{m_0^2\sigma a^{2-n}}\,. \tag{3.7}$$

Note that from (2.10) the coefficient of ξ^j in Re $(R_1(\xi)-R(\xi))$ is bounded by

$$\text{Const. } a^{-j}(|\log(a^{n-2})|+1)^{(n-j)/2} \tag{3.8a}$$

for $1\leqq j\leqq n-2$, and is zero if $j=n-1$ or n.

For $j=0$ we use the fact that $R(\xi)$ has no quadratic term to bound the constant term of $\mathrm{Re}(R_1(\xi)-R(\xi))$ by

$$\frac{\text{Const.}}{a^2}[(\log(\sigma a^{2-n})+1]^{n/2}\,. \tag{3.8b}$$

Also note that the transformations (α), (β), and (γ) yield

$$\int \mathscr{E}_\Lambda(P_\mu)\,dq_{\Lambda,m^2}=e^{-P_\mu(a)|\Lambda|}\int \mathscr{E}_{\Lambda'}(R_1)\,dq_{\Lambda',m^2}\,. \tag{3.9}$$

For large μ_0, $a^{k-n}\sim -\dfrac{n}{\mu_0 k}$ and so $b_2\sim\left[\binom{n}{2}-\binom{k}{2}\dfrac{n}{k}\right]>0$. Hence if we choose ε sufficiently small in Theorem 2.1, we see that the coefficients of σR_1 are small for large values of μ: $\mu\geqq\mu_0(\sigma)$.

We now prove Theorem 2.1 assuming the reader is familiar with the cluster expansion in [3].

Proof of Theorem 2.1. The proof follows from the convergence of the cluster expansion. It is convenient to make a slight technical modification in the expansion. We define the Wick order of the interaction to agree with the Gaussian measures $dq_{C(s)}$ which arise in the expansion. The covariances $C(s)$ are convex combinations of $(-\Delta_{\Lambda,\Gamma}+m_0^2)^{-1}$ where the subscript Γ indicates that the Laplacian has Dirichlet boundary conditions on Γ. The dependence of the Wick order on s merely yields extra terms when evaluating derivatives

$$\frac{d}{ds}\int :A:_{C(s)}\,\mathscr{E}(R_1)\,dq_{C(s)}$$

which come from differentiating

$$:R_1:_{C(s)} \quad \text{and} \quad :A:_{C(s)}$$

with respect to s.

With the above modification we apply the cluster expansion in the transformed variables. The cluster property (2.6) becomes

$$|\langle A_1'\, T^t A_2'\rangle_{\Lambda'} - \langle A_1'\rangle_{\Lambda'}\langle T^t A_2'\rangle_{\Lambda'}| \leqq \text{Const.}\, e^{-m't}$$

where

$$\langle\ \rangle_{\Lambda'} = \langle\ \rangle_{\Lambda',\sigma R_1, m_1^2}$$

and Const. is independent of Λ for Λ sufficiently large. By (3.9) the partition function is a nonzero multiple of

$$\int \mathscr{E}_{\Lambda'}(\sigma R_1)\, dq_{\Lambda', m_1^2}\,.$$

The convergence (and Euclidean invariance) of (2.5) follows as in [2] from a more general cluster property obtained by replacing R_1 by $R_1 g$ above where $g(x)$ is a measurable function of compact support such that $0 \leqq g \leqq 1$.

First we note that for fixed g

$$\lim_{L\to\infty} \langle A'\rangle_{\Lambda_L', R_1\sigma g, m_1^2}$$

exists for any expanding sequence of rectangles and is just the expectation corresponding to free boundary conditions. This follows easily from a single application of formula (1.7) in [3].

By the cluster property we have (see [3] pg 161-2)

$$\begin{aligned}
&|\langle A'\rangle_{\Lambda_L', R_1\sigma, m_1^2} - \langle A'\rangle_{\Lambda_L', R_1\sigma g, m_1^2}| \\
&\quad = \left|\int_0^1 ds\{\langle A' R_1 \sigma(1-g)\rangle_s - \langle A'\rangle_s \langle R_1\sigma(1-g)\rangle_s\}\right| \\
&\quad \leqq O(e^{-m'd}\, d)
\end{aligned}$$

where d is the distance of the support of $1-g$ to the localization of A' and

$$\langle\ \rangle_s \equiv \langle\ \rangle_{A'_L, R_1 \sigma(s+(1-s)g), m_1^2}\,.$$

Hence

$$\langle A \rangle_{P_\mu} = \lim_{\substack{L\to\infty \\ g\to 1}} \langle A' \rangle_{A'_L, R_1 \sigma g, m_1^2}$$

exists and is seen to be Euclidean covariant.

To prove the convergence of the cluster expansion we first choose $\sigma = \sigma(\lambda_0)$ sufficiently large in order to produce a large bare mass. Then we choose $\mu_0(\sigma)$ and $\varepsilon = \varepsilon(\mu_0, \lambda_0)$. There are only three estimates whose proofs require some change. The first such estimate is on derivatives of the covariances $C(s)$ with respect to the interpolating parameters (s_b). In the proof [3] we must replace Brownian motion on R^2 with Brownian motion on the torus because of our use of periodic boundary conditions. If we replace distances on the plane by distance on the torus the proof is as in [3].

The second estimate on derivatives in s may be written

$$\left|\int \mathcal{M}\, \mathcal{E}_V(\sigma R_1)\, dq_C\right| \leqq e^{-K|V|}$$

where $\mathcal{M}$ is a localized monomial and V is a union of lattice squares of area $|V|$. We bound the magnitude of the above integral by

$$\|\mathcal{M}\|_{p'}\, \|\mathcal{E}_V(\sigma R_1)\|_p$$

where $\|\ \|_p$ is the $L^p(dq_C)$ norm and p' is the dual index. The norm of $\mathcal{M}$ may be estimated as in [3] by $e^{-(K+1)|V|}$ for m_1^2 sufficiently large. The constant $p>1$ is chosen in the following section and is independent of μ. Given σ large we choose μ_0 large and $\varepsilon(\mu_0)$ small so that the coefficients of R are bounded by 1 and

$$\|\mathcal{E}_V(\sigma R_1)\|_p \leqq e^{|V|}\,. \tag{3.10}$$

This bound is proved in § 4. The third estimate we must check is

$$\left|\int \mathcal{E}_\Delta(\sigma R_1)\, dq_C\right| \geqq \tfrac{1}{2} \tag{3.11}$$

for $\mu(\sigma)$ large and $\mu \in D_{\mu_0}(\varepsilon)$, $\mu \in D_{\lambda_0}(\varepsilon)$. Here Δ is a lattice square. This estimate follows easily from (3.10) as we shall see in § 4.

§ 4. Bounds on the Vacuum Energy

To prove (3.10) we show that for $\sigma = 1$ and large μ_0

$$\|\mathcal{E}_V(R_1)\|_p \leqq e^{\alpha(\mu)|V|} \tag{4.1}$$

where $\alpha(\mu)\to 0$ as $\mu\to\infty$. The desired bound for general σ follows from scaling and choosing μ large so that $\alpha(\mu)\cdot\sigma = 1$. In this section we set $h=\sigma=1$ for notational convenience.

Using Jensen's inequality and the fact that

$$0 \leqq C \leqq (-\Delta_\Lambda + m_1^2)^{-1}$$

it follows that (see Lemma II.20 of [4])

$$\|\mathscr{E}_V(R_1)\|_p^p = \int \mathscr{E}_V(p\,\mathrm{Re}\,R_1)\,dq_C \leqq \int \mathscr{E}_V(p\,\mathrm{Re}\,R_1)\,dq_{\Lambda,m_1^2}\,. \tag{4.2}$$

To bound the left side of (4.2) we apply (γ) to obtain[1]

$$\int \mathscr{E}_V(p\,\mathrm{Re}\,R_1)\,dq_{\Lambda,m_1^2} = \frac{\int \mathscr{E}_V\left(pR_2 + \left(\frac{m_1^2}{2} - \beta\right)\xi^2\right) dq_{\Lambda,2\beta}}{\int \mathscr{E}_\Lambda\left(\left(\frac{m_1^2}{2} - \beta\right)\xi^2\right) dq_{\Lambda,2\beta}}\,, \tag{4.3}$$

where $\beta > 0$ is defined in Lemma 4.2, is independent of μ and

$$:R_2:_{2\beta} = :\mathrm{Re}\,R_1:_{m_1^2}\,. \tag{4.4}$$

The denominator of (4.3) is greater than 1 by Jensen's inequality. The next two lemmas enable us to bound the numerator.

Lemma 4.1. *Let μ and λ be real and let $a(\mu)$ be the value of ξ which minimizes $P_\mu(\xi)$. There is a constant $c_0 > 0$ independent of μ such that*

$$P_\mu(\xi + a) - P_\mu(a) \geqq c_0\, a^{n-i}\,\xi^i \tag{4.5}$$

for $2 \leqq i \leqq n$ and for μ sufficiently large.

Proof. For simplicity we assume $P_\mu(\xi) = \xi^n + \mu\xi^k$. Note that

$$\mu = -\frac{n}{k}\,a^{n-k} \quad \text{and} \quad P(a) = a^n\left(1 - \frac{n}{k}\right).$$

Setting $\xi = ay$ (4.5) is equivalent to

$$f(y) \equiv (y+1)^n - \frac{n}{k}(y+1)^k + \frac{n}{k} - 1 \geqq c_0 y^i\,. \tag{4.6}$$

Since $n \geqq i$ is even, (4.5) holds for $|y| \geqq K$ for large K. If $|y| \leqq \varepsilon$, (4.5) holds for small $\varepsilon > 0$ because the coefficient of y^2 on the left side of (4.6) equals $f''(0) = n(n-k)/2 > 0$. Now since $f(y) \geqq 0$ and has a unique zero at $y = 0$, there exists a constant $\delta > 0$ such that $f(y) > \delta$, when $\varepsilon \leqq |y| \leqq K$. Now by choosing $c_0 \leqq \delta K^{-i}$ the inequality follows for $\varepsilon \leqq |y| \leqq K$. This completes the proof.

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[1] For notational simplicity we define the covariance of $dq_{\Lambda,2\beta}$ to be

$$(-\Delta + 2\beta\chi_V + m_1^2(1 - x_V))^{-1}\,.$$

Lemma 4.2. *Let λ and μ be real and let μ be large. There are constants $M, \beta > 0, p > 1$ independent of μ such that*

$$-M(1+|\log r|)^{n/2} \leqq p^2 :R_2(q_r(x)):_{2\beta} + (b_2 - \beta):q_r(x)^2:_{2\beta}$$

Proof. By (3.5) and Lemma 4.1 there are constants δ and $\beta > 0$ such that

$$0 \leqq p^2 R(\xi) + (b_2 - \beta)\,\xi^2 - D(\xi) + \text{const.} \tag{4.7}$$

where

$$D(\xi) = \delta \sum_{i=1}^{n/2} |a|^{2-2i}\xi^{2i}.$$

By (2.9), (3.8) and the definition of $:R_2:$ the coefficient of $q_r(x)^j$ in $:R_2(q_r): - R(q_r)$ is bounded by

$$\text{const.}\ a^{-j}(|\log(ar)|+1)^{(n-j)/2}$$

for j such that $1 \leqq j \leqq n-2$.

The coefficients are zero for $j=n$ or $n-1$. The $j=0$ coefficient is bounded by

$$\text{Const.}\ \frac{1}{a^2}(|\log(ar)|+1)^{n/2}.$$

Hence,

$$\begin{aligned}-M(1+|\log(r)|)^{n/2} \leqq D(q_r(x)) + p^2 :R_2(q_r(x)):_\beta - p^2 R(q_r(x)) \\ + (b_2-\beta)\,(:q_r^2(x):_\beta - q_r(x)^2).\end{aligned} \tag{4.8}$$

The lemma now follows by adding (4.7) and (4.8).

By Lemma 4.2 and the uniform boundedness of the coefficients of the interaction, it follows from the proof in [1, Theorem 2.10] that

$$\int \mathscr{E}_\Lambda(p^2 R_2)\, dq_{\Lambda',m_1^2} \leqq e^{K|V|}. \tag{4.9}$$

To obtain the sharper bound (4.1) we proceed as follows.

Let $Y \subset V$ be a union of lattice squares $\Delta \subset Y$. Define $\psi_\Delta = \mathscr{E}_\Delta(p^2 R_2) - 1$. Then

$$\begin{aligned}\int \mathscr{E}_V(p^2 R_2)\, dq_{\Lambda',m_1^2} &= \int \prod_\Delta (\psi_\Delta + 1)\, dq_{\Lambda',m_1^2} \\ &= \sum_{Y\subset V} \int \prod_{\Delta\subset Y} \psi_\Delta dq_{\Lambda',m_1^2}.\end{aligned}$$

Lemma 4.3. *Given $\varepsilon > 0$, there is a μ_0 such that if $\mu > \mu_0$ then*

$$\left|\int \prod_{\Delta_i \subset Y} \psi_{\Delta_i} dq_{\Lambda',m_1^2}\right| \leqq \varepsilon^{|V|}.$$

Proof. Let $R_{2i} = \int_{\Delta_i} :R_2(q_r(x)):dx$. By the fundamental theorem of calculus

$$\psi_{\Delta_i} = \int_0^1 ds_i p^2 R_{2i}\, \mathscr{E}_{\Delta_i}(s_i p^2 R_2).$$

We estimate $\Pi_{\Delta_i \subset Y} \psi_{\Delta_i}$ by applying a Hölder inequality to separate the exponential from the polynomial

$$\left|\int \prod_{\Delta_i \subset Y} \psi_{\Delta_i} dq_{\Lambda',m_1^2}\right| \leqq \left\|\prod_i R_{2i}\right\|_{p'} \sup_{0 \leqq s_i \leqq 1} \left\|\mathscr{E}\left(p \sum_i s_i R_i\right)\right\|_p .$$

By (4.9)

$$\|\mathscr{E}(pR_2)\|_p \leqq e^{K|V|} .$$

Using standard estimates, see [1, Theorem 2.5], and the fact that the coefficients of R_2 are $O(a^{-1})$ it follows that

$$\left\|\prod_i R_{2i}\right\|_{p'} \leqq \left|\frac{\text{Const.}}{a}\right|^{|Y|} . \tag{4.10}$$

Hence the proof is complete since $a^{-1} \to 0$ as $\mu \to \infty$. By Lemma 4.3 we can bound (4.9) by

$$\begin{aligned} \sum_{Y \subset V} \varepsilon^{|Y|} &= \sum_{m=0}^{|V|} \binom{|V|}{m} \varepsilon^m \\ &= (1+\varepsilon)^{|V|} \\ &\leqq e^{|V|\varepsilon} . \end{aligned}$$

This completes the proof of 3.10.

To establish (3.11) we apply Lemma 4.3 to a single lattice square Δ. Hence

$$\begin{aligned} \left|\int \mathscr{E}_\Delta(\sigma R_1)\, dq_{\Lambda',m_1^2}\right| &= \left|1 + \int \mathscr{E}_\Delta(\sigma R_1)\, dq_{\Lambda',m_1^2} - 1\right| \\ &\geqq 1 - \varepsilon . \end{aligned}$$

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§ 5. A Correction

In this section we correct a gap in the proof of the estimate

$$\int_0^\infty e^{-m_0^2 T} \int_{W(l')} dz_{xy}^T dT \leqq \text{Const.}\, e^{-m_0|l|} \tag{5.1}$$

which appeared in [3] p. 229. We thank Joel Feldman for pointing out this gap and H. P. McKean for helpful suggestions.

For each path $z(\cdot)$ let $\tau_i(z(\cdot))$ be the first time $t \geqq 0$ such that $z(t) \in b_i'$. We index the b_i' so that $W(l') = \{z \,|\, \tau_i \leqq \tau_{i+1} \leqq \dots T\}$, see [3, p. 229]. Let

P_x be the Wiener probability given that paths start at x at time zero. Then the left side of (5.1) equals

$$\int_0^\infty \int \int_{\substack{\Sigma t_i \geqq T \\ t_i \geqq 0}} e^{-m_0^2 T} P_x\{\tau_1 \leqq \tau_2 \leqq \cdots \tau_m \leqq T; z(T) \in dy'\} \, \delta(y-y')\, dT$$
$$= \int_0^\infty \int \int e^{-m_0^2 T} P_x\{\tau_1 \in dt_1, \tau_2 - \tau_1 \in dt_2 \cdots; z(T) \in dy'\} \, \delta(y-y')\, dT. \tag{5.2}$$

By the strong Markov property and the fact that τ_i depends only on paths up to time $\Sigma t_i \equiv T - T'$, (5.2) is bounded by

$$\int_{T', t_i \geqq 0} e^{-m_0^2 (T' + \Sigma t_i)} P_x\{\tau_1 \in dt_1, \tau_2 - \tau_1 \in dt_2 \ldots\} \sup_{\xi_m \in b'_m} p(T', \xi_m, y)\, dT'$$
$$\leqq E_x\{e^{-m^2 \tau_1} e_+^{-m_0^2(\tau_2 - \tau_1)} \ldots\} \sup_{\xi_m \in b'_m} \hat{p}(m_0^2, \xi_m, y). \tag{5.3}$$

where E_x is the expectation corresponding to P_x, $\hat{p}$ denotes the Laplace transform of the transition probability and $e_+^x \equiv e^x$ for $x \geqq 0$ and $\equiv 0$ for $x < 0$. Let $T_1 = \tau_1$ and let

$$T_i = \inf\{t \mid z(t + \tau_{i-1}) \in b'_i\}, \qquad i \geqq 2.$$

Note that $T_i = \tau_i - \tau_{i-1}$ when $\tau_i - \tau_{i-1} \geqq 0$. Using this fact and the strong Markov property (5.3) is bounded by

$$E_x[e^{-m_0^2 \tau_1} e_+^{-m_0^2(\tau_2 - \tau_1)} \ldots E_{z(\tau_{n-1})} e_+^{-m_0^2 T_m}] \sup_{\xi_m \in b'_m} \hat{p}$$
$$\leqq E_x\{e^{-m_0^2 T_1} E_{z(\tau_2)}\{e^{-m_0^2 T_2} \ldots\} \ldots\} \sup_{\xi_m \in b'_m} \hat{p}$$
$$\leqq \sup_{\substack{\xi_i \in b'_i \\ \xi_0 = x}} \Pi E_{\xi_{i-1}}(e^{-m_0^2 T_i}) \hat{p}.$$

Since $\hat{p}$ is the resolvent kernel of $(-\Delta + m_0^2)$ it is bounded as before. To bound the expectations we define T_i^* to be the first time a path hits an infinite line separating b'_{i-1} and b'_i. Note that for paths starting at $\xi_{i-1} \in b'_{i-1}$, $T_i^* \leqq T_i$ and that T_i^* is effectively a stopping time for a one-dimensional Brownian motion whose expectation may be computed. Thus

$$E_{\xi_{i-1}}(e^{-m_0^2 T_i}) \leqq E_{\xi_{i-1}}(e^{-m_0^2 T_i^*}) = e^{-m_0 r_i}$$

where r_i is the distance between ξ_{i-1} and the infinite line. See [12, p. 27]. By optimizing r_i our proof is complete.

Remark. We have used $-\Delta$ as the generator for Brownian motion rather than $-\frac{1}{2}\Delta$.

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