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Irreducible Multiplier Corepresentations of the Extended Poincaré Group

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Abstract. The irreducible multiplier corepresentations of the extended Poincaré group $\mathscr{P}$ are, for positive and zero mass, determined by generalized inducing from a generalized little group. This approach is compared with the previous one of Wigner. For $m>0$, and any spin j, a particular realization is noted which is manifestly covariant on all four components of $\mathscr{P}$. The choice of covering group for $\mathscr{P}$ is discussed, and reasons are given for preferring a group for which S and T generate the quaternion group of order 8.

§ 1. Introduction

1.1. In this paper we consider, following Parthasarathy [7], Lever [4] and Shaw and Lever [10], a new approach* to the problem of determining all the physically relevant irreducible *multiplier corepresentations* (see, for example, [10]) of the extended Poincaré group $\mathscr{P}$ (and hence of determining the corresponding irreducible PUA-representations – see [7] – of $\mathscr{P}$). By "physically relevant" we mean those representations such that $p^2 \geqq 0, p_4 > 0$ and, in the case $p^2 = 0$ of zero mass, such that the spin is finite.

As all physicists know, the positive energy condition $p_4 > 0$ entails that time reversal T and space-time inversion $ST = -I$ must be represented by antiunitary operators, and space inversion S by a unitary operator. In other words, in the terminology of [7], we consider only those PUA-representations associated with the particular UA-decomposition

$$\mathscr{P} = \mathscr{P}^{\uparrow} \cup \mathscr{P}^{\downarrow} . \tag{1.1}$$

In this paper we will not at all discuss the problems (see [14], [3]) of the physical interpretation or existence of the discrete symmetry operators. Our object instead is to clarify the possible mathematical approaches to the problem alluded to in the opening paragraph. In particular we will describe a new method of attack on the problem which

* *Note Added in Proof:* See [15] for a simplified account of the approach in the case of non-zero mass.

has considerable virtues of clarity and simplicity over the more customary approach.

In § 2 we outline the usual way (essentially that of Wigner [12], [14]) of tackling the problem. First of all representations of the restricted Poincaré group $\mathscr{P}_+^\uparrow$ are obtained, using for mass $m>0$ the little group $SU(2)$ and for $m=0$ the little group $\mathscr{E}$ [as defined in Eq. (2.7)]. The difficulty in this method now comes in adjoining the reflection operators.

The virtue of the new method, as described in § 3, is that the reflections are incorporated already at the little group level. In particular there is no need to induce up to the Poincaré group level in order to determine (see § 4.4 of [10]) the Wigner type (also the commutant $[U]$) of the corepresentation, and so settle the question of whether or not a doubling (or even, in the case $m=0$, $j\neq 0$, a quadrupling) of spin states occurs for a given 4-momentum.

Physically it should already be obvious what the generalized little groups must be. For under the usual Wigner-Lüders interpretation of time reversal as motion reversal, the representatives of both S and T will send a state of 4-momentum $p_0=(0,0,0,m)$ into another one of 4-momentum p_0. Bearing in mind that S, T commute with spatial rotations, the generalized little group L_{p_0} in the case $m>0$ can be taken to be the direct product $SU(2)\times F_4$, where F_4, the discrete group generated by S and T, can be identified[1] with the Klein 4-group $\{e, a, b, c\}$:

$$e=I\,, \qquad a=ST, \qquad b=T, \qquad c=S\,. \tag{1.2}$$

Similarly in the case $m=0$ we see that the generalized little group of $p_0=(0,0,1,1)$ is generated by $\mathscr{E}$ together with the y-reversal $(x_1, x_2, x_3, x_4)\mapsto(x_1, -x_2, x_3, x_4)$ and space-time inversion $x\mapsto -x$, and hence is given as in Eq. (3.16).

It is quite an easy matter to determine the relevant irreducible multiplier corepresentations of these generalized little groups. By applying Mackey's theory ([5], [6]) of group representations, generalized as in [4], [7], and [10] so as to apply to the case when some group elements are represented antiunitarily, we thereby obtain all the desired irreducible multiplier corepresentations of $\mathscr{P}$. (It should be noted that § 5 of [7] contains an oversight, in that an incorrect action upon the characters is employed, leading to the incorrect choice $SU(2)\times F_2$ for the generalized little group in the physically relevant – Eq. (1.1) – $m>0$ case.)

Whether one adjoints the reflection operators at the Poincaré or at the little group level, "Clifford's Theorem" and its generalizations

[1] The notation is chosen so that the element a in Eqs. (1.2) can be set equal to the fixed element $a\in G^-$ used throughout [10].

will be found useful[2], in that it relates representations of a group to those of a subgroup of index 2. As usually quoted (see for example Theorem 13.3 in Boerner [1]) it refers to ordinary representations; its generalization to multiplier representations is easily obtained (see § 3.4 of [10]). Of more point is the generalization to ordinary corepresentations – this was in part sketched already in § 6 of Clifford's paper [2], but given in full detail by Wigner ([13], Chapter 26), with a resulting classification of ordinary irreducible corepresentations into three types I, II, and III. Actually what is needed is a combination of both generalizations, so as to apply to irreducible multiplier corepresentations; this was carried out in [10] – see especially Theorem B – and again there are three Wigner types I, II, and III.

Before describing the two methods in more detail, let us first of all describe the extended Poincaré group $\mathscr{P}$, and also a certain covering group $\tilde{\mathscr{P}}$; we will also take this opportunity to note the possible multipliers for the Klein 4-group $\mathscr{P}/\mathscr{P}_+^\uparrow$ and for $\tilde{\mathscr{P}}$, and to determine all the irreducible multiplier corepresentations of the Klein 4-group.

In § 4 we will make out a case for preferring another covering group of $\mathscr{P}$. However $\tilde{\mathscr{P}}$ is good enough for most purposes.

1.2. The extended Poincaré group $\mathscr{P}$, and its identity component $\mathscr{P}_+^\uparrow$, are semi-direct products

$$\mathscr{P} = \mathscr{T} \odot \mathscr{L} , \qquad \mathscr{P}_+^\uparrow = \mathscr{T} \odot \mathscr{L}_+^\uparrow , \tag{1.3}$$

of the abelian invariant subgroup $\mathscr{T}$, consisting of all the spacetime translations, with the appropriate homogeneous Lorentz group $\mathscr{L}$ or $\mathscr{L}_+^\uparrow$. As usual, it helps instead to work with the simplyconnected covering group of $\mathscr{P}_+^\uparrow$, namely

$$\tilde{\mathscr{P}}_+^\uparrow = \mathscr{T} \odot SL(2, \mathbf{C}) , \tag{1.4}$$

with multiplication law defined by

$$(x, A)(x', A') = (x + \Lambda(A)\, x', AA') , \tag{1.5}$$

where $A \mapsto \Lambda(A)$ denotes the familiar covering homomorphism from $SL(2, \mathbf{C})$ onto $\mathscr{L}_+^\uparrow$, with kernel $Z_2 = \{I, -I\}$.

Now the extended Lorentz group is a semi-direct product

$$\mathscr{L} = \mathscr{L}_+^\uparrow \odot F_4 , \qquad F_4 = \{I, ST, T, S\} , \tag{1.6}$$

where F_4 is isomorphic to the Klein 4-group, as in Eq. (1.2), and where $F \in F_4$ acts upon $\mathscr{L}_+^\uparrow$ by inner automorphism: $\Lambda \mapsto F\Lambda F^{-1}$. Since

[2] Of course – see § 4 of [10] – Clifford's Theorem, and its generalization to corepresentations, can be viewed as very simple instances of the ordinary, and generalized, Mackey theory.

$SL(2, \mathbf{C})$ is the universal covering group of $\mathscr{L}_+^\uparrow$, there is a corresponding unique automorphic action $A \mapsto F(A)$ of $F \in F_4$ upon $SL(2, \mathbf{C})$ which satisfies

$$\Lambda(F(A)) = F\Lambda(A)F^{-1}. \tag{1.7}$$

Hence we may define a covering group $\tilde{\mathscr{L}}$ of $\mathscr{L}$ by

$$\tilde{\mathscr{L}} = SL(2, \mathbf{C}) \odot F_4. \tag{1.8}$$

Explicitly the action of F_4 upon $SL(2, \mathbf{C})$ is given by

$$S(A) = T(A) = (A^\dagger)^{-1}, \quad ST(A) = A. \tag{1.9}$$

We can now define a covering group $\tilde{\mathscr{P}}$ of $\mathscr{P}$ by

$$\tilde{\mathscr{P}} = \mathscr{T} \odot \tilde{\mathscr{L}}, \tag{1.10}$$

where the action of $(A, F) \in \tilde{\mathscr{L}}$ upon $\mathscr{T}$ is given by

$$(A, F)x = \Lambda(A)Fx. \tag{1.11}$$

In full detail the multiplication law for the group $\tilde{\mathscr{P}} = \mathscr{T} \odot (SL(2, \mathbf{C}) \odot F_4)$ thus reads

$$(x, A, F)(x', A', F') = (x + \Lambda(A)Fx', AF(A'), FF'), \tag{1.12}$$

and the covering homomorphism $\Lambda: \tilde{\mathscr{P}} \to \mathscr{P}$ is given by

$$(x, A, F) \mapsto (x, \Lambda(A), F). \tag{1.13}$$

It is known (see for example Corollary 1 on page 51 of [8], or Theorem 10.40 of [11]) that every multiplier of $\tilde{\mathscr{P}}_+^\uparrow$ is trivial, and that accordingly the projective unitary representations of $\mathscr{P}_+^\uparrow$ are in $1-1$ correspondence with the ordinary unitary representations of $\tilde{\mathscr{P}}_+^\uparrow$. It is also known (see Table 1 in [14], or Table (ii) in [7]) that every multiplier for the Klein 4-group $\{e, a, b, c\}$, with respect to the UA-decomposition $\{e, c\} \cup \{a, b\}$ (see Eq. (1.2) in [10]), is equivalent to one of the four multipliers $\sigma^{\alpha\beta}$ (where $\alpha = \pm 1, \beta = \pm 1$) given by Table 1.

Piecing together these two items of information, and using the facts that $\tilde{\mathscr{P}}$ can also be viewed as the semi-direct product $\tilde{\mathscr{P}}_+^\uparrow \odot F_4$, and that $\tilde{\mathscr{P}}_+^\uparrow$ has no non-trivial characters, one can prove (see Lemma 5.2 of [7])

Table 1. The inequivalent multipliers of the Klein 4-group (with $G^+ = \{e, c\}$)

$\sigma^{\alpha\beta}$	e	c	b	a
e	1	1	1	1
c	1	1	1	1
b	1	$\alpha\beta$	α	β
a	1	$\alpha\beta$	α	β

that every multiplier for $\tilde{\mathscr{P}}$, with respect to the UA-decomposition $\tilde{\mathscr{P}}^{\uparrow} \cup \tilde{\mathscr{P}}^{\downarrow}$, is equivalent to one of the four multipliers $\sigma^{\alpha\beta}$ defined by

$$\sigma^{\alpha\beta}((x, A, F), (x', A', F')) = \sigma^{\alpha\beta}(F, F') . \tag{1.14}$$

1.3. Using Theorem B of [10], it is an easy matter to determine, for each choice of multiplier $\sigma^{\alpha\beta}$ in Table 1, all the irreducible $\sigma^{\alpha\beta}$-corepresentations of the Klein 4-group, up to unitary equivalence. [There are of course only two choices $D^{\pm}$ (both of dimension 1) for the irreducible representation D of $H \equiv \{e, c\}$, namely D^{+}, D^{-}, where $D^{\pm}(c) = \pm 1$.] The results are displayed in the following table:

Table 2. The irreducible $\sigma^{\alpha\beta}$-corepresentations $U^{\eta}_{\alpha\beta}$ of the Klein 4-group (the label $\eta = \pm 1$ being required only in the two cases $\alpha = \beta$)

Value of α	Value of β	Wigner type	Dimension	$U^{\pm}_{\alpha\beta}(c)$	$U^{\pm}_{\alpha\beta}(b)$	$U^{\pm}_{\alpha\beta}(a)$
1	1	I	1	$\pm I$	$\pm\kappa$	κ
-1	-1	II	2	$\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\pm\begin{pmatrix}0 & -\kappa\\ \kappa & 0\end{pmatrix}$	$\begin{pmatrix}0 & -\kappa\\ \kappa & 0\end{pmatrix}$
1 / -1	-1 / 1	III	2	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}0 & \beta\kappa\\ -\kappa & 0\end{pmatrix}$	$\begin{pmatrix}0 & \beta\kappa\\ \kappa & 0\end{pmatrix}$

The carrier space has been taken to be $\boldsymbol{C}$ in the case $\alpha = \beta = 1$ and to be $\boldsymbol{C} \oplus \boldsymbol{C}$ in the other cases, with $\kappa: \boldsymbol{C} \to \boldsymbol{C}$ denoting complex conjugation $\lambda \mapsto \bar{\lambda}$.

Observe that each of the four choices $\sigma^{\alpha\beta}$ of multiplier gives rise to just one irreducible PUA-representation $\boldsymbol{U}_{\alpha\beta}$. In the two cases $\alpha = \beta$, but *not* in the two cases $\alpha = -\beta$, observe that $\boldsymbol{U}_{\alpha\beta}$ possesses two unitarily inequivalent versions $U^{+}_{\alpha\beta}$, $U^{-}_{\alpha\beta}$ (obtained by choosing the upper and lower signs in the first two entries in the table) – both $U^{+}_{\alpha\beta}$, $U^{-}_{\alpha\beta}$ having the same multiplier $\sigma^{\alpha\beta}$.

Let us analyse this last situation a little further. As discussed in § 1.3 of [10, for a given multiplier σ for a group G one can be interested in classifying the irreducible σ-corepresentations of G up to (ordinary) unitary equivalence – and not merely up to projective unitary equivalence. In general we obtain thereby a finer classification, since (as we have just seen) a PUA-representation $\boldsymbol{U}$ may possess versions U_1, U_2, having moreover the same multiplier $\sigma_1 = \sigma_2 = \sigma$, which are unitarily inequivalent.

Now two σ-corepresentations U_1, U_2 are versions of the same PUA-representation $\boldsymbol{U}$ if and only if they satisfy

$$U_2(g) = \lambda(g)\, U_1(g) \tag{1.15}$$

for some *generalized character* λ for G:

$$\lambda(g)\, \lambda(g')^g = \lambda(gg')\,, \qquad (|\lambda(g)| = 1)\,. \tag{1.16}$$

It may or may not be the case that U_1 and U_2 are unitarily equivalent. When λ is a *trivial generalized character*, i.e. one of the form λ_α (for some $\alpha \in \boldsymbol{C}$ of unit modulus) given by

$$\lambda_\alpha(g) = \begin{cases} 1, & \text{if} \quad g \in G^+\,, \\ \alpha, & \text{if} \quad g \in G^-\,, \end{cases} \tag{1.17}$$

then U_1 and U_2 are always unitarily equivalent, since we obtain

$$U_2(g) = P U_1(g)\, P^{-1}\,, \tag{1.18}$$

upon taking $P = \beta I$, with $\beta^2 = \alpha$. Clearly we are only interested in determining the generalized characters of a group G up to *equivalence*, two characters λ, λ' being defined to be equivalent if $\lambda' = \lambda_\alpha \lambda$ for some trivial character λ_α.

Up to equivalence the Klein 4-group has just 2 generalised characters (with respect to the UA-decomposition $\{e, c\} \cup \{a, b\}$) λ^+, λ^-, given by

$$\lambda^\pm(a) = 1\,, \qquad \lambda^\pm(b) = \pm 1\,, \qquad \lambda^\pm(c) = \pm 1\,. \tag{1.19}$$

As indicated in the above table, an irreducible multiplier corepresentation U of $\{e, a, b, c\}$ of type I or II, but not of type III, is unitarily inequivalent to $\lambda^- U$.

§ 2. Irreducible Multiplier Corepresentations of $\tilde{\mathscr{P}}$ — Method 1

2.1. If U is an irreducible $\sigma^{\alpha\beta}$-corepresentation of $\tilde{\mathscr{P}}$, then (by considering $U \downarrow \tilde{\mathscr{P}}^\uparrow$ – see Theorem A in [10] – and using Schur's lemma) we find that the element $-I \in SL(2, \boldsymbol{C})$, which belongs to the centre of $\tilde{\mathscr{P}}$, is represented by $\pm I$. It follows that the irreducible PUA-representations $\boldsymbol{V}$ of $\mathscr{P}$ are in $1-1$ correspondence with the irreducible PUA-representations $\boldsymbol{U}$ of $\tilde{\mathscr{P}}$ by means of the relation $\boldsymbol{U} = \boldsymbol{V} \circ \Lambda$, with Λ as in Eq. (1.13).

Now the only representations of $\tilde{\mathscr{P}}_+^\uparrow$ of dimension 1 is the trivial one. Hence, by § 1.3, if U_1, U_2 are distinct $\sigma^{\alpha\beta}$-corepresentations of $\tilde{\mathscr{P}} = \tilde{\mathscr{P}}_+^\uparrow \odot F_4$ which are versions of the PUA-representation $\boldsymbol{U}$ of $\tilde{\mathscr{P}}$, then, up to unitary equivalence, the only possibility is that U_2 and U_1 are related by

$$U_2(x, A, F) = \lambda^-(F)\, U_1(x, A, F).$$

Of course it *may* still be the case that U_1 and U_2 are unitarily equivalent. Corresponding to U_1, U_2 we can find versions V_1, V_2, having the same multiplier $\omega^{\alpha\beta}$, say, of the PUA-representation V of $\mathscr{P}$ such that

$$U_1 = V_1 \circ \Lambda, \qquad U_2 = V_2 \circ \Lambda \qquad \text{and} \qquad V_2(x, \Lambda, F) = \lambda^-(F)\, V_1(x, \Lambda, F).$$

Since the only representation of $\mathscr{P}_+^\uparrow$ of dimension 1 is the trivial one, up to unitary equivalence there are no other versions of V with multiplier $\omega^{\alpha\beta}$ other than V_1, V_2. Thus in order to determine the irreducible multiplier corepresentations of $\mathscr{P}$, it suffices to determine the irreducible $\sigma^{\alpha\beta}$-corepresentations of $\tilde{\mathscr{P}}$ for each of the four choices $\alpha = \pm 1$, $\beta = \pm 1$ of $\sigma^{\alpha\beta}$.

2.2. Irreducible Unitary Representations of $\tilde{\mathscr{P}}_+^\uparrow$. Restricting our attention now to $\tilde{\mathscr{P}}_+^\uparrow$, and recalling that $\sigma^{\alpha\beta}$, thus restricted, is $\equiv 1$, we see that the PU-representations of $\mathscr{P}_+^\uparrow$ are in $1-1$ correspondence with the ordinary unitary representations of $\tilde{\mathscr{P}}_+^\uparrow$. Since the latter group is a regular semi-direct product $\mathscr{T} \odot SL(2, \boldsymbol{C})$, its unitary representations are most powerfully determined by applying Mackey's theory of induced representations. Although the details are very well known, we repeat them here so as to allow the generalization in § 3 to stand out in full clarity.

The characters $\chi_p \in \hat{\mathscr{T}}$ (= the dual group of $\mathscr{T}$) are of the form $\chi_p(x) = \exp(ip \cdot x)$. The natural action of $A \in SL(2, \boldsymbol{C})$ upon $\mathscr{T}$ is $\chi \mapsto A\chi$, where

$$(A\chi)(x) = \chi(\Lambda(A^{-1})\, x); \tag{2.1}$$

in terms of the 4-momentum p rather the character χ_p it reads simply

$$p \mapsto \Lambda(A)\, p. \tag{2.2}$$

The isotropy group G_p of χ_p is accordingly

$$G_p = \mathscr{T} \odot L_p, \tag{2.3}$$

where the little group L_p is given by

$$\begin{aligned} L_p &= \{A : A\chi_p = \chi_p, A \in SL(2, \boldsymbol{C})\} \\ &= \{A : \Lambda(A)\, p = p, A \in SL(2, \boldsymbol{C})\}. \end{aligned} \tag{2.4}$$

On each of the physically relevant ($m \geqq 0$) orbits $p^2 = m^2$, $p_4 > 0$ we choose a particular 4-momentum p_0. The $[m, j]$ representation of $\tilde{\mathscr{P}}_+^\uparrow$ is then obtained as the induced representation

$$U^{m,j} = (\chi_{p_0} U^j) \uparrow \tilde{\mathscr{P}}_+^\uparrow, \tag{2.5}$$

where U^j is a (finite-dimensional) irreducible unitary representation of $L_{p_0} = G_{p_0}/\mathcal{T}$. If $m > 0$ and we take $p_0 = (0, 0, 0, m)$, then the little group is

$$L_{p_0} = SU(2) \tag{2.6}$$

and U^j is the familiar spin j representation D^j of dimension $2j+1$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. If $m = 0$ and we take $p_0 = (0, 0, 1, 1)$, then the little group is

$$L_{p_0} = \mathcal{E} = \left\{ \begin{pmatrix} \omega & \bar{\omega}\zeta \\ 0 & \bar{\omega} \end{pmatrix}, \omega, \zeta \in \boldsymbol{C}, |\omega| = 1 \right\} \tag{2.7}$$

and the (physically relevant) irreducible unitary representations U^j are the 1-dimensional ones V^j given by

$$V^j(A_{\zeta,\omega}) = \omega^{2j} \quad (j = 0, \pm\tfrac{1}{2}, \pm 1, \pm\tfrac{3}{2}, \ldots), \tag{2.8}$$

where $A_{\zeta,\omega}$ denotes the $SL(2, \boldsymbol{C})$ matrix in Eq. (2.7).

[The group $\mathcal{E}/Z_2$ is isomorphic to the non-compact group of proper Euclidean motions in the plane; only when the "translations" $A_{\zeta,1}$ are represented trivially can we obtain a finite- (in fact a one-) dimensional irreducible unitary representation; the infinite-dimensional representations of $\mathcal{E}$ are ruled out by the physical requirement that, for a given 4-momentum, only a finite number of linearly independent spin states should be possible.]

2.3. Adjoining the Reflections. The problem now is to construct all those irreducible multiplier corepresentations of $\tilde{\mathscr{P}}$ which decompose on restriction to $\tilde{\mathscr{P}}_+^\uparrow$ into direct sums of the representations [m, j]. This problem was solved by Wigner [14], who first of all adjoined space inversion S – using essentially "Cliffords Theorem" (see § 3.4 of [10]) – and then carried on to adjoin time reversal T and space-time inversion ST – using essentially Theorem B of [10]. However, operating as Wigner does at the Poincaré group level, the problem is far from trivial; even the adjoining of S can involve a "surprising amount of computation" (Wigner [14], § 8). In § 3 we will demonstrate how much simpler it is to adjoint the reflections at the little group level, using "generalized inducing" from a "generalized little group".

2.4. Irreducible Unitary Representations of $\tilde{\mathscr{P}}^\uparrow$. Actually if we are only interested in the lesser problem of adjoining space inversion, then only ordinary inducing is involved; it may therefore be worthwhile seeing how simple this lesser problem becomes at the little group level before moving on to generalized inducing. For $m > 0$ the little group is now

$$L_{p_0} = SU(2) \times F_2\,, \quad (F_2 = \{I, S\})\,, \tag{2.9}$$

with irreducible representions[3] $D^{j\pm}$ given by

$$D^{j\pm}(A, F) = D^j(A)\, D^\pm(F)\,, \quad \text{where} \quad D^\pm(S) = \pm 1\,, \tag{2.10}$$

[3] According to Eq. (1.14) and Table 1, $\sigma \equiv 1$ for $\tilde{\mathscr{P}}^\uparrow$.

and having therefore the same dimension $2j+1$ as D^j. Thus by operating at the little group level it is immediate that no doubling of dimension occurs upon incorporating space inversion. Upon inducing we obtain the required representations $U^{m,j,\pm}$, say, of $\tilde{\mathscr{P}}^{\uparrow}$:

$$U^{m,j,\pm} = (\chi_{p_0} D^{j\pm}) \uparrow \tilde{\mathscr{P}}^{\uparrow} . \tag{2.11}$$

Of course $U^{m,j,+}$ is projectively equivalent to $U^{m,j,-}$, and so both representations define the same PUA-representation $\boldsymbol{U}^{m,j}$ of $\tilde{\mathscr{P}}^{\uparrow}$.

For $m=0$ the little group is now

$$L_{p_0} = \mathscr{E} \cup Y\mathscr{E} , \tag{2.12}$$

where $\Lambda(Y)$ is the "y-reversal" $(x_1, x_2, x_3, x_4) \to (x_1, -x_2, x_3, x_4)$. Thus

$$Y = (\Pi_y, S) = (i\sigma_y, S) \tag{2.13}$$

where $\Lambda(\Pi_y)$ has to be the π-rotation $(x_1, x_2, x_3, x_4) \to (-x_1, x_2, -x_3, x_4)$, and so $\Pi_y = \pm i\sigma_y$. Take note of the properties

$$Y^2 = (\Pi_y^2, S^2) = (-I, I) \tag{2.14}$$

and, writing $(A, I) \in SL(2, \boldsymbol{C}) \times F_4$ simply as A,

$$Y^{-1} A Y = \Pi_y^{-1} (A^{\dagger})^{-1} \Pi_y = \bar{A} , \qquad A \in SL(2, \boldsymbol{C}) , \tag{2.15}$$

(assuming the usual choice of Pauli matrices, viz. σ_x, $i\sigma_y$, σ_z all real).

The conjugate[3] by Y of the representation V^j of $\mathscr{E}$ is therefore V^{-j}:

$$(Y V^j)(A) = V^j(\bar{A}) = V^{-j}(A) , \qquad \text{for} \qquad A \in \mathscr{E} . \tag{2.16}$$

Hence, for non-zero helicity j, a doubling must occur upon adjoining Y, the relevant irreducible representations[3] being the 2-dimensional ones W^j, $j = \frac{1}{2}, 1, \frac{3}{2}, \ldots$, (of Clifford type "III" – see [10], § 3.4) defined by

$$W^j(A_{\zeta,\omega}) = \begin{pmatrix} \omega^{2j} & 0 \\ 0 & \bar{\omega}^{2j} \end{pmatrix} , \qquad W^j(Y) = \begin{pmatrix} 0 & (-)^{2j} \\ 1 & 0 \end{pmatrix} , \tag{2.17}$$

the sign $(-)^{2j}$ coming from $V^j(Y^2) = V^j(-I) = (-)^{2j}$. For zero helicity no doubling occurs (i.e. Clifford type "$I\pm$"), the relevant irreducible representations being the 1-dimensional ones W^{0+}, W^{0-} defined by

$$W^{0\pm}(A) = 1 , \qquad W^{0\pm}(Y) = \pm 1 . \tag{2.18}$$

§ 3. Irreducible Multiplier Corepresentations of $\tilde{\mathscr{P}}$ — Method 2

3.1. The Action of $\tilde{\mathscr{L}}$ upon $\hat{\mathscr{T}}$. In order to carry out our plan of obtaining the irreducible $\sigma^{\alpha\beta}$-corepresentations of $\tilde{\mathscr{P}}$ by generalized inducing from a generalised little group, we need to be aware of the correct generalization[4] of Eqs. (2.1), (2.2) – i.e. we need to determine the relevant action of $g \in \tilde{\mathscr{L}}$ upon $\hat{\mathscr{T}}$. As noted in § 4.2 of [10], for the given UA-decomposition $\tilde{\mathscr{P}}^{\uparrow} \cup \tilde{\mathscr{P}}^{\downarrow}$ the relevant action is mathematically forced upon us, and – setting $\sigma = \sigma^{\alpha\beta}$ in Eq. (4.1) of [10] – reads

$$(g\chi)(x) = \begin{cases} \chi(\Lambda(g^{-1})\,x), & g \in \tilde{\mathscr{L}}^{\uparrow}, \\ \overline{\chi(\Lambda(g^{-1})\,x)}, & g \in \tilde{\mathscr{L}}^{\downarrow}, \end{cases} \tag{3.1}$$

where, for $g = (A, F) \in \tilde{\mathscr{L}} = SL(2, \boldsymbol{C}) \odot F_4$, we have written $\Lambda(g) = \Lambda(A) \cdot F \in \mathscr{L}$. In terms of p rather than χ_p it reads

$$p \mapsto \varepsilon(g)\,\Lambda(g)\,p \tag{3.2}$$

where $\varepsilon(g) = 1$, if $g \in \tilde{\mathscr{L}}^{\uparrow}$, and $= -1$, if $g \in \tilde{\mathscr{L}}^{\downarrow}$.

On account of the presence of $\varepsilon(g)$ in the last equation [i.e. of complex conjugation in Eq. (3.1)], note that the $\tilde{\mathscr{L}}$ – orbits in $\hat{\mathscr{T}}$ coincide with the $\tilde{\mathscr{L}}\updownarrow$-orbits, and hence satisfy $p_4 > 0$ in the physically relevant cases. Physically of course the argument is in the reverse direction – the requirement $p_4 > 0$ of positive energy forces us to adopt the UA-decomposition $\mathscr{P} = \mathscr{P}^{\uparrow} \cup \mathscr{P}^{\downarrow}$.

The (generalized) isotropy group G_p of χ_p is accordingly

$$G_p = \mathscr{T} \odot L_p, \tag{3.3}$$

where the *generalized little group* L_p is given by

$$L_p = \{g : \Lambda(g)\,p = \varepsilon(g)\,p, \quad g \in \tilde{\mathscr{L}}\}. \tag{3.4}$$

Corresponding to Eq. (3.5), we now obtain the irreducible $\sigma^{\alpha\beta}$-corepresentation $U^{mj\eta}_{\alpha\beta}$ of $\tilde{\mathscr{P}}$ by *generalized inducing* (cf. § 4 of [10]):

$$U^{mj\eta}_{\alpha\beta} = (\chi_{p_0} U^{j\eta}_{\alpha\beta}) \uparrow \tilde{\mathscr{P}} \tag{3.5}$$

where $U^{j\eta}_{\alpha\beta}$ is an irreducible $\sigma^{\alpha\beta}$-corepresentation of $L_{p_0} = G_{p_0}/\mathscr{T}$, the label j having the usual (spin or helicity) significance, as in § 2.2, and the label η (when it is required – see later) taking the values $+1, -1$. We will now determine the generalized little group L_{p_0}, and its irreducible-corepresentations $U^{j\eta}_{\alpha\beta}$, in the cases of physical interest.

3.2. The Case $m > 0$. Taking $p_0 = (0, 0, 0, m)$ again, it follows from Eq. (3.4) that the generalized little group is

$$L_{p_0} = SU(2) \times F_4, \tag{3.6}$$

the product being direct since, by Eq. (1.9),

$$F(A) = A, \quad \text{for} \quad F \in F_4, \quad A \in SU(2). \tag{3.7}$$

[4] Parthasarathy (in § 5 of [7]) incorrectly adheres to the ungeneralized form of the action – i.e. omits the complex conjugation in Eq. (3.1).

One method of determining the $U^{j\eta}_{\alpha\beta}$ is to set $G=SU(2)\times F_4$, $H=SU(2)\times F_2$ [see Eq. (2.9)], $D=D^{j\eta}$ [see Eq. (2.10)], $a=ST$, $\sigma=\sigma^{\alpha\beta}$, in Theorem B of [10]. By Eq. (2.15) of [10] we need also to set $E=E^{j\pm}$, where

$$\begin{aligned} &\text{(a)}\ E^{j\pm}(A)=D^{j\pm}(A)\,(=D^j(A))\,, \qquad A\in SU(2)\,,\\ &\text{(b)}\ E^{j\pm}(S)=\alpha\beta D^{j\pm}(S)\,(=\pm\alpha\beta I)\,. \end{aligned} \tag{3.8}$$

If $\alpha\beta=-1$, then Eq. (3.8b) shows that E and D are not antiunitarily equivalent, so that $U=U^j_{\alpha\beta}$ is necessarily of Wigner type III, being given as in Eq. (2.18) of [10], with $\sigma(a,a)=\sigma^{\alpha\beta}(ST,ST)=\beta$. If $\alpha\beta=+1$, then it follows from Eq. (3.8) that $E^{j\pm}$ and $D^{j\pm}$ are antiunirarily equivalent by means of the well-known antiunitary operator K which satisfies, for $A\in SU(2)$,

$$\text{(a)}\ KD^j(A)K^{-1}=D^j(A)\,, \qquad \text{(b)}\ K^2=(-)^{2j}I\,. \tag{3.9}$$

Thus when $\alpha=\beta$ the $\sigma_{\alpha\beta}$-corepresentation $U=U^{j\pm}_{\alpha\beta}$ is necessarily of Wigner type I or II; since in our case $\sigma(a,a)D(a^2)=\beta I$, we see from Eq. (3.9b) that $U^{j\pm}_{\alpha\beta}$ is of type I if $\alpha=\beta=(-)^{2j}$ and of type II if $\alpha=\beta=-(-)^{2j}$, being given respectively by Eqs. (2.16) and (2.17) of [10].

These results are displayed in Table 3. Of course the restriction of $U^{j\eta}_{\alpha\beta}$ to $SU(2)$ is D^j (type I) and $D^j\oplus D^j$ (types II and III). Our results (after inducing up to $\tilde{\mathscr{P}}$) agree with those of Wigner (see Table 3 of [14]) – the link-up of notation being

$$\varepsilon_\theta=\sigma^{\alpha\beta}(T,T)=\alpha\,, \qquad \varepsilon_I=\sigma^{\alpha\beta}(ST,ST)=\beta\,. \tag{3.10}$$

Observe that each of the four choices $\sigma^{\alpha\beta}$ of multiplier gives rise (for a given spin j) to just one irreducible PUA-representation $\boldsymbol{U}^j_{\alpha\beta}$. In each of the two cases $\alpha=\beta=(-)^{2j}$ and $\alpha=\beta=-(-)^{2j}$, observe that two unitarily inequivalent $\sigma^{\alpha\beta}$-corepresentations $U^{j+}_{\alpha\beta}$, $U^{j-}_{\alpha\beta}$ belong to the same PUA-representation $\boldsymbol{U}^j_{\alpha\beta}$. On the other hand in the two cases $\alpha=-\beta=(-)^{2j}$ and $\alpha=-\beta=-(-)^{2j}$, $\boldsymbol{U}^j_{\alpha\beta}$ is realized (up to

Table 3. The irreducible $\sigma^{\alpha\beta}$-corepresentations $U^{j\eta}_{\alpha\beta}$ of the generalized little group $SU(2)\times F_4$

Value of		Wigner type	Dimension	Value of $U^{j\eta}_{\alpha\beta}(g)$ for g equal to		
$\alpha(-)^{2j}$	$\beta(-)^{2j}$			S	T	ST
1	1	I	$2j+1$	ηI	ηK	K
-1	-1	II	$2(2j+1)$	$\eta\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}$	$\eta\begin{pmatrix} 0 & -K\\ K & 0\end{pmatrix}$	$\begin{pmatrix} 0 & -K\\ K & 0\end{pmatrix}$
1 -1	-1 1	III	$2(2j+1)$	$\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix}$	$\begin{pmatrix} 0 & \beta K^{-1}\\ -K & 0\end{pmatrix}$	$\begin{pmatrix} 0 & \beta K^{-1}\\ K & 0\end{pmatrix}$

unitary equivalence) by just one $\sigma^{\alpha\beta}$-corepresentation $U^j_{\alpha\beta}$ (the label η not being required in this case). These facts are of course consistent with remarks made previously at the end of § 1.3 and in § 2.1.

A second method of determining the $U^{j\eta}_{\alpha\beta}$ is to set $G = G' = SU(2) \times F_4$, $H = SU(2)$, $D = D^j$, $\sigma = \sigma^{\alpha\beta}$ in § 4.4 of [10]. [That $G = G'$ – i.e. that the representation D^j is self-conjugate with respect to G: $FD^j \sim D^j$, for $F \in F_4$ – follows immediately from Eqs. (3.7), (3.9), since the relevant multipliers (in Eq. (4.1) of [10]) equal 1.] According to Eqs. (4.18), (4.19), and (4.27) of [10], we obtain $U^{j\eta}_{\alpha\beta}$ in the form

$$U^{j\eta}_{\alpha\beta} = T^j \otimes \Omega^\eta_{\alpha\beta} \tag{3.11}$$

where the irreducible τ^j-corepresentation T^j of G is given in our case by

$$T^j(A, F) = D^j(A) \begin{cases} I, & \text{if} \quad F \in F_4^\uparrow = \{I, S\}, \\ K, & \text{if} \quad F \in F_4^\downarrow = \{T, ST\}, \end{cases} \tag{3.12}$$

and where $\Omega^\eta_{\alpha\beta}$ is an irreducible $\omega^{\alpha\beta}_j$-corepresentation of F_4, with multiplier $\omega^{\alpha\beta}_j = \sigma^{\alpha\beta}/\tau^j$. Now, by Eqs. (3.9), (3.12), the multiplier of T^j is

$$\tau^j = \sigma^{(-)^{2j}(-)^{2j}}$$

and hence that of $\Omega^\eta_{\alpha\beta}$ is

$$\omega^{\alpha\beta}_j = \sigma^{\alpha\beta}/\tau^j = \sigma^{\alpha_j\beta_j} \tag{3.13}$$

where α_j, β_j are defined in terms of α, β by

$$\alpha_j = (-)^{2j}\alpha\,, \qquad \beta_j = (-)^{2j}\beta\,. \tag{3.14}$$

Hence the possible choices of the $\Omega^\eta_{\alpha\beta}$ in Eq. (3.11) are

$$\Omega^\eta_{\alpha\beta} = U^\eta_{\alpha_j\beta_j} \qquad \text{(as defined in Table 2)}\,, \tag{3.15}$$

and we thereby immediately derive the results of Table 3 from those of Table 2.

3.3. The Case $m = 0$. Taking $p_0 = (0, 0, 1, 1)$, the generalized little group is now

$$L_{p_0} = \mathscr{E} \cup Y\mathscr{E} \cup Y'\mathscr{E} \cup YY'\mathscr{E} \tag{3.16}$$

where $\Lambda(Y)$ is, as previously, y-reversal and where $\Lambda(Y') = -\Lambda(Y)$ and $\Lambda(YY') = -I = ST$. Thus we can take

$$Y = (\Pi_y, S)\,, \qquad Y' = (\Pi_y^{-1}, T)\,, \qquad YY' = (I, ST)\,, \tag{3.17}$$

with $\Pi_y(= -\Pi_y^{-1})$ as in Eq. (2.13).

One way of determining the $U^{j\eta}_{\alpha\beta}$ is to set $H = \mathscr{E} \cup Y\mathscr{E}$, $G = H \cup aH$, $a = YY'$, $D = W^{j\eta}$ [see Eqs. (2.17), (2.18)], $\sigma = \sigma^{\alpha\beta}$ in Theorem B of [10]. Consider first of all the case $j > 0$ of non-zero helicity (in which case the label η is not required). By Eq. (2.15) of [10] we need also to set $E = E^j$, where

$$\begin{aligned} &\text{(a)}\ E^j(A) = W^j(A)\,, \qquad A \in \mathscr{E}\,, \\ &\text{(b)}\ E^j(Y) = \alpha\beta W^j(Y)\,. \end{aligned} \tag{3.18}$$

It follows that, for all four choices of α, β, E^j, and W^j are antiunitarily equivalent by means of the antiunitary operator

$$K = \begin{pmatrix} 0 & (-)^{2j}\alpha\beta\kappa \\ \kappa & 0 \end{pmatrix} \quad (\kappa = \text{complex conjugation})\,. \tag{3.19}$$

Hence $U = U^j_{\alpha\beta}$ is always of Wigner type I or II. Since $K^2 = (-)^{2j}\alpha\beta$, while $\sigma^{\alpha\beta}(YY', YY')\, W^j((YY')^2) = \beta I$, we deduce that $U^j_{\alpha\beta}$ is of type I or II according as α is equal to $(-)^{2j}$ or $-(-)^{2j}$. On referring to Eqs. (2.16), (2.17) of [10], we obtain the results displayed in Table 4(a). (As in Table 3, we give, in the first two columns, the values of α_j, β_j [see Eq. (3.14)] rather than α, β; however in the matrices in the last three columns we have for reasons of space used α, β rather than their $\pm(-)^{2j}$ equivalents.) We repeat the the label η is not required in the cases $j = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ of non-zero helicity just discussed. Of course the restriction of $U^j_{\alpha\beta}$ to $\mathscr{E}$ is W^j (type I) and $W^j \oplus W^j$ (type II).

In the case $j = 0$ of zero helicity, the subgroup $\mathscr{E}$ of L_{p_0} is represented trivially, so that $U^{0\eta}_{\alpha\beta}$ is the $\sigma^{\alpha\beta}$-corepresentation $U^{\eta}_{\alpha\beta}$ of the Klein 4-group $L_{p_0}/\mathscr{E}$, as determined previously in Table 2. The results in the first four columns of Table 4 agree with those of Wigner (see Table 4 of [14]).

Table 4. The irreducible $\sigma^{\alpha\beta}$-corepresentations $U^{j\eta}_{\alpha\beta}$ of the generalized little group $\mathscr{E} \cup Y\mathscr{E} \cup Y'\mathscr{E} \cup YY'\mathscr{E}$

Value of α_j	Value of β_j	Type	Dimension	Value of $U^{j\eta}_{\alpha\beta}(g)$ for g equal to Y	Y'	$YY' = ST$
(a) *The case* $j \neq 0$:						
1	1	I	2	$\begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \kappa & 0 \\ 0 & \alpha\beta\kappa \end{pmatrix}$	$\begin{pmatrix} 0 & \beta\kappa \\ \kappa & 0 \end{pmatrix}$
1	-1					
-1	1	II	4	$\begin{pmatrix} \cdot & -\alpha & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -\alpha \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & -\kappa & \cdot \\ \cdot & \cdot & \cdot & -\alpha\beta\kappa \\ \kappa & \cdot & \cdot & \cdot \\ \cdot & \alpha\beta\kappa & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \beta\kappa \\ \cdot & \cdot & -\kappa & \cdot \\ \cdot & -\beta\kappa & \cdot & \cdot \\ \kappa & \cdot & \cdot & \cdot \end{pmatrix}$
-1	-1					
(b) *The case* $j = 0$						
1	1	I	1	($U^{0\eta}_{\alpha\beta}$ equals $U^{\eta}_{\alpha\beta}$ as given in Table 2)		
-1	-1	II	2			
1	-1	III	2			
-1	1	III	2			

§ 4. Covering Groups for $\mathscr{P}$ and Manifestly Covariant Representations

4.1. The Choice of Covering Group for the Extended Poincaré Group $\mathscr{P}$. Let $X, Y \in \mathscr{L}_-^\uparrow$ denote respectively x-reversal, y-reversal; then $\Pi_z = XY \in \mathscr{L}_+^\uparrow$ is the π-rotation $(x_1, x_2, x_3, x_4) \mapsto (-x_1, -x_2, x_3, x_4)$. Define also $X' \in \mathscr{L}_-^\downarrow$ and $\Pi_z' \in \mathscr{L}_+^\downarrow$ by $X' = -X, \Pi_z' = -\Pi_z$. Then $\check{F}_4 = \{I, \Pi_z', X', Y\}$ is a group, which can be identified with the Klein 4-group $\{e, a, b, c\}$ by

$$e = I\,, \qquad a = \Pi_z'\,, \qquad b = X'\,, \qquad c = Y\,. \tag{4.1}$$

Since $\mathscr{L}$ is the semi-direct product

$$\mathscr{L} = \mathscr{L}_+^\uparrow \odot \check{F}_4\,, \tag{4.2}$$

(where $F \in \check{F}_4$ acts by inner automorphism: $\Lambda \mapsto F\Lambda F^{-1}$) we can [cf. Eqs. (1.6)–(1.9)] define a covering group $\check{\mathscr{L}}$ of $\mathscr{L}$ by

$$\check{\mathscr{L}} = SL(2, \boldsymbol{C}) \odot \check{F}_4\,, \tag{4.3}$$

where the action of $\check{F}_4$ upon $SL(2, \boldsymbol{C})$ is determined by Eq. (1.7) to be

$$Y(A) = \bar{A}\,, \qquad X'(A) = \sigma_z \bar{A} \sigma_z^{-1}\,, \qquad \Pi_z'(A) = \sigma_z A \sigma_z^{-1}\,, \tag{4.4}$$

and where the covering map $\check{\mathscr{L}} \to \mathscr{L}$ is $(A, F) \to \Lambda(A) F$.

Since $S = \Pi_y Y$, $T = \Pi_x X'$, $ST = \Pi_z \Pi_z'$, the corresponding elements of $\check{\mathscr{L}}$ are given (up choices of signs) by

$$\check{S} = (i\sigma_y, Y)\,, \qquad \check{T} = (-i\sigma_x, X')\,, \qquad \check{S}\check{T} = (i\sigma_z, \Pi_z')\,. \tag{4.5}$$

Writing $(-I, I) \in \check{\mathscr{L}}$ simply as $-I \in SL(2, \boldsymbol{C})$, note that $\check{S}$, $\check{T}$, $\check{S}\check{T}$ form an anticommuting triad whose squares are all equal to $-I$:

$$\begin{aligned} &(\check{S})^2 = (\check{T})^2 = (\check{S}\check{T})^2 = -I \in SL(2, \boldsymbol{C})\,,\\ &\check{S}\check{T} = -\check{T}\check{S}\,, \qquad \text{etc.} \end{aligned} \tag{4.6}$$

In other words, $\check{S}$, $\check{T}$ generate the quaternion group (of order 8). Clearly then the cover $\check{\mathscr{L}}$ of $\mathscr{L}$ is not isomorphic to the previously used cover $\tilde{\mathscr{L}}$, since previously S, T, ST formed a commuting triad with square equal to $+I$ – i.e. S, T previously generated the Klein 4-group.

Introducing the four multipliers $\sigma^{\alpha\beta}$ [defined as in Eq. (1.14) but with F_4 replaced by $\check{F}_4$] for the cover

$$\check{\mathscr{P}} = \tilde{\mathscr{P}}_+^\uparrow \odot \check{F}_4$$

of $\mathscr{P}$, then on the same lines as in § 3 we can determine the irreducible $\sigma^{\alpha\beta}$-corepresentations $\check{U}_{\alpha\beta}^{mj\eta}$ of $\check{\mathscr{P}}$, with results of course in agreement with

Tables 3 and 4. The main point to note is that the pair α, β now switch roles with the pair α_j, β_j. For example, instead of

$$U(T)^2 = \sigma^{\alpha\beta}(T, T)\, U(T^2) = \alpha U(I) = \alpha I \tag{4.7}$$

we now have

$$\check{U}(\check{T})^2 = \sigma^{\alpha\beta}(\check{T}, \check{T})\, \check{U}(\check{T}^2) = \alpha \check{U}(-I) = \alpha_j I\,, \tag{4.8}$$

since $\check{U}^j(-I) = (-)^{2j} I$. Thus Tables 3 and 4 will list the $\sigma^{\alpha\beta}$-corepresentations $\check{U}^{j\eta}_{\alpha\beta}$ of the relevant little groups provided that elsewhere in the tables α, β are replaced by α_j, β_j. A further point is that extra phase factors i^{2j} and $(-i)^{2j}$ are required in $\check{U}(\check{S})$, $\check{U}(\check{T})$ [also in $\check{U}(\check{Y})$, $\check{U}(\check{Y}')$], arising from the switch from $U^j(S)^2 = I$ to $\check{U}^j(\check{S})^2 = \check{U}^j(-I) = (-)^{2j} I$ [and from $U^j(Y)^2 = (-)^{2j} I$ to $\check{U}^j(\check{Y})^2 = +I$].

Other covering groups $\tilde{\mathscr{P}}$ of $\mathscr{P}$ exist, all satisfying $\tilde{\mathscr{P}}/Z_2 \simeq \mathscr{P}$ and $\tilde{\mathscr{P}}/\tilde{\mathscr{P}}^{\uparrow}_{+} \simeq \mathscr{P}/\mathscr{P}^{\uparrow}_{+} \simeq F_4$. Is there any reason why we should prefer one covering group to another? In partial answer to this question, we note here several respects in which $\check{\mathscr{P}}$ is to be preferred to $\bar{\mathscr{P}}$.

(a) According to Eq. (3.2) the group $\check{F}_4$ leaves fixed both $(0, 0, 0, m)$ and $(0, 0, 1, 1)$. Hence if one proceeds using $\check{\mathrm{P}}$, the little groups for $m > 0$ and $m = 0$ emerge on a somewhat equal footing, being respectively $SU(2) \odot \check{F}_4$ and $\mathscr{E} \odot \check{F}_4$.

(b) The normal assumption concerning U^{mj}, $(m > 0)$, is that no doubling of states occurs, i.e. U is of Wigner type I. By Table 3 this means that U^{mj} is a multiplier corepresentation of $\bar{\mathscr{P}}$ which has the non-trivial multiplier σ^{--} whenever $2j$ is odd. In contrast, this normal assumption corresponds to choosing the *ordinary* corepresentation $\check{U}^{mj}_{++}$ (i.e. $\alpha = \beta = 1$) of $\check{\mathscr{P}}$ for all values of the spin j.

(c) As introduced above, via Eqs. (4.1)–(4.4), the group $\check{\mathscr{L}}$ hardly appears worthy of especial attention. Nevertheless, as we will point out in § 4.2, a group $\mathscr{L}(C_2)$ isomorphic to $\check{\mathscr{L}}$ is forced upon us if we demand that the covering group $\mathscr{L}(C_2)$ of $\mathscr{L}$ should – like that $SL(2, \boldsymbol{C})$ of $\mathscr{L}^{\uparrow}_{+}$ – consist of linear or antilinear operators on a 2-dimensional complex space C_2 (= the space of 2-component Lorentz spinors). The group $\check{\mathscr{L}} \simeq \mathscr{L}(C_2)$ thereby appears in a much more favourable (and coordinate-free) light.

Actually – see § 4.3 – virtues (b) and (c) are not entirely unrelated.

4.2. The Group $\mathscr{L}(C_2)$. In recent years one of us has been putting together a rather thorough coordinate-free account (now nearing completion [9]) of Minkowski space M and of associated spaces and groups. In the course of carrying out this project, the group $\mathscr{L}(C_2)$ was discovered, as sketched below, and applied in several different contexts – for example $\mathscr{L}^{\uparrow}_{+}(C_2)$ is useful in treating the symmetry properties of the Wigner $3-j$ symbols. Of relevance to the subject matter of this article is the use of $\mathscr{L}(C_2)$ in arriving at manifestly covariant[5] realizations of the representations $\boldsymbol{U}^{mj}$ of the extended Poincaré group, as we will sketch briefly in § 4.3.

Let C_2 denote a complex 2-dimensional vector space which is equipped with sympletic geometry by means of a (non-degenerate) skew symmetric bilinear form $[\,,\,]$. Let $AL(C_2)$ and $GAL(C_2)$ denote respectively all the antilinear mappings and antilinear isomorphisms $C_2 \to C_2$. Then

$$GALL(C_2) = GL(C_2) \cup GAL(C_2) \tag{4.9}$$

is a group. If A is a linear or antilinear mapping $C_2 \to C_2$, its adjoint $\tilde{A}$ is defined by

$$[\tilde{A}\xi, \eta] = [\xi, A\eta]^A, \qquad \xi, \eta \in C_2. \tag{4.10}$$

Minkowski space M is now introduced as the (real) vector space $AL\,Sk(C_2)$ consisting of all the skew symmetric elements of $AL(C_2)$:

$$M = AL\,Sk(C_2) = \{A : A \in AL(C_2), \tilde{A} = -A\}, \tag{4.11}$$

the Lorentz scalar product on M being defined by

$$p \cdot q = -\tfrac{1}{2} \operatorname{tr}(p \circ q) \tag{4.12}$$

and having signature $(- - - +)$. Each time-like vector $p \in M$ gives rise to a $U(2)$-geometry on C_2 by means of the hermitian inner product $(\,,\,)_p$ defined by

$$(\xi, \eta)_p = [p\xi, \eta], \qquad \xi, \eta \in C_2, \tag{4.13}$$

which is positive or negative definite according as p is future – or past – pointing.

The group

$$\mathscr{L}(C_2) = \mathscr{L}^{\uparrow}_{+}(C_2) \cup \mathscr{L}^{\uparrow}_{-}(C_2) \cup \mathscr{L}^{\downarrow}_{+}(C_2) \cup \mathscr{L}^{\downarrow}_{-}(C_2) \tag{4.14}$$

is defined as follows:

$$\begin{aligned} &\mathscr{L}^{\uparrow}_{+}(C_2) = Sp(C_2) = \{A : A \in GL(C_2), \tilde{A}A = I\}, \\ &\mathscr{L}^{\uparrow}_{-}(C_2) = ALSp(C_2) = \{A : A \in GAL(C_2), \tilde{A}A = I\}, \\ &\mathscr{L}^{\downarrow}_{+}(C_2) = iSp(C_2) = \{A : A \in GL(C_2), \tilde{A}A = -I\}, \\ &\mathscr{L}^{\downarrow}_{-}(C_2) = iALSp(C_2) = \{A : A \in GAL(C_2), \tilde{A}A = -I\}. \end{aligned} \tag{4.15}$$

Hence

$$\mathscr{L}(C_2) = \{A : A \in GALL(C_2), \tilde{A}A = \pm I\}. \tag{4.16}$$

Define, for $A \in \mathscr{L}(C_2)$, the linear operator $\Lambda(A)$ on M by

$$\Lambda(A)\,p = A \circ p \circ A^{-1}, \qquad p \in M. \tag{4.17}$$

[5] i.e. on all four components of $\mathscr{P}$.

Then $\pm A \to \Lambda(A) = \Lambda(-A)$ defines a $2-1$ homomorphism $\mathcal{L}(C_2) \to \mathcal{L}$ whose restriction to the identity component is the familar homomorphism $Sp(C_2)\,(\simeq SL(2, \boldsymbol{C})) \to \mathcal{L}_+^\uparrow$. We thus obtain a group isomorphism

$$\mathcal{L}(C_2)/Z_2 \simeq \mathcal{L}\,. \tag{4.18}$$

Note that $\Lambda(iI) = -I$, since $i \circ p \circ i^{-1} = -p$. The property

$$iA = \pm Ai\,, \quad A \in \mathcal{L}_\pm(C_2)\,, \tag{4.19}$$

corresponds to the commutativity of $-I \in \mathcal{L}$ with every $\Lambda \in \mathcal{L}$. [In fact one easily sees that the choice of operator $A \in GALL(C_2)$ which corresponds to the element $-I \in \mathcal{L}$ is forced upon us to be $A = \pm iI$, and hence determines our definition of $\mathcal{L}_+^\downarrow(C_2)$ in Eq. (4.14).]

One can check that the $\mathcal{L}(C_2)$-versions $\check{S}$, $\check{T}$, $\check{S}\check{T} = iI$ of S, T, ST satisfy Eq. (4.6), and so we find that $\mathcal{L}(C_2)$ is isomorphic to $\check{\mathcal{L}}$. Nevertheless the virtues of $\mathcal{L}(C_2)$ as defined in Eq. (4.14) are more manifest than those of $\check{\mathcal{L}}$ as defined in § 4.1. [Incidentally any other choice of skew-symmetric form on C_2 is a scalar multiple of the original choice, and leads to a group isomorphic to $\mathcal{L}(C_2)$.]

Remark. The intersection of $\mathcal{L}_+^\uparrow(C_2)$ with the group $U(C_2)_p$ of the linear isometries of $(\,,)_p$ is a group $SU(C_2)_p\,(\simeq SU(2))$ consisting of those $SL(C_2)$-transformations A which commute with $p: A \circ p = p \circ A$. [Moreover, by Eq. (4.17), the image of $SU(C_2)_p$ under Λ is the group $SO(3)_p\,(\simeq SO(3))$ of those $\mathcal{L}_+^\uparrow$-transformations which preserve the preferred time axis defined by p.] The full commutant of p thus consists of all the real scalar multiples of $SU(C_2)_p$, and so is isomorphic to the quatemions $\boldsymbol{H}$. Since a unit time-like vector $k = p/m$, $p \cdot p = m^2 > 0$, satisfies[6] $k^2 = -I$, we can use Eq. (2.5′) of [10] to deduce directly the result that a corepresentation of Wigner type II has commutant $\simeq \boldsymbol{H}$.

4.3. Manifestly Covariant Corepresentations of $\mathcal{P}(C_2)$. Let V^j denote the $2j$th symmetrized tensorial power $\vee^{2j} C_2$ of C_2, and, for any $A \in GALL(C_2)$, define $D^j(A)$ to be the restriction to V^j of $\otimes^{2j} A$. The map $A \mapsto D^j(A)$ defines a corepresentation of $GALL(C_2)$, and hence also of $\mathcal{L}(C_2)$, with carrier space the $(2j+1)$-dimensional space V^j; its restriction to $A \in \mathcal{L}_+^\uparrow(C_2)$ is of course the familiar spin j representation of $SL(2, \boldsymbol{C})$.

For $m > 0$, let $H_m^\pm$ denote the two sheets $\pm p_4 > 0$ of the momentum space hyperboloid $p \cdot p = m^2$, and let $\mathcal{H}^\pm$ denote the Hilbert space of functions $\phi: H_m^\pm \to V^j$ which have finite norm with respect to the inner product defined below in Eq. (4.20). The corresponding configuration space functions ψ (using a covariant Fourier transform) are the positive

[6] Here k^2 denotes $k \circ k$ (and not $k . k$).

and negative energy solutions of the Klein-Gordan equation $(\square + m^2) \cdot \psi = 0$. We are now going to define a decomposable corepresentation $U = U^+ \oplus U^-$ of

$$\mathscr{P}(C_2) = \mathscr{T} \odot \mathscr{L}(C_2)$$

having carrier space $\mathscr{H} = \mathscr{H}^+ \oplus \mathscr{H}^-$.

The inner product on $\mathscr{H}$ is defined to be

$$(\phi_1, \phi_2) = \int (\phi_1, \phi_2)_p \, d\Omega_m(p) \tag{4.20}$$

where $d\Omega_m(p) = d^3 \boldsymbol{p}/|p_4|$ and where – cf. Eq. (4.13) –

$$(\phi_1, \phi_2)_p = [D^j(\varepsilon(p)\, p/m)\, \phi_1(p), \phi_2(p)]\,, \tag{4.21}$$

with $\varepsilon(p) = \operatorname{sgn} p_4$.

The corepresentation U of $\mathscr{P}(C_2)$ is now defined by the (configuration space) transformation law $\psi \to \psi' = U(x, A)\, \psi$ given by

$$\psi'(\Lambda(A)\, y + x) = D^j(A) \begin{cases} \psi(y), & \text{if} \quad \Lambda \in \mathscr{L}_+^\uparrow \cup \mathscr{L}_-^\downarrow\,, \\ \psi^c(y), & \text{if} \quad \Lambda \in \mathscr{L}_-^\uparrow \cup \mathscr{L}_+^\downarrow\,, \end{cases} \tag{4.22}$$

where the map $C : \mathscr{H} \to \mathscr{H}$ is defined by

$$\phi \mapsto \phi^c\,, \quad \text{where} \quad \phi^c(p) = D^j(-ip/m)\, \phi(-p)\,. \tag{4.23}$$

Note that charge conjugation C is (at the present first quartization level) antilinear, and satisfies

$$\begin{aligned} &\text{(a)}\ C^2 = I \quad \text{(b)}\ C^\dagger C = I \quad \text{(c)}\ CU(x, A) = U(x, A)\, C\,, \\ &(d)\ C \text{ gives rise to bijections } \mathscr{H}^\pm \to \mathscr{H}^\mp\,. \end{aligned} \tag{4.24}$$

Bearing in mind that $D^j(A)$ is antilinear on the coset $\mathscr{L}_-(C_2)$, the presence of C in Eq. (4.22) is essential in order to produce a corepresentation U of $\mathscr{P}(C_2)$ with respect to the physically relevant UA-decomposition $\mathscr{P}^\uparrow \cup \mathscr{P}^\downarrow$.

The subspaces $\mathscr{H}^\pm$ of $\mathscr{H}$ are invariant under U and so carry corepresentations $U^\pm$ which are interwined by C (suitably restricted):

$$CU^\pm(x, A) = U^\mp(x, A)\, C\,, \quad x \in \mathscr{T}\,, \quad A \in \mathscr{L}(C_2)\,. \tag{4.25}$$

We thus arrive at a manifestly covariant realization U^+ of the (positive mass) PUA-representation $\boldsymbol{U}^{mj}$ of $\mathscr{P}$ of Wigner type I. We repeat that U^+ is an *ordinary* corepresentation of $\mathscr{P}(C_2)$.

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