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Haag-Ruelle Approximation of Collision States

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Abstract. We investigate the rate of convergence of the Haag-Ruelle approximation $\Psi(t)$ at large times t for arbitrary collision states Ψ with finite energy. An improved estimate of the norm distance $\|\Psi - \Psi(t)\|$ is given. In particular for states Ψ with smooth asymptotic wave functions it turns out that $\|\Psi - \Psi(t)\|$ approaches 0 almost like $t^{-3/4}$.

I. Introduction

The fundamental work of Haag [1] and Ruelle [2] established the existence of states which can be interpreted as asymptotic particle configurations within the framework of quantum field theory. Since this ingenious construction is by now well known it may suffice to sketch the procedure briefly: given any incoming or outgoing particle configuration one can construct sequences of vectors $\Psi(t)$ in a Hilbert space $\mathscr{H}$ by applying products of suitably chosen almost local one-particle creation operators at time t to the vacuum vector. These Haag-Ruelle approximations converge strongly in the limit of large negative and positive times and the limit vectors Ψ^{in} and Ψ^{out} correspond to the given incoming and outgoing particle configuration respectively.

One might think that one can forget about the approximations $\Psi(t)$ once one has constructed the collision states Ψ^{in}, Ψ^{out} since all the information relevant for physics is contained in matrix elements which are computable from these vectors. However, for some problems it is sufficient and much simpler to consider the approximations of a given collision state instead of the state itself. Several interesting results in collision theory have been derived from the well known kinematical properties of the vectors $\Psi(t)$ at finite times t and the convergence behaviour of the sequences $\Psi(t)$. For example, Araki and Haag showed that local observables provide a direct interpretation of scattering states as asymptotic particle configurations [3]. For a detailed summary of results of more technical nature see the lecture notes of Araki [4] and Hepp [5].

In all these investigations, the crucial point is to derive an adequate estimate for the norm distances $\|\Psi^{\text{in}} - \Psi(t)\|$ and $\|\Psi^{\text{out}} - \Psi(t)\|$ at large negative and positive times t respectively. In the papers quoted above such estimates are given for asymptotic configurations which are well behaved: technically speaking, the wavefunctions corresponding to the asymptotic particle configurations are supposed to be elements of the Schwartz-space $\mathscr{S}$. Unfortunately, such configurations are not the only important ones in collision theory. Under the influence of interaction even those states with a smooth incoming wavefunction can have an outgoing configuration which, in many cases, can no longer be described by a function from $\mathscr{S}$. Thus one is forced to consider wavefunctions which are not continuously differentiable if, for example, one is interested in the spatio-temporal behaviour of collision states. For this reason we investigate the convergence behaviour of the Haag-Ruelle approximations $\Psi(t)$ for arbitrary asymptotic configurations.

In this paper we give an estimate which clarifies the connection *between the degree of singularity of the wavefunctions in momentum* space and the decrease of the norm distances $\|\Psi^{\text{in}} - \Psi(t)\|$ and $\|\Psi^{\text{out}} - \Psi(t)\|$ for asymptotic t. Moreover, for smooth wavefunctions our estimate is an improvement on previous known ones: it turns out that $\|\Psi^{\text{out}} - \Psi(t)\|$ approaches 0 almost like $t^{-3/4}$ for large t. (An analogous statement holds for the states Ψ^{in}.) This is only a slight modification of the asymptotic $t^{-1/2}$ behaviour which was proved by Haag and Ruelle [1], [2]. Yet we shall show in a forthcoming paper that our result is the optimal one to be expected within the general framework of quantum field theory. There should be models compatible with the basic postulates in which our result can not be improved.

Our methods of proof can easily be carried over to the case of non-relativistic potential scattering theory for short range potentials. We want also to point out that our arguments can be applied in a model-world of arbitrary spatial dimension. We do not need to assume that the dimension of space is greater than 2.

II. Assumptions and Definitions

Our arguments do not depend very sensitively on the framework in which the basic postulates of quantum field theory are expressed. Since we want to avoid all unnecessary complications we shall formulate our assumptions in terms of the field-algebra[1] instead of the field-operators (Wightman-fields) themselves. Thus we shall never have to worry about domain questions etc. Yet at the expense of increased technicality we could have derived our results in the Wightman-frame-

[1] Roughly speaking, the field algebra is generated by all bounded functions of the selfadjoint components of the fields.

work [6] as well. For the sake of simplicity we shall furthermore restrict our attention to a theory with only one kind of particle which we shall assume to be a massive neutral boson with spin 0.

With these simplifications, the Hilbert space $\mathscr{H}$ of all physical states is a direct sum of the 1-dimensional space $\{c \cdot \Omega\}$ corresponding to the vacuum, the one particle space $\mathscr{H}_1$ and the space of the collision states. We assume furthermore that $\mathscr{H}$ carries a continuous unitary representation of the translations $x=(t, \boldsymbol{x}) \rightarrow U(x)$ and that the vacuum Ω is the only vector invariant under the action of $U(x)$. In this paper we neither make explicitly use of the Lorentz-covariance nor of the local properties of the field algebras nor of the positivity of the energy. (For a review of the basic postulates see [4].) But we shall use some properties of a relativistic quantum theory which are consequences of these fundamental assumptions. It is crucial for our argument that there exists a set $\mathscr{P}$ of almost local 1-particle creation operators with properties specified in the following theorem [3], [7]:

Theorem. *For each compact set* $\mathbb{K} \subset \mathbb{R}^3$ *there exists a bounded operator* $A \in \mathscr{P}$ *such that*

i) $A\Omega$ *is a one particle state,* $A\Omega \in \mathscr{H}_1$, *and* $A^*\Omega = 0$.

ii) *the 1-particle wavefunction of* $A\Omega$ *in momentum space is constant on* $\mathbb{K}$, *more precisely:*

$$(\boldsymbol{p}, A\Omega) = (2\pi)^{-3/2} \quad \text{for} \quad \boldsymbol{p} \in \mathbb{K}\,.$$

iii) *the Fourier transform* $\tilde{A}(p)$ *of* $A(x) = U(x)\,A\,U^{-1}(x)$ *has compact support (in the sense of distributions). As a consequence, the derivatives of* $A(x)$ *with respect to the translations exist in the norm topology.*

iv) *A together with its derivatives is almost local. By this we mean the following: let* $A_1, \ldots, A_n$ *stand for A, a derivative of A or their adjoints. Then the truncated vacuum expectation value of the translated operators has the following decrease property:*

$$|(\Omega, A_1(\boldsymbol{x}_1) \ldots A_n(\boldsymbol{x}_n)\,\Omega)_T| \leqq \varphi\Big(\sum_{i<j} |\boldsymbol{x}_i - \boldsymbol{x}_j|\Big)\,.$$

φ *is a continuous, monotone function which decreases faster than any inverse power of its argument,* $\lim_{v \to \infty} v^N \varphi(v) = 0$ *for all* $N \in \mathbb{N}$.

The set $\mathscr{P}$ will be used in the following to construct the collision states: let $\Psi_{f_n}^{\text{out}}$ be a vector in $\mathscr{H}$ corresponding to an outgoing n-particle configuration with a momentum distribution $\tilde{f}_n(\boldsymbol{p}_1 \ldots \boldsymbol{p}_n)$. ($\Psi_{f_n}^{\text{in}}$ is defined analogously.) If $\tilde{f}_n(\boldsymbol{p}_1 \ldots \boldsymbol{p}_n)$ has compact support in $\mathbb{R}^{3n}$ we can find a compact set $\mathbb{K} \subset \mathbb{R}^3$ such that $\operatorname{supp} \tilde{f}_n \subset \mathbb{K}^n$. ($\mathbb{K}^n$ is the n-fold Cartesian product of $\mathbb{K}$.) Corresponding to $\mathbb{K}$ we take an operator A from $\mathscr{P}$

with properties listed in the theorem and construct the Haag-Ruelle approximations of $\Psi_{f_n}^{\text{out}}$,

$$\Psi_{f_n}(t) = \int \prod_{l=1}^{n} d^3x_l f_n(t|\boldsymbol{x}_1 \dots \boldsymbol{x}_n) A(t, \boldsymbol{x}_1) \dots A(t, \boldsymbol{x}_n) \Omega . \tag{1}$$

Here we have introduced the configuration space wavefunction at time t,

$$f_n(t|\boldsymbol{x}_1 \dots \boldsymbol{x}_n) = (2\pi)^{-3n/2} \int \prod_{l=1}^{n} d^3p_l e^{-it\omega_l + i\boldsymbol{x}_l\boldsymbol{p}_l} \cdot \tilde{f}_n(\boldsymbol{p}_1 \dots \boldsymbol{p}_n) \tag{2}$$

using the abbreviation $\omega_l = \omega_{\boldsymbol{p}_l} = (|\boldsymbol{p}_l|^2 + \mu^2)^{1/2}$; μ is the mass of the particle. As we remarked in the introduction, the work of Haag and Ruelle implies that the sequence $\Psi_{f_n}(t)$ converges strongly towards $\Psi_{f_n}^{\text{out}}$ and that $\|\Psi_{f_n}^{\text{out}} - \Psi_{f_n}(t)\| \leqq c \cdot t^{-1/2}$ for large positive t if the wavefunction $\tilde{f}_n(\boldsymbol{p}_1 \dots \boldsymbol{p}_n)$ is smooth. For a special class of configurations which are characterized by smooth wavefunctions f_n with "non-overlapping momenta"[2] Hepp [5] and independently Araki and Haag [3] could give a much better estimate: for $t > 0$ and any $N \in \mathbb{N}$

$$\|\Psi_{f_n}^{\text{out}} - \Psi_{f_n}(t)\| \leqq c_N \cdot t^{-N} . \tag{3}$$

This observation will be the starting point for our work.

III. Approximation of Arbitrary Collision States

We shall show now how the Haag-Ruelle approximation $\Psi_f(t)$ converges if f[3] is an arbitrary square-integrable wavefunction with compact support in momentum space. The underlying idea of proof is very simple: we split the function f into the sum of a smooth function g with non-overlapping momenta and the (possibly) singular remainder $\Delta f = f - g$. According to Eq. (1) in the preceding chapter we get then a decomposition of $\Psi_f(t)$: $\Psi_f(t) = \Psi_g(t) + \Psi_{\Delta f}(t)$. Now $\Psi_g(t)$ converges very rapidly towards Ψ_g^{out} for large t (see relation (3) above) and $\Psi_{\Delta f}(t)$, $\Psi_{\Delta f}^{\text{out}}$ both have small norms if g is a reasonable approximation of f. So one has only to find an appropriate decomposition of f at each time t in order to get a good estimate for the rate of convergence of $\|\Psi_f^{\text{out}} - \Psi_f(t)\|$.

For a quantitative result we need an estimate of the asymptotic behaviour of $\Psi_g(t)$ which is better than the one given by inequality (3). We have to control how the constant c_N in this relation depends on the properties of the function g. For this purpose we introduce a set of

[2] The function f_n is said to have "non-overlapping momenta" if the support of $\tilde{f}_n(\boldsymbol{p}_1 \dots \boldsymbol{p}_n)$ has a finite distance from the planes $\boldsymbol{p}_i - \boldsymbol{p}_j = 0$, $i \neq j$ *in* $\mathbb{R}^{3n}$, $n \geqq 2$.

[3] Here we have omitted the index n of f since we shall consider throughout this chapter only n-particle states and their approximations.

norms on the linear subspace $\mathring{\mathscr{S}}(\mathbb{R}^{3n})$ of $\mathscr{S}(\mathbb{R}^{3n})$ which is spanned by the functions with non-overlapping momenta.

Definition. Let g be an element of $\mathring{\mathscr{S}}(\mathbb{R}^{3n})$. Then for each $N \in \mathbb{N}$ we define a norm $\|g\|_N$ of g by

$$\|g\|_N^2 = \sum_{i+|j| \leqq N} \int \prod_{l=1}^{n} d^3 p_l \left| \left(1 + \sum_{l'=1}^{n} |\boldsymbol{p}_{l'}|\right)^{3N} \cdot \left(1 + \sum_{m<m'} \frac{1}{|\boldsymbol{p}_m - \boldsymbol{p}_{m'}|}\right)^{N+i} \cdot \tilde{g}^{(j)}(\boldsymbol{p}_1 \dots \boldsymbol{p}_n) \right|^2 .$$

Here we have introduced the multi-index $(j) = j_1 \dots j_{3n}$; $|j| = \sum_{l=1}^{3n} j_l$ and

$$\tilde{g}^{(j)}(\boldsymbol{p}_1 \dots \boldsymbol{p}_n) = \frac{\partial^{j^1}}{\partial(p_1)_1^{j^1}} \cdots \frac{\partial^{j_{3n}}}{\partial(p_n)_3^{j_{3n}}} \tilde{g}(\boldsymbol{p}_1 \dots \boldsymbol{p}_n) .$$

Besides the norms $\|g\|_N$ we shall also consider the norm $\|g\|$ of g in $L^2(\mathbb{R}^{3n})$,

$$\|g\|^2 = \int \prod_{l=1}^{n} d^3 p_l \cdot |\tilde{g}(\boldsymbol{p}_1 \dots \boldsymbol{p}_n)|^2 .$$

With this notation we are now able to state our main result:

Theorem. *Let f be an element of $L^2(\mathbb{R}^{3n})$ with support in a given compact region $\mathbb{K}^n$ of momentum space*[4]. *Then the following inequality holds for arbitrary elements $g \in \mathring{\mathscr{S}}(\mathbb{R}^{3n})$ and $N \in \mathbb{N}$, $N > 1$:*

$$\|\Psi_f^{\text{out}} - \Psi_f(t)\| \leqq c_N \cdot \|g\|_N \cdot t^{-N+1} + c \cdot \|f - g\| \quad \textit{for} \quad t > 0 . \tag{4}$$

The constants c_N and c neither depend on f nor on g but they may depend on the size and location of $\mathbb{K}$. (An analogous result holds for Ψ_f^{in}.)

Before we start to prove this theorem let us discuss some consequences. It is obvious from the theorem and from the definition of the norms $\| \,.\, \|_N$ that there are two independent sources for a slow decrease of the norm distance $\|\Psi_f^{\text{out}} - \Psi_f(t)\|$ for large t: a bad momentum space behaviour and possible threshold contributions (at the points $\boldsymbol{p}_i = \boldsymbol{p}_j$, $i \neq j$) of the wavefunction f. Corresponding to this fact one splits up the function f into a singular and a non-singular part and then decomposes the non-singular part into a threshold contribution and a function with non-overlapping momenta. The last term can then be taken as a suitable approximation of f.

The first step of this procedure may be performed by a convolution of f with smooth approximations of the δ-function: let φ be any element

[4] We do not need to mention explicitly that $\tilde{f}(\boldsymbol{p}_1 \dots \boldsymbol{p}_n)$ must be totally symmetric in $\boldsymbol{p}_1 \dots \boldsymbol{p}_n$ since it is an asymptotic wavefunction.

of $\mathscr{S}(\mathbb{R}^{3n})$ with compact support in momentum space and satisfying the normalisation condition $\int \prod_{l=1}^{n} d^3p_l \tilde{\varphi}(\boldsymbol{p}_1 \dots \boldsymbol{p}_n) = 1$. Then the functions

$(s \geqq 1)$

$$\tilde{f}_s(\boldsymbol{p}_1 \dots \boldsymbol{p}_n) = \int \prod_{l=1}^{n} d^3q_l \cdot s^{3n} \tilde{\varphi}(s[\boldsymbol{p}_1 - \boldsymbol{q}_1] \dots s[\boldsymbol{p}_n - \boldsymbol{q}_n]) \cdot \tilde{f}(\boldsymbol{q}_1 \dots \boldsymbol{q}_n)$$

are elements of $\mathscr{S}(\mathbb{R}^{3n})$ with support within a fixed compact region of momentum space. It is easy to verify that the norms of the derivatives $\tilde{f}_s^{(j)}(\boldsymbol{p}_1 \dots \boldsymbol{p}_n)$ in $L^2(\mathbb{R}^{3n})$ have the bounds $\|f_s^{(j)}\| \leqq c_{|j|} \cdot (1 + s^{|j|})$ and that $\|f - f_s\|$ approaches 0 if s becomes large. We divide the wavefunctions f into classes with the same degree of singularity in momentum space by looking at the rate of convergence of $\|f - f_s\|$.

Definition. The linear space $M_\alpha(\mathbb{R}^{3n})$, $\alpha > 0$ is generated by the wavefunctions f (with compact support in momentum space) for which one can find an approximating sequence f_s, $s \geqq 1$ [5] with the properties:

i) $\tilde{f}_s(\boldsymbol{p}_1 \dots \boldsymbol{p}_n)$ has (for all $s \geqq 1$) its support within a fixed compact region of $\mathbb{R}^{3n}$.

ii) $\|f_s^{(j)}\| \leqq c_{|j|}(1 + s^{|j|})$ for all multi-indizes (j).

iii) $\|f - f_s\| \leqq c \cdot s^{-\alpha}$.

The set of those functions which are not contained in any of the spaces $M_\alpha(\mathbb{R}^{3n})$, $\alpha > 0$ will be denoted by $M_0(\mathbb{R}^{3n})$ [6].

The second step is to remove the threshold contributions from f_s. To this end we take a function $\mathring{h} \in \mathscr{C}^\infty(\mathbb{R}^3)$; $\mathring{h}(\boldsymbol{u}) = 0$ for $|\boldsymbol{u}| \leqq 1$ and $\mathring{h}(\boldsymbol{u}) = 1$ for $|\boldsymbol{u}| \geqq 2$. This function will be used to construct the smooth approximations $f_{r,s}$ with non-overlapping momenta:

$$\tilde{f}_{r,s}(\boldsymbol{p}_1 \dots \boldsymbol{p}_n) = \prod_{m > m'} \mathring{h}(r[\boldsymbol{p}_m - \boldsymbol{p}_{m'}]) \cdot \tilde{f}_s(\boldsymbol{p}_1 \dots \boldsymbol{p}_n), \qquad r \geqq 1 .$$

It follows from the support properties of $(1 - \mathring{h})$ and the smoothness of f_s and $\mathring{h}$ in momentum space that $\|f - f_{r,s}\| \leqq \|f - f_s\| + c \cdot \left(\frac{s}{r}\right)^{3/2}$, c being independent of r and s. A straightforward calculation also shows that $\|f_{r,s}\|_N \leqq c_N \cdot r^N \cdot (r+s)^N$ for $r, s \geqq 1$; the constant c_N again does not depend on r and s.

Now we replace g in relation (4) by the 2-parameter family $f_{r,s}$. The following inequality which holds for arbitrary $N \in \mathbb{N}$, $N > 1$ is

[5] The explicit construction of f_s given above need not be optimal.

[6] To illustrate this somewhat technical characterisation of the wavefunctions let us give an example: f is an element of $M_\alpha(\mathbb{R}^{3n})$ if the configuration space integral $\int \prod_{l=1}^{n} d^3x_l \left| \left(\sum_{l'=1}^{n} |\boldsymbol{x}_{l'}| \right)^\alpha f(\boldsymbol{x}_1 \dots \boldsymbol{x}_n) \right|^2$ exists.

then a consequence of the estimates just given:

$$\|\Psi_f^{\text{out}} - \Psi_f(t)\| \leqq c_N \cdot r^N \cdot (r+s)^N \cdot t^{-N+1} + c \cdot \|f - f_s\| + c' \cdot \left(\frac{s}{r}\right)^{3/2} \quad \text{for} \quad t > 0$$

with constants c_N, c, c' not depending on r, s or t. It is easy to minimize the right hand side of this inequality with respect to r and s and one obtains:

Corollary. i) *Let f be an element of $M_\alpha(\mathbb{R}^{3n})$, $\alpha > 0$. Then for any positive $\gamma < \dfrac{3\alpha}{6+4\alpha}$ there exists a constant c_γ such that*

$$\|\Psi_f^{\text{out}} - \Psi_f(t)\| \leqq c_\gamma \cdot t^{-\gamma} \quad \textit{for} \quad t > 0 .$$

ii) *For $f \in M_0(\mathbb{R}^{3n})$ and $\gamma < \frac{1}{2}$ we get the estimate*

$$\|\Psi_f^{\text{out}} - \Psi_f(t)\| \leqq c_\gamma \cdot d(t^\gamma) \quad \textit{for large } t .$$

(Here we have introduced the distance function $d(s) = \|f - f_s\|$.)

Loosely speaking, $\|\Psi_f^{\text{out}} - \Psi_f(t)\|$ approaches 0 almost like $t^{-3\alpha/(6+4\alpha)}$ for $\alpha < \infty$ and like $t^{-3/4}$ for $\alpha = \infty$. It is the threshold contributions that prevent the norm distances $\|\Psi_f^{\text{out}} - \Psi_f(t)\|$ from decreasing faster than $t^{-3/4}$, even for smooth wavefunctions f. For wavefunctions vanishing at the thresholds $\boldsymbol{p}_i = \boldsymbol{p}_j$, $i \neq j$ one can get better estimates (depending on the order of the zeros and α) for the rate of convergence of the Haag-Ruelle approximations $\Psi_f(t)$. Since the argument is the same as above we leave the details to the reader.

Now we come to the postponed proof of the theorem. We start with a trivial lemma clarifying the connection between the difference of the momenta $(\boldsymbol{p} - \boldsymbol{q})$ and the difference of the corresponding velocities $\left(\dfrac{\boldsymbol{p}}{\omega_{\boldsymbol{p}}} - \dfrac{\boldsymbol{q}}{\omega_{\boldsymbol{q}}}\right)$:

Lemma 1. i) *The 3×3-matrix $R(\boldsymbol{p}, \boldsymbol{q})$ defined by*

$$R(\boldsymbol{p}, \boldsymbol{q})_{ik} = 2\frac{\omega_{\boldsymbol{p}}\omega_{\boldsymbol{q}}}{\omega_{\boldsymbol{p}} + \omega_{\boldsymbol{q}}} \cdot \left\{\delta_{ik} + \frac{(\boldsymbol{p}+\boldsymbol{q})_i \cdot (\boldsymbol{p}+\boldsymbol{q})_k}{(\omega_{\boldsymbol{p}} + \omega_{\boldsymbol{q}})^2 - |\boldsymbol{p}+\boldsymbol{q}|^2}\right\} \quad i, k = 1, 2, 3$$

transforms $\left(\dfrac{\boldsymbol{p}}{\omega_{\boldsymbol{p}}} - \dfrac{\boldsymbol{q}}{\omega_{\boldsymbol{q}}}\right)$ into $(\boldsymbol{p} - \boldsymbol{q})$.

ii) *The components of this matrix are arbitrarily often differentiable with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$ and the derivatives are bounded for all (j) according to*

$$|R^{(j)}(\boldsymbol{p}, \boldsymbol{q})_{ik}| \leqq c_{|j|} \cdot (1 + |\boldsymbol{p}| + |\boldsymbol{q}|)^3 .$$

(The proof of this statement is trivial and can be omitted.) We need the matrix $R(\boldsymbol{p}, \boldsymbol{q})$ to construct some operators which will occur later in our estimates. These operators are defined on the functions $g \in \mathring{\mathscr{S}}(\mathbb{R}^{3n})$: let $a \neq b$ be any two elements of the index set $\{1 \dots n\}$, then

$$(\widetilde{\boldsymbol{S}_{ab} g})(\boldsymbol{p}_1 \dots \boldsymbol{p}_n) = R(\boldsymbol{p}_a, \boldsymbol{p}_b) \cdot \frac{(\boldsymbol{p}_a - \boldsymbol{p}_b)}{|\boldsymbol{p}_a - \boldsymbol{p}_b|^2} \, \tilde{g}(\boldsymbol{p}_1 \dots \boldsymbol{p}_n)$$

and

$$(\widetilde{T_{ab} g})(\boldsymbol{p}_1 \dots \boldsymbol{p}_n) = (\boldsymbol{V}_a - \boldsymbol{V}_b) \cdot (\widetilde{\boldsymbol{S}_{ab} g})(\boldsymbol{p}_1 \dots \boldsymbol{p}_n) .$$

It is obvious that the function $(T_{ab} g)$ and the components of the vector-valued function $(\boldsymbol{S}_{ab} g)$ are again elements of $\mathring{\mathscr{S}}(\mathbb{R}^{3n})$. Since the definition of the norms $\| \, . \, \|_N$ has been adjusted to the properties of the operators $\boldsymbol{S}_{ab}$ and T_{ab} one can easily verify the following proposition:

Lemma 2. *Let g be an element of $\mathring{\mathscr{S}}(\mathbb{R}^{3n})$. Then there exist for all $N \in \{0, \mathbb{N}\}$ certain constants c_N and c'_N (not depending on g) such that*

$$\|(\boldsymbol{S}_{ab} g)_l\|_N \leqq c_N \cdot \|g\|_{N+1} \quad \text{for} \quad l = 1, 2, 3 \quad \text{and} \quad \|(T_{ab} g)\|_N \leqq c'_N \cdot \|g\|_{N+1} \, .$$

(Here $\| \, . \, \|_0$ stands for the $L^2(\mathbb{R}^{3n})$ norm $\| \, . \, \|$.)

In the next lemma an important estimate is given for the asymptotic behaviour of the configuration-space wavefunction $g(t \,|\, \boldsymbol{x}_1 \dots \boldsymbol{x}_n)$ [7] at large times t.

Lemma 3. *Let g be an element of $\mathring{\mathscr{S}}(\mathbb{R}^{3n})$. Then for two arbitrary elements $a \neq b$ from the index set $\{1 \dots n\}$ and any $N \in \{0, \mathbb{N}\}$ the following inequality holds:*

$$|t^N g(t \,|\, \boldsymbol{x}_1 \dots \boldsymbol{x}_n)| \leqq (1 + |\boldsymbol{x}_a - \boldsymbol{x}_b|)^N \cdot g_{N,t}(\boldsymbol{x}_1 \dots \boldsymbol{x}_n) \, .$$

The function $g_{N,t}(\boldsymbol{x}_1 \dots \boldsymbol{x}_n)$ is square-integrable and

$$\|g_{N,t}\| \leqq c_N \cdot \|g\|_N$$

with a constant c_N not depending on g and t.

Proof. Since

$$(\boldsymbol{V}_a - \boldsymbol{V}_b) \, e^{-it \sum_{l=1}^{n} \omega_l} = (-it) \cdot \left(\frac{\boldsymbol{p}_a}{\omega_a} - \frac{\boldsymbol{p}_b}{\omega_b} \right) \cdot e^{-it \sum_{l=1}^{n} \omega_l}$$

one gets immediately after partial integration:

$$it \cdot g(t \,|\, \boldsymbol{x}_1 \dots \boldsymbol{x}_n) = \int \prod_{l=1}^{n} d^3 p_l e^{-it\omega_l} \cdot (\widetilde{T_{ab} e_{\boldsymbol{x}_1 \dots \boldsymbol{x}_n} \cdot g})(\boldsymbol{p}_1 \dots \boldsymbol{p}_n) \, .$$

Here we have introduced the function $e_{\boldsymbol{x}_1 \dots \boldsymbol{x}_n}$,

$$\widetilde{e_{\boldsymbol{x}_1 \dots \boldsymbol{x}_n}}(\boldsymbol{p}_1 \dots \boldsymbol{p}_n) = (2\pi)^{-3n/2} \cdot e^{i \sum_{l=1}^{n} \boldsymbol{x}_l \boldsymbol{p}_l} \, .$$

[7] See relation (2) of the preceding chapter.

It follows from the definition of the operators S_{ab} and T_{ab} that

$$\widetilde{(T_{ab}e_{x_1\dots x_n}\cdot g)}(p_1 \dots p_n) = i(x_a - x_b)\cdot \widetilde{(e_{x_1\dots x_n}\cdot S_{ab}g)}(p_1 \dots p_n) + \widetilde{(e_{x_1\dots x_n}\cdot T_{ab}g)}(p_1 \dots p_n)$$

and therefore

(a) $it\cdot g(t|x_1 \dots x_n) = i(x_a - x_b)\cdot (S_{ab}g)(t|x_1 \dots x_n) + (T_{ab}g)(t|x_1 \dots x_n)$.

Now we can prove the lemma by induction. Since

$$\int \prod_{l=1}^{n} d^3x_l \cdot |g(t|x_1 \dots x_n)|^2 = \|g\|^2$$

the statement is true for $N=0$. Let us therefore assume that the lemma holds for N. It follows from relation (a) that

$$|t^{N+1}g(t|x_1 \dots x_n)| \leqq 2(1+|x_a - x_b|) \cdot \left(\sum_{l=1}^{3} |t^N(S_{ab}g)_l(t|x_1 \dots x_n)| + |t^N(T_{ab}g)(t|x_1 \dots x_n)|\right).$$

According to our assumption we get

$$|t^N(S_{ab}g)_l(t|x_1 \dots x_n)| \leqq (1+|x_a - x_b|)^N \hat{g}_{l,N,t}(x_1 \dots x_n) \quad \text{with} \quad \|\hat{g}_{l,N,t}\| \leqq c_N \cdot \|(S_{ab}g)_l\|_N$$

and

$$|t^N(T_{ab}g)(t|x_1 \dots x_n)| \leqq (1+|x_a - x_b|)^N \hat{\hat{g}}_{N,t}(x_1 \dots x_n) \quad \text{with} \quad \|\hat{\hat{g}}_{N,t}\| \leqq c_N \cdot \|(T_{ab}g)\|_N.$$

But we know from Lemma 2 that $\|(S_{ab}g)_l\|_N \leqq c'_N \cdot \|g\|_{N+1}$ and $\|T_{ab}g\|_N \leqq c''_N \cdot \|g\|_{N+1}$ and this proves our statement.

Finally, some additional remarks about the 1-particle creation operators are necessary: if A is an element from the set $\mathscr{P}$ one can define another almost local operator j by

$$j = (2\pi)^{-2}\cdot \int d^4p\, i(p_0 - \omega_p)\, \tilde{A}(p).$$

Since A creates a 1-particle state from the vacuum it is obvious that $j\Omega = 0$. From the fact that $\tilde{A}(p)$ has compact support it follows that j is a bounded operator. Thus one can easily deduce from the decrease properties of the truncated vacuum expectation values of the operators A and j the following lemma:

Lemma 4. *Let A be any operator from the set $\mathscr{P}$ and j the operator defined above. If f is an arbitrary element from $L^2(\mathbb{R}^{3n})$, then*

$$\left\|\int \prod_{l=1}^{n} d^3x_l f(x_1 \dots x_n)\, A(x_1) \dots A(x_n)\, \Omega\right\|^2 \leqq c\cdot \int \prod_{l=1}^{n} d^3x_l |f(x_1 \dots x_n)|^2$$

and

$$\left\| \int \prod_{l=1}^{n} d^3x_l f(\boldsymbol{x}_1 \ldots \boldsymbol{x}_n)\, A(\boldsymbol{x}_1) \ldots j(\boldsymbol{x}_a) \ldots A(\boldsymbol{x}_n)\, \Omega \right\|^2$$

$$\leqq c_N \cdot \sum_{b=a+1}^{n} \int \prod_{l=1}^{n} d^3x_l (1 + |\boldsymbol{x}_a - \boldsymbol{x}_b|)^{-2N} \cdot |f(\boldsymbol{x}_1 \ldots \boldsymbol{x}_n)|^2$$

for all $N \in \mathbb{N}$. *The constants* c *and* c_N *in these inequalities do not depend on* f.

Now we are almost finished: let f be any wavefunction with compact support in momentum space and g an element of $\mathring{\mathscr{S}}(\mathbb{R}^{3n})$. Then one gets for the norm distance $\|\Psi_f(t'') - \Psi_f(t')\|$, $t'' \geqq t' > 0$ the estimate:

$$\|\Psi_f(t'') - \Psi_f(t')\| \leqq \|\Psi_g(t'') - \Psi_g(t')\| + \|\Psi_{f-g}(t'')\| + \|\Psi_{f-g}(t')\|\,.$$

It follows from the definition of the Haag-Ruelle approximations and Lemma 4 that $\|\Psi_{f-g}(t'')\| \leqq c \cdot \|f-g\|$ and $\|\Psi_{f-g}(t')\| \leqq c\|f-g\|$. Clearly, $\|\Psi_g(t'') - \Psi_g(t')\| \leqq \int_{t'}^{t''} dt \|\partial_t \Psi_g(t)\|$ and therefore one has only to consider the state $\partial_t \Psi_g(t)$. It is standard to prove the relation

$$\partial_t \Psi_g(t) = \sum_{a=1}^{n} \int \prod_{l=1}^{n} d^3x_l g(t|\boldsymbol{x}_1 \ldots \boldsymbol{x}_n)\, A(t, \boldsymbol{x}_1) \ldots j(t, \boldsymbol{x}_a) \ldots A(t, \boldsymbol{x}_n)\, \Omega\,,$$

j being the operator defined above. Because of the translation invariance of the vacuum and Lemma 4 one concludes that for all $N \in \mathbb{N}$

$$\|\partial_t \Psi_g(t)\|^2 \leqq c''_N \sum_{a=1}^{n} \sum_{b=a+1}^{n} \int \prod_{l=1}^{n} d^3x_l (1 + |\boldsymbol{x}_a - \boldsymbol{x}_b|)^{-2N} |g(t|\boldsymbol{x}_1 \ldots \boldsymbol{x}_n)|^2\,.$$

Applying Lemma 3 to the right hand side of this inequality one gets

$$\|\partial_t \Psi_g(t)\| \leqq c'_N \cdot \|g\|_N \cdot t^{-N}$$

with a constant c'_N not depending on g. This shows (after integration) that

$$\|\Psi_f(t'') - \Psi_f(t')\| \leqq c_N \cdot \|g\|_N \cdot ((t'')^{-N+1} + (t')^{-N+1}) + c\|f-g\|$$

for all $N \in \mathbb{N}$, $N > 1$. If one now puts $t'' = \infty$ and $t' = t$ it follows

$$\|\Psi_f^{\text{out}} - \Psi_f(t)\| \leqq c_N \cdot \|g\|_N \cdot t^{-N+1} + c \cdot \|f-g\|$$

and this, finally, proves the theorem.

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