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Remarks on the FKG Inequalities

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Abstract. The FKG inequalities are generalized to two probability distributions. A theorem is proved which shows how one distribution dominates the other and makes it clear why expectation values of increasing functions with respect to one distribution are larger than with respect to the other.

Let Γ be a finite distributive lattice and let μ_1 and μ_2 be probability distributions on Γ. One of the most common applications of the FKG inequalities [3] is to find conditions which guarantee that for all functions f on Γ such that $x \leqq y$ implies $f(x) \leqq f(y)$ one has

$$\text{(1)} \qquad \sum_{x \in \Gamma} f(x)\, \mu_1(x) \geqq \sum_{x \in \Gamma} f(x)\, \mu_2(x)\,.$$

The inequality (1) seems to be saying that somehow the distribution μ_1 is situated higher up on the lattice than μ_2 is. We prove here a theorem making this precise. The theorem is also strong enough to imply inequalities such as (1) and the FKG inequalities [Corollary (12) below].

(2) **Lemma.** *Let Γ be a finite distributive lattice. Then Γ is isomorphic to a sublattice, $\tilde{\Gamma}$, of the lattice of subsets of a finite set Λ. Moreover, Λ and $\tilde{\Gamma}$ can be chosen in such a way that $\emptyset$ and Λ are in $\tilde{\Gamma}$ and for all $A, B \in \tilde{\Gamma}$ there is a sequence $A = A_0, A_1, \ldots, A_n = B$ in $\tilde{\Gamma}$ such that*

$$|A_i \triangle A_{i+1}| = 1 \quad \textit{for all } i\,.$$

Here $|A|$ denotes the cardinality of A, and

$$A \triangle B = (A \backslash B) \cup (B \backslash A)\,.$$

Lemma (2) is essentially Corollary (2), Page 59 in [1]. Actually Corollary (2) in [1] is not phrased this way; however, it is easily seen from the proof to be equivalent to Lemma (2).

For the rest of this note Λ will be a fixed finite set and Γ will be a sublattice of the lattice of subsets of Λ. It will also be assumed that Γ has the properties of $\tilde{\Gamma}$ mentioned in Lemma (2).

Let μ be a strictly positive probability distribution on Γ. That is, $\mu(A) > 0$ for all $A \in \Gamma$ and $\sum_{A \in \Gamma} \mu(A) = 1$.

For $x \in A$ and $A \in \Gamma$ define a function $c(x, A)$ as follows:

(3) $$c(x,A)=\begin{cases}1 & \text{if} \quad x \notin A \quad \text{and} \quad A \cup \{x\} \in \Gamma \\ \mu(A \backslash \{x\})/\mu(A) & \text{if} \quad x \in A \quad \text{and} \quad A \backslash \{x\} \in \Gamma \\ 0 & \text{otherwise}.\end{cases}$$

If $A, B \in \Gamma$ and $A \neq B$ define

(4) $$\Omega(A,B)=\begin{cases}c(x,A) & \text{if} \quad A \triangle B = \{x\} \\ 0 & \text{otherwise},\end{cases}$$

and define $\Omega(A, A)$ so that $\sum_{B \in \Gamma} \Omega(A, B) = 0$. We think of Ω as a matrix and let Ω^n be the n^{th} power of that matrix (Ω^0 is the identity matrix).

Now let
$$P_t(A,B)=\sum_{n=0}^{\infty} \frac{t^n}{n!} \Omega^n(A,B).$$

Since $\Omega(A, B) \geqq 0$ if $A \neq B$ and $\sum_{B \in \Gamma} \Omega(A, B) = 0$, it follows that $P_t(\cdot, \cdot)$ is the transition function of a Markov process on Γ (see [2], Chapter 6, Paragraph 1). The matrix Ω is called the generator of the process.

(5) **Lemma.** *The Markov process on Γ with generator Ω has μ as a stationary distribution*

$$\left(\text{i.e. } \sum_{A \in \Gamma} \mu(A)\, P_t(A,B) = \mu(B) \textit{ for all } t \geqq 0\right).$$

Moreover, this Markov process has only one stationary distribution.

Proof. One easily checks from the definition of Ω that

$$\sum_{A \in \Gamma} \mu(A)\, \Omega(A,B) = 0 \qquad \text{for all} \qquad B \in \Gamma.$$

The stationarity of μ follows immediately from this. The Markov process has only one stationary distribution if $P_t(A, B) > 0$ for all $A, B \in \Gamma$ and all $t > 0$ (see [2], Chapter 5, Paragraph 2). In order to prove that $P_t(A, B) > 0$ for all $A, B \in \Gamma$ and all $t > 0$ it suffices to show that for all $A, B \in \Gamma$ there is a sequence $A = A_0, A_1, \ldots, A_n = B$ in Γ such that

$$\prod_{i=1}^{n} \Omega(A_{i-1}, A_i) > 0.$$

But again this follows from the definition of Ω and the assumed structure of Γ.

(6) **Theorem.** *Let μ_1 and μ_2 be two strictly positive probability distributions on Γ satisfying*

(7) $$\mu_1(A \cup B)\, \mu_2(A \cap B) \geqq \mu_1(A)\, \mu_2(B).$$

Then there is a probability distribution, ν, on the subsets of $\Gamma \times \Gamma$ such that

$$\begin{aligned} &\sum_{B\in\Gamma} \nu(A,B) = \mu_1(A) \quad &&\textit{for all} \quad A\in\Gamma \\ (8) \qquad &\sum_{A\in\Gamma} \nu(A,B) = \mu_2(B) \quad &&\textit{for all} \quad B\in\Gamma \\ &\textit{and}\ \ \nu(A,B)=0 \quad &&\textit{unless} \quad A\supset B\,. \end{aligned}$$

Proof. Let $c_i(x, A)$ be defined as in (3) with μ_i in place of μ and let Ω_i be defined as in (4) with c_i in place of c. For $A_1, A_2, B_1, B_2 \in \Gamma$ with either $A_1 \neq B_1$ or $A_2 \neq B_2$ define

$$\bar{\Omega}(A_1, A_2; B_1, B_2)$$

$$=\begin{cases} \min(c_1(x,A_1), c_2(x,A_2)) \text{ if } x\in(A_1\cap A_2)\cup(A_1^c\cap A_2^c) \\ \qquad\qquad \text{and } A_1 \triangle B_1 = A_2 \triangle B_2 = \{x\} \\ c_1(x,A_1) - \min(c_1(x,A_1), c_2(x,A_2)) \text{ if } x\in(A_1\cap A_2)\cup(A_1^c\cap A_2^c) \\ \qquad\qquad \text{and } A_1 \triangle B_1 = \{x\}, A_2 = B_2 \\ c_2(x,A_2) - \min(c_1(x,A_1), c_2(x,A_2)) \text{ if } x\in(A_1\cap A_2)\cup(A_1^c\cap A_2^c) \\ \qquad\qquad \text{and } A_1 = B_1, A_2 \triangle B_2 = \{x\} \\ \Omega_1(A_1,B_1) \text{ if } A_1 \triangle B_1 \subset A_1 \triangle A_2 \text{ and } A_2 = B_2 \\ \Omega_2(A_2,B_2) \text{ if } A_2 \triangle B_2 \subset A_1 \triangle A_2 \text{ and } A_1 = B_1 \\ 0 \text{ otherwise}. \end{cases}$$

Define $\bar{\Omega}(A_1, A_2; A_1, A_2)$ so that

$$\sum_{B_1, B_2 \in \Gamma} \bar{\Omega}(A_1, A_2; B_1, B_2) = 0\,.$$

We think of $\bar{\Omega}(A_1, A_2; B_1, B_2)$ as a matrix with rows indexed by (A_1, A_2) and columns indexed by (B_1, B_2). Then $\bar{\Omega}^n$ is just the matrix $\bar{\Omega}$ raised to the n^{th} power. Similarly Ω_i^n is the matrix Ω_i raised to the n^{th} power.

The following facts are easily checked by induction on n:

i) $\sum_{B_2\in\Gamma} \bar{\Omega}^n(A_1, A_2; B_1, B_2) = \Omega_1^n(A_1, B_1)$ for all $A_1, B_1 \in \Gamma$.

ii) $\sum_{B_1\in\Gamma} \bar{\Omega}^n(A_1, A_2; B_1, B_2) = \Omega_2^n(A_2, B_2)$ for all $A_2, B_2 \in \Gamma$.

iii) If $A_1 \supset A_2$, then $\bar{\Omega}^n(A_1, A_2; B_1, B_2) = 0$ unless $B_1 \supset B_2$.

In checking iii) one needs the lattice structure of Γ to guarantee for example that if $A_1 \supset A_2$ and $x \notin A_1$ but $A_2 \cup \{x\} \in \Gamma$, then $A_1 \cup \{x\} \in \Gamma$. Also the only place where the inequality (7) is used is in checking iii).

Now $\bar{\Omega}$ is the generator of a Markov process on $\Gamma \times \Gamma$. We take $(\Lambda, \emptyset)$ as the initial state and apply the ergodic theorem (see Theorem (1.1) in Chapter 6 of [2]) to this Markov process to conclude the existence of

$$\nu(B_1, B_2) = \lim_{t \to \infty} \sum_{n=0}^{\infty} \frac{t^n}{n!} \bar{\Omega}^n(\Lambda, \emptyset; B_1, B_2). \tag{9}$$

The ergodic theorem together with Lemma (5) applied to the Markov processes on Γ with generators Ω_i imply that

$$\begin{aligned} \mu_1(B_1) &= \lim_{t \to \infty} \sum_{n=0}^{\infty} \frac{t^n}{n!} \Omega_1^n(\Lambda, B_1) \\ \mu_2(B_2) &= \lim_{t \to \infty} \sum_{n=0}^{\infty} \frac{t^n}{n!} \Omega_2^n(\emptyset, B_2). \end{aligned} \tag{10}$$

Since $\Lambda \supset \emptyset$, iii) applied to (9) implies that $\nu(B_1, B_2) = 0$ unless $B_1 \supset B_2$. Finally by applying i), ii) and (10) we get

$$\begin{aligned} \sum_{B_2 \in \Gamma} \nu(B_1, B_2) &= \sum_{B_2} \lim_{t \to \infty} \sum_{n=0}^{\infty} \frac{t^n}{n!} \bar{\Omega}^n(\Lambda, \emptyset; B_1, B_2) \\ &= \lim_{t \to \infty} \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{B_2} \bar{\Omega}^n(\Lambda, \emptyset; B_1, B_2) \\ &= \lim_{t \to \infty} \sum_{n=0}^{\infty} \frac{t^n}{n!} \Omega_1^n(\Lambda, B_1) = \mu_1(B_1). \end{aligned}$$

Similarly $\sum_{B_1 \in \Gamma} \nu(B_1, B_2) = \mu_2(B_2)$.

This completes the proof of the theorem.

We conclude with two corollaries. The second one is the FKG inequality.

(11) **Corollary.** *Let Γ, μ_1, and μ_2 be as in Theorem (6). Let f be an increasing function on Γ. Then*

$$\sum_{A \in \Gamma} f(A)\, \mu_1(A) \geqq \sum_{B \in \Gamma} f(B)\, \mu_2(B).$$

Proof. Let ν be as in the conclusion of Theorem (6). Then

$$\begin{aligned} \sum_{A \in \Gamma} f(A)\, \mu_1(A) &= \sum_{A, B \in \Gamma} f(A)\, \nu(A, B) = \sum_{A \supset B} f(A)\, \nu(A, B) \\ &\geqq \sum_{A \supset B} f(B)\, \nu(A, B) = \sum_{A, B \in \Gamma} f(B)\, \nu(A, B) = \sum_{B \in \Gamma} f(B)\, \mu_2(B). \end{aligned}$$

(12) **Corollary** (*FKG*). *Let Γ be a finite distributive lattice and let μ be a probability distribution on Γ satisfying*

$$\mu(A \cup B)\, \mu(A \cap B) \geqq \mu(A)\, \mu(B). \tag{13}$$

Then if f and g are two increasing functions on Γ,

$$\sum_{A \in \Gamma} f(A)\, g(A)\, \mu(A) \geqq \sum_{A \in \Gamma} f(A)\, \mu(A) \sum_{B \in \Gamma} g(B)\, \mu(B)\,. \tag{14}$$

Proof. (13) implies that the $A \in \Gamma$ for which $\mu(A) > 0$ form a sublattice. By restricting our attention to this sublattice we may assume, without loss of generality, that μ is strictly positive. Also, by adding a constant if necessary, we may assume that $g > 0$. Now let

$$\mu_1(A) = g(A)\, \mu(A) / \sum_{B \in \Gamma} g(B)\, \mu(B)\,,$$

and let $\mu_2(A) = \mu(A)$.

Using (13) and the monotonicity of g one easily checks that μ_1 and μ_2 satisfy (7). Thus applying Corollary (11) to μ_1, μ_2 and f we conclude that

$$\sum_{A \in \Gamma} f(A)\, g(A)\, \mu(A) / \sum_{B \in \Gamma} g(B)\, \mu(B) \geqq \sum_{A \in \Gamma} f(A)\, \mu(A)\,,$$

which is the same as (14) since g is positive.

References

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