Fluctuating Magnetic Fields

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Abstract. Some problems pertaining to the behaviour of a classical spin under the influence of a random Gaussian magnetic field are discussed. It is shown that, in agreement with simple expectations, the magnetic moment is effectively decreased to lowest order. Various physical applications and connections with group theory are pointed out.

1. Introduction

The present investigation was motivated by the quest for a simple explanation of the size and sign of lowest order radiative corrections. It has often been proposed to understand qualitatively these effects by assuming that test charges be submitted in addition to external fields to random ones arising from vacuum quantum fluctuations [1]. Applied to a charged non-relativistic particle moving in a fixed external potential and to a fluctuating electric field, one is led to a formula for the Lamb shift in striking qualitative agreement with the exact result. A similar calculation performed for the anomalous magnetic moment yields a value of the correct order of magnitude but with *wrong* sign. This was generally attributed to added fluctuations arising from a relativistic treatment including negative energy states of the Dirac electron. Koba [2] pointed out that these added contributions were in the right direction. The fact that fluctuating magnetic fields tend to reduce the effective magnetic moment is suggested by the following heuristic argument. Consider the coupling of the magnetic moment μ to the external field B_0:

$$- \mu \cdot B_0 = - |\mu| \, |B_0| \cos \theta \, ,$$

we can write $\cos \theta$ as $\cos \bar{\theta} \cos \tilde{\theta} + \sin \bar{\theta} \sin \tilde{\theta} \cos(\bar{\varphi} - \tilde{\varphi})$, where $\bar{\theta}$ and $\bar{\varphi}$ are the polar angles of the mean direction of μ and $\tilde{\theta}$, $\tilde{\varphi}$ represent fluctuations. Averaging $\langle \cos \theta \rangle$ using the fact that $\tilde{\varphi}$ is uniformly distributed and that fluctuations are small, yields

$$\langle \cos \theta \rangle \cong \cos \bar{\theta} \left(1 - \frac{\langle \tilde{\theta} \rangle}{2} \right).$$

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Hence $|\boldsymbol{\mu}|$ is replaced by an effective smaller quantity $|\boldsymbol{\mu}| \, (1 - (\langle \tilde{\theta}^2 \rangle / 2))$. This argument might seem slightly oversimplified. It will be proved in essence correct for the realistic cases. Hence the whole idea of explaining in simple terms the lowest order radiative corrections by fluctuating fields requires real elaboration and deserves still more work to be fully elucidated.

Nevertheless, it was felt that the subject of the motion of a classical non-relativistic spin under the influence of a random magnetic field is, by itself, a non-trivial matter, the application of which is not limited to the above-mentioned problem. Some domains of application are the study of depolarization of a spin $\frac{1}{2}$ particle in a medium, the behaviour of two-level systems under random Hamiltonians, and, by generalizing to other groups than the ordinary three-dimensional rotation group, and other manifolds than the unit sphere, it can be cast in a group theoretic frame-work of generalized Brownian motion on Riemanian manifolds.

We shall refrain from doing so to keep the language simple and will phrase the discussion in terms of the motion of the spin s, a three-vector of fixed length, which without loss of generality can be taken equal to unity.

Section 2 gathers the necessary tools and presents a discussion of the general case. Sections 3 and 4 will elaborate two extreme situations for which a complete solution can be found and which both exhibit intrinsic geometric properties. Finally, in Section 5, we examine to lowest order the effect of the introduction of an external fixed magnetic field.

2. Preliminaries

The motion to be studied is described by the simple equation

$$\frac{d s}{d t} = \frac{g e}{2 m c} \boldsymbol{B}_t \times \boldsymbol{s} \, .$$

The notations are: e charge, m mass, c velocity of light, $\boldsymbol{B}_t$ magnetic field, s spin which is taken of unit length, g dimensionless gyromagnetic ratio (2 for a point Dirac particle). It simplifies matters to choose a field dimension in such a way that $(g e / 2 m c) = 1$, in which case $\boldsymbol{B}_t$ has the dimension of a (Larmor) frequency. From now on we write in matrix form:

$$\frac{d s}{d t} = \boldsymbol{B}_t \cdot \vec{\tau} s \tag{1}$$

where s is thought of as a column vector and the three τ_i matrices (representing the Lie algebra of rotations and corresponding to the repre-

sentation of spin 1) are defined as $(\tau_i)_{jk} = \varepsilon_{jik}$ and satisfy

$$[\tau_i, \tau_j] = -\varepsilon_{ijk}\tau_k \qquad \sum_i \tau_i^2 = -2$$

$$\tau_i = \tau_i^* = -\tau_i^T \qquad (\vec{\tau} \cdot \boldsymbol{p})^3 = -p^2 \vec{\tau} \cdot \boldsymbol{p} \quad \forall \boldsymbol{p}. \tag{2}$$

$$\sum_i \tau_i A \tau_i = A^T - \operatorname{tr} A \qquad \text{for any } 3 \times 3 \text{ matrix } A.$$

The random magnetic field B_t will be taken to be Gaussian, its correlation functions being specified by a generating functional

$$\left\langle \exp i \int_{-\infty}^{+\infty} dt\, \boldsymbol{\varphi}(t) \cdot \boldsymbol{B}(t) \right\rangle$$

$$= \exp -\frac{1}{2\theta^2} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2\, \varphi_i(t_1) Q_{ij}(t_1, t_2) \varphi_j(t_2). \tag{3}$$

A time scale θ has been introduced in order to make Q dimensionless. Its definition implies, of course, some specific normalization of Q.

We require the above expectation value to satisfy the following requirements:

(i) to be stationary in time, i.e., $Q_{ij}(t_1, t_2) = Q_{ij}(t_1 - t_2)$;

(ii) to be rotationally invariant, i.e., $Q_{ij}(t_1 - t_2) = \delta_{ij} Q(t_1 - t_2)$;

(iii) to be such that the measure, which serves to define the expectation values, be really a probability measure, i.e.,

$$\left| \left\langle \exp i \int_{-\infty}^{+\infty} dt\, \boldsymbol{\varphi}(t), \boldsymbol{B}(t) \right\rangle \right| \leqq 1$$

or

$$\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2\, \boldsymbol{\varphi}(t_1) \cdot \boldsymbol{\varphi}(t_2) Q(t_1 - t_2) \geqq 0.$$

According to a known theorem of Bochner, this means that $Q(t)$ is the Fourier transform of a measure. Furthermore, without loss of generality, $Q(t_1 - t_2) = Q(t_2 - t_1)$. With this at hand, we readily find:

$$\langle B_i(t_1) B_j(t_2) \rangle = \frac{1}{\theta^2} \delta_{ij} Q(t_1 - t_2). \tag{4}$$

The expectation value of an odd number of B fields vanishes while, for an even number, we have a Wick theorem

$$\langle B_{i_1}(t_1) \dots B_{i_{2n}}(t_{2n}) \rangle = \frac{1}{\theta^{2n}} \sum_{\substack{\text{all possible} \\ \text{pairings}}} \langle B_{i_{\alpha_1}}(t_{\alpha_1}) B_{i_{\alpha_2}}(t_{\alpha_2}) \rangle \dots \tag{5}$$

$$\dots \langle B_{i_{\alpha_{2n-1}}}(t_{\alpha_{2n-1}}) B_{i_{\alpha_{2n}}}(t_{\alpha_{2n}}) \rangle.$$

The number of terms occurring on the right-hand side is $(2n!/2^n n!)$ $= (2n-1)!!$. This right-hand side is sometimes called the haffnian of order n of the matrix operator $Q_{ij}(t-t')$.

Let the spin start at position s_1 at time t_1. Its evolution in time is dictated by Eq. (1), the solution of which we denote by $s(t; s_1, t_1)$. The function

$$G(s_2, t_2 | s_1, t_1) = \langle \delta(s(t_2; s_1, t_1), s_2) \rangle \qquad t_2 > t_1 \tag{6}$$

is the fundamental object we want to study. It is the probability of finding the spin at s_2 at time t_2 knowing that it was at s_1 at time t_1.

From translational invariance in time and rotational invariance, we have

$$G(s_2, t_2 | s_1, t_1) = G(s_1 \cdot s_2, t_2 - t_1)$$

$$\lim_{t_2 \to t_1} G(s_2 \cdot s_1, t_2 - t_1) = \delta_{sp}(s_2, s_1) \tag{7}$$

$$= \frac{1}{4\pi} \sum_0^\infty (2l + 1) P_l(s_2 \cdot s_1)$$

where $\delta_{sp}(s_2, s_1)$ is the δ function on the sphere. Furthermore

$$G(s_2 \cdot s_1, t) \geqq 0 \qquad \int d\Omega_2 \, G(s_2 \cdot s_1, t) = 1 \, .$$

Finally the definition (6) implies that

$$G(s_2 \cdot s_1, t_2 - t_1) = \frac{1}{4\pi} \sum_0^\infty (2l + 1) < P_l(s(t_2; s_1, t_1) \cdot s_2) \rangle$$

$$= \frac{1}{4\pi} \sum_0^\infty (2l + 1) P_l(s_2 \cdot s_1) g_l(t_2 - t_1) \, . \tag{8}$$

Indeed the formal solution of Eq. (1) is, with T standing for the time ordering symbol,

$$s(t; s_1, t_2) = T \left\{ \exp \int_{t_1}^t dt' \, \boldsymbol{B}(t') \cdot \vec{\tau} \right\} s_1 \, .$$

The operator standing in front of s_1 is an orthogonal real matrix depending functionally on B. We shall abbreviate it by

$$s(t; s_1, t_1) = U(t, t_1) s_1 \, .$$

Hence

$$\langle P_l(s(t_2; s_1, t_1) \cdot s_2) \rangle = \langle P_l(s_2 U(t_2, t_1) s_1) \rangle \, .$$

Now in the space of functions defined on the sphere, we introduce the ordinary angular momentum basis states $|lm)$ [we note the Dirac bras and kets $|)$ and $(|$ not to be confused with the expectation values $\langle \, \rangle$, over our random magnetic field]. Then

$$\frac{1}{4\pi} (2l + 1) P_l(s_2 \cdot s_1) = \sum_m (s_2 | lm)(lm | s_1) \, .$$

Consequently

$$\frac{2l+1}{4\pi} \langle P_l(s_2 U(t_2, t_1) s_1) \rangle$$

$$= \sum_{m, m'} (s_2 | lm) \langle (lm | U(t_2, t_1) | lm') \rangle (lm' | s_1).$$

Since U is a rotation it commutes with L^2, the square of the angular momentum operator. Now

$$\langle (lm | U(t_2, t_1) | lm') \rangle = g_l(t_2 - t_1) \delta_{mm'}.$$

Indeed, the translational invariance in time is clear from the property of expectation values. Furthermore, the expectation values of a rotated $B_t \to B_t' = R B_t$ are identical with those of B_t if the rotation R is time independent. But this is clearly equivalent to replacing U by $R^{-1} U R$. Hence

$$\langle (lm | U(t_2, t_1) | lm') \rangle = \int dR \langle (lm | R^{-1} U(t_2, t_1) R | lm') \rangle$$

where dR is the normalized measure on the rotation group. The right-hand side is, by the orthogonality property of matrix elements of irreducible representations:

$$\frac{1}{2l+1} \delta_{mm'} \left\langle \sum_\mu (l\mu | U(t_2, t_1) | l\mu) \right\rangle = \delta_{mm'} g_l(t_2 - t_1).$$

Then we have indeed

$$\langle P_l(s_2 U(t_2, t_1) s_1) \rangle = P_l(s_2 \cdot s_1) g_l(t_2 - t_1)$$

$$g_l(t_2 - t_1) = \frac{1}{2l+1} \left\langle \sum_m (lm | U(t_2, t_1) | lm) \right\rangle.$$

(9)

The quantities $(lm | U(t_2, t_1) | lm')$ are nothing but the matrix elements in the lth representation of the operator

$$T \left\{ \exp \int_{t_1}^{t_2} dt \, \mathbf{L} \cdot \mathbf{B}(t) \right\}$$

understood for an abstract $\mathbf{L}$ satisfying only

$$[L_i, L_j] = -\varepsilon_{ijk} L_k \qquad L^+ = -L$$

with the states $|lm)$ such that

$$L^2 | lm) = -l(l+1) | lm) \qquad L_3 | lm) = im | lm),$$

$$-l \leqq m \leqq +l$$

(10)

the functions of L being envisaged in the enveloping algebra of the Lie algebra, of commutation relations. We use a real notation, in contradistinction to physicist's habits not to be bothered by unnecessary i's. We note in passing that we could as well discuss the evolution of a spin $\frac{1}{2}$, ψ, under the influence of a random Hamiltonian $\frac{1}{2}\boldsymbol{B}_t \cdot \boldsymbol{\sigma}$ ($\boldsymbol{\sigma}$: Pauli matrices). Then, if its density matrix at time zero is $\frac{1}{2}(1 + \boldsymbol{P} \cdot \boldsymbol{\sigma})$, its polarization at time t would be, according to the above discussion, $\boldsymbol{P}(t) = g_1(t) \boldsymbol{P}$ and g_1 would describe its depolarization in time (in a non-isotropic medium it could happen that Q_{ij} is non-diagonal and a more general discussion is necessary; see Section 5).

In order to motivate the following sections, let us qualitatively sketch our expectations for the behaviour of G for an arbitrary correlation function $Q(t)$. To be specific, assume $Q(t)$ to be the Fourier transform of a smooth positive function decreasing fast at infinity [we can have in mind an example like $(1/a) e^{-bt^2}$]. For short times, if $Q(t)$ is sufficiently smooth, the behaviour should be very similar to the one for the case where Q is simply constant, which we call, for lack of a better word, "black noise" (since its spectrum is characterized by the zero frequency only) and represents the strong correlation case. On the other hand, for any reasonable system, $Q(t)$ will have a correlation time t_c and for $t \gg t_c$ we expect the behaviour to be very much the same as if $Q(t)$ were proportional to a δ function – which we call "white noise" since its spectrum is flat. These two extreme cases are discussed in detail in the following sections.

We have introduced above the conditional probability

$$G(s_2, t_2 | s_1, t_1) \quad t_2 > t_1 .$$

We could unfold completely the process by studying more general multi-time probabilities of the type

$$G_3(s_3, t_3; s_2, t_2 | s_1, t_1) = \langle \delta(s(t_3; s_1, t_1), s_3) \delta(s(t_2; s_1, t_1), s_2) \rangle \quad t_3 > t_2 > t_1$$

etc. Clearly,

$$G(s_3, t_3 | s_1, t_1) = \int d\Omega_2 G_3(s_3, t_3; s_2, t_2 | s_1, t_1) .$$

If a correlation time t_c exists, we expect that for $t_3 - t_2 \gg t_c$ $t_2 - t_1 \gg t_c$ we have a quasi factorization

$$G_3(s_3, t_3; s_2, t_2 | s_1, t_1) \approx G(s_3, t_3 | s_2, t_2) G(s_2, t_2 | s_1, t_1) .$$

If this is not an asymptotic equality but valid at all times then the process is Markovian and the semi-group law applies

$$G(s_3, t_3 | s_1, t_1) = \int d\Omega_2 G(s_3, t_3 | s_2, t_2) G(s_2, t_2 | s_1, t_1) \quad t_3 > t_2 > t_1 .$$

It is suggested that this is the case when $Q(t)$ is a δ function. This will be proved in Section 3.

On the other hand, when no correlation time exists, $Q(t)$ is decreasing very slowly or tending to a constant at infinity [it cannot grow since positivity implies in particular $|Q(t)| \leq Q(0)$]. We have then on the sphere a strongly correlated process and we do not even expect the motion to be ergodic at large times, i.e., for $t_2 \to \infty$, $G(s_2, t_2; s_1, t_1)$ does not tend to a uniform distribution $1/4\pi$ but, on the contrary, to some stable configuration $G_\infty(s_2, s_1)$.

We can give a very heuristic argument leading to a formula for $G_\infty(s_2, s_1)$. In Section 4, a calculation of $G(s_2, t_2 | s_1, t_1)$ will be given for $Q(t) =$ constant. If $Q(t) \to$ constant then, even for large times, the magnetic field is strongly correlated. It is "as if" we were discussing the motion under the influence of a constant magnetic field of unknown direction (isotropy of Q_{ij}). Consequently, we can guess that the average value $\langle \ \rangle$ can be interpreted for large times as meaning average over time and direction

$$\langle \ \rangle \to \lim_{t_2 - t_1 \to \infty} \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} dt \int \frac{d\Omega_B}{4\pi}.$$

Now under the action of a constant field, the evolution simplifies to $U(t_2, t_1) = e^{\boldsymbol{B} \cdot \vec{\tau}(t_2 - t_1)}$ with no time ordering required, and we expect G_∞ to be given by

$$G_\infty(s_2 \cdot s_1) = \frac{1}{4\pi} \sum_0^\infty (2l + 1) P_l(s_2 \cdot s_1)$$

$$\cdot \lim_{t \to \infty} \frac{1}{t} \int_0^t dt' \int \frac{d\Omega_B}{4\pi} \frac{1}{(2l+1)} \sum_m (lm | e^{t \boldsymbol{L} \cdot \boldsymbol{B}} | lm).$$

Only the terms corresponding to $m = 0$ survive the time averaging and we find

$$G_\infty(s_2 \cdot s_1) = \frac{1}{4\pi} \sum_0^\infty P_l(s_2 \cdot s_1) = \frac{1}{4\pi} \frac{1}{\sqrt{2(1 - s_1 \cdot s_2)}} = \frac{1}{4\pi} \cdot \frac{1}{|s_1 - s_2|} \quad (11)$$

where $|s_1 - s_2|$ is the Euclidean distance in the ambient Euclidean space. As it should, $\int d^2 s_2 \, G_\infty(s_2, s_1) = 1$, and G_∞ is a good candidate for a probability distribution on the sphere. This is indeed what will emerge from the more exact calculation. Note that $g_1(\infty) = \frac{1}{3}$.

The fact that this equilibrium distribution appears as a Coulomb potential created by a unit charge at point s_1 is at first sight rather intriguing.

Finally the reader will have no difficulty to prove the following positivity property of the function $G_2(s_1 \cdot s_2, t)$. Write $G_2(s_1 \cdot s_2, -t) = G_2(s_1 \cdot s_2, t)$ and let $\varphi(s, t)$ be a real function decreasing fast enough for $|t| \to \infty$, then

$$\int dt_1 \int dt_2 \int d\Omega_1 \int d\Omega_2 \, \varphi(s_2, t_2) \, G(s_2 \cdot s_1, t_2 - t_1) \, \varphi(s_1 t_1) > 0.$$

If φ is expanded in spherical harmonics, we find in particular

$$\int dt_1 \int dt_2 \, \psi(t_2) \, g_l(t_2 - t_1) \, \psi(t_1) > 0 \,.$$

Hence $g_l(t)$ extended as an even function of t is the Fourier transform of a measure, with total measure one since $g_l(0) = 1$. In particular, $|g_l(t)| < 1$. This can be verified on the examples worked out later on.

3. White Noise

This case corresponds to $Q(t_1 - t_2) = \theta \delta(t_1 - t_2)$. We have to take the mean value

$$\left\langle T \left\{ \exp \int_0^t dt' \, \boldsymbol{B}(t') \cdot \boldsymbol{L} \right\} \right\rangle \,.$$

Quite generally, this quantity equals one for $t = 0$ and its derivative is

$$\frac{d}{dt} \left\langle T \left\{ \exp \int_0^t dt' \, \boldsymbol{B}(t') \cdot \boldsymbol{L} \right\} \right\rangle$$

$$= \frac{1}{\theta^2} \cdot \int_0^t dt_1 \, Q(t - t_1) \sum_{k=1}^3 \left\langle L_k \, T \left\{ \exp \int_{t_1}^t dt_1' \, \boldsymbol{B}(t_1') \cdot \boldsymbol{L} \right\} L_k \right.$$

$$\left. \cdot T \left\{ \exp \int_0^{t_1} dt_1'' \, \boldsymbol{B}(t_1'') \cdot \boldsymbol{L} \right\} \right\rangle \,.$$

Now the integral will be trivial if $Q(t - t_1)$ is concentrated at $t = t_1$. We cannot set $Q = \theta \delta$ at once since $\int_0^a \delta(t) \, dt$ is meaningless. However, δ can be approximated as close as we wish by Gaussians for instance, which satisfy all the constraints imposed on a correlation function. In particular, they are symmetric in time. In the limit, this introduces a factor $\frac{1}{2}$ and we see that:

$$\frac{d}{dt} \left\langle T \left\{ \exp \int_0^t dt' \, \boldsymbol{B}(t') \cdot \boldsymbol{L} \right\} \right\rangle = \frac{L^2}{2\theta} \left\langle T \left\{ \exp \int_0^t dt' \, \boldsymbol{B}(t') \cdot \boldsymbol{L} \right\} \right\rangle \,.$$

Taking into account the boundary condition at $t = 0$ one can solve and obtain:

$$\left\langle T \left\{ \exp \int_0^t dt' \, \boldsymbol{B}(t') \cdot \boldsymbol{L} \right\} \right\rangle = \exp \frac{t}{2\theta} L^2 \,.$$

And

$$G(s_2, t_2 \,|\, s_1, t_1) = \frac{1}{4\pi} \sum_0^\infty (2l + 1) \, P_l(s_2 \cdot s_1) \, e^{-\frac{l(l+1)(t_2 - t_1)}{2\theta}} \,. \qquad (12)$$

As an operator on functions defined on the sphere, L^2 is identical with the Laplace operator Δ^{sph}, and we see that G satisfies the diffusion equation

of Brownian motion

$$\left(2\theta\frac{\partial}{\partial t_2} - \Delta_{s_2}^{sph}\right) G(s_2, t_2 \,|\, s_1, t_1) = 0 \quad \lim_{t_2 \to t_1} G(s_2, t_2 \,|\, s_1, t_1) = \delta^{sp}(s_2, s_1). \quad (13)$$

Clearly, the Markovian character of the motion follows from the fact that Q is a δ function:

$$G(s_3, t_3 \,|\, s_1, t_1) = \int d\Omega_2 \, G(s_3, t_3 \,|\, s_2, t_2) \, G(s_2, t_2 \,|\, s_1, t_1)$$

while the generator of the semi-group of evolution is $\Delta^{sph}/2\theta$. The series solution (12) exhibits simply the large time behaviour of G which tends to $1/4\pi$ for $t \gg \theta$ the only time scale of the problem. This is characteristic of a Brownian motion on a compact space. The function G is quite remarkable and we shall briefly list below some of its properties. In a sense, it is a generalization of the well-known Jacobi Θ functions, introduced in the theory of elliptic functions. Indeed for the corresponding problem of diffusion on a circle instead of a sphere, we would find, writing $s_{1,2} = e^{i\varphi_{1,2}}$

$$\sum_{-\infty}^{+\infty} e^{in(\varphi_2 - \varphi_1)} e^{-\frac{n^2 t}{2\theta}} = 1 + 2\sum_{1}^{\infty} \cos 2n\pi \frac{(\varphi_2 - \varphi_1)}{2\pi} \left(e^{-\frac{t}{2\theta}}\right)^{n^2}$$

$$\equiv \Theta_3\left(\frac{\varphi_2 - \varphi_1}{2\pi}, e^{-\frac{t}{2\theta}}\right)$$

in the conventional notation [3] [1].

In fact there is a connection, by means of Abel's transforms, between our G and Θ functions. This can be exploited as we shall see to give the short-time behaviour of G which we expect to be essentially given by a Gaussian of width proportional to $\sqrt{t}$ around $s_1 = s_2$, i.e., $|s_1 - s_2|$ proportional to $\sqrt{t}$. Finally, there exists a reproducing property up to a factor when wandering on the complex sphere reminiscent of the corresponding property of Θ functions:

$$\Theta_3\left(\frac{\varphi + it/\theta}{2\pi}, e^{-\frac{t}{2\theta}}\right) = e^{\frac{t}{2\theta} - i\varphi} \, \Theta_3\left(\frac{\varphi}{2\pi}, e^{-\frac{t}{2\theta}}\right).$$

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[1] $\Theta_1(v, q) = 2q^{\frac{1}{4}} \sum_0^\infty (-1)^n q^{n(n+1)} \sin(2n+1)\pi v,$

$\Theta_2(v, q) = 2q^{\frac{1}{4}} \sum_0^\infty q^{n(n+1)} \cos(2n+1)\pi v,$

$\Theta_3(v, q) = 1 + 2\sum_1^\infty q^{n^2} \cos 2n\pi v,$

$\Theta_4(v, q) = 1 + 2\sum_1^\infty (-1)^n q^{n^2} \cos 2n\pi v.$

We should, of course, have remarked that $G(s_1 \cdot s_2, t)$ is an entire analytic function of order zero in $s_1 \cdot s_2$. Recalling the Mehler-Dirichlet integral representation of Legendre polynomials [3]

$$P_l(\cos\varphi) = \frac{\sqrt{2}}{\pi} \int_0^\varphi \frac{dv}{\sqrt{\cos v - \cos\varphi}} \cos(l+\tfrac{1}{2}) v$$

and writing $s_1 \cdot s_2 = \cos\varphi$

$$G(-\cos\varphi, t)$$

$$= \frac{\sqrt{2}}{4\pi^2} \int_0^\varphi \frac{dv}{\sqrt{\cos v - \cos\varphi}} \sum_0^\infty (2l+1)(-1)^l \cos(l+\tfrac{1}{2})v\, e^{-l(l+1)\frac{t}{2\theta}}$$

$$= \frac{\sqrt{2}}{4\pi^2} \int_0^\varphi \frac{dv}{\sqrt{\cos v - \cos\varphi}} \frac{d}{dv} 2\sum_0^\infty (-1)^l \sin(l+\tfrac{1}{2})v\, e^{-l(l+1)\frac{t}{2\theta}}$$

$$= \frac{\sqrt{2}}{4\pi^2} \int_0^\varphi \frac{dv}{\sqrt{\cos v - \cos\varphi}} \frac{d}{dv} \Theta_2\left(\frac{v}{2\pi}, e^{-\frac{t}{2\theta}}\right) e^{\frac{t}{8\theta}}.$$

Hence

$$G(\cos\varphi, t) = -\frac{\sqrt{2}}{(2\pi)^3} \int_\varphi^\pi \frac{dv}{\sqrt{\cos\varphi - \cos v}} \Theta_2'\left(\frac{v}{2\pi}, e^{-\frac{t}{2\theta}}\right) e^{\frac{t}{8\theta}} \quad (14)$$

where prime denotes derivatives with respect to the first argument. This Eq. (14) is the required connection between G and Θ function.

Instead of working out the complete series for the behaviour of G close to $t=0$ we will content ourselves with the leading term. Now using the Poisson summation formula

$$\sum_{-\infty}^{+\infty} f(n)\, e^{in\varphi} = \sum_{-\infty}^{+\infty} \frac{1}{2\pi} \int \tilde{f}(\alpha)\, e^{in(\varphi-\alpha)}\, d\alpha = \sum_{-\infty}^{+\infty} \tilde{f}(\varphi + 2n\pi)$$

$$\Theta_2'\left(\frac{v}{2\pi}, e^{-\frac{t}{2\theta}}\right) = \left(\frac{2\pi\theta}{t}\right)^{\frac{3}{2}} \sum_{-\infty}^{+\infty} (-1)^{n+1}(v+2n\pi)\, e^{-\frac{\theta}{2t}(v+2n\pi)^2}.$$

Hence the leading term of G for t small is

$$\frac{1}{\sqrt{2}} \left(\frac{\theta}{2\pi t}\right)^{\frac{3}{2}} \int_\varphi^{2\pi-\varphi} \frac{dv}{\sqrt{\cos\varphi - \cos v}} \left[v e^{-\frac{\theta}{2t}v^2} + (2\pi - v) e^{-\frac{\theta}{2t}(2\pi-v)^2}\right] e^{\frac{t}{8\theta}}.$$

The factor $e^{t/8\theta}$ goes to one as $t\to 0$. Although this gives, in principle, an answer valid for all angles, it is hard to extract from it the required information. Returning to the very definition (12) of G we note that the factor $e^{-l(l+1)(t/2\theta)}$ is very slowly varying for $t \ll \theta$. Hence large terms will predominantly contribute to the sum which will be concentrated at very

small φ. Thus we can approximate $P_l(\cos\varphi)$ by $J_0((2l+1)\sin\varphi/2)$ and replace the summation by an integral, the well-known impact parameter approximation. This yields

$$G(\cos\varphi, t) = \frac{1}{(2\pi)^2} \int d^2 b\, e^{i\boldsymbol{b}\cdot(\boldsymbol{s}_2 - \boldsymbol{s}_1) - \frac{b^2 t}{2\theta}} = \frac{\theta}{2\pi t} e^{-|s_1 - s_2|^2 \frac{\theta}{2t}} \qquad t\to 0,\ \varphi\ \text{small}$$

a mildly surprising result and obtainable by approximating the sphere close to s_1 by a plane. This is, of course, not exact for φ close to π where the above integral formula can be used and yields

$$G(-1, t) \cong \sqrt{\frac{\pi}{2}} \left(\frac{\theta}{t}\right)^{\frac{3}{2}} e^{-\frac{\theta\pi^2}{2t}} \qquad t\to 0\,.$$

The last comment we would like to make has to do with analytic continuation on the complex sphere or complex $\cos\varphi$ plane. This is best expressed in the language of group theory. The translations in the complex φ plane of magnitude it/θ discussed for the automorphic Θ_i functions should in the present case be replaced by complex rotations as follows.

Let R be a rotation. Choose s_1 along the z axis, s_2 in the (x, z) plane and decompose as usual R in the product $R_\gamma R_\beta R_\alpha$ of three rotations of angle α around the z axis, β around the $y = (s_1 \times s_2)/(|s_1 \times s_2|)$ axis and γ around the z axis again. Then $s_2 \cdot R s_1 = \cos\beta \cos\varphi + \sin\beta \sin\varphi \cos\gamma$ and

$$\int\limits_0^{2\pi} \int\limits_0^{2\pi} \frac{d\alpha\, d\gamma}{(2\pi)^2} P_l(s_2 R s_1) = P_l(s_2 \cdot s_1) P_l(\cos\beta)\,.$$

Now one has the following Abel transformation

$$\frac{(2l+1)}{2\sqrt{2}} \int\limits_1^{\mathrm{ch}\, a} \frac{dx\, P_l(x)}{\sqrt{\mathrm{ch}\, a - x}} = \mathrm{sh}(l + \tfrac{1}{2})\, a$$

which can easily be proved using the generating function of Legendre polynomials [3].

This means that if we take β complex

$$\frac{1}{8\pi^2} \int\limits_0^{2\pi} d\alpha \int\limits_0^{2\pi} d\gamma \int\limits_0^a \mathrm{sh}\,\beta\, d\beta (2l+1)\, P_l(s_2 R_{\gamma, i\beta, \alpha} s_1) \cdot \frac{1}{\sqrt{2(\mathrm{ch}\, a - \mathrm{ch}\, \beta)}}$$

$$= P_l(s_2 \cdot s_1)\, \mathrm{sh}(l + \tfrac{1}{2})\, a\,.$$

Now

$$\left(\frac{2d}{da} - 1\right) \mathrm{sh}(l + \tfrac{1}{2})\, a = -\,\mathrm{sh}(l + \tfrac{1}{2})\, a + (2l+1)\, \mathrm{ch}(l + \tfrac{1}{2})\, a$$

$$= l\, e^{(l + \frac{1}{2})a} + (l+1)\, e^{-(l + \frac{1}{2})a}\,.$$

We cannot, without precautions, take the derivative d/da of the kernel $\theta(\mathrm{ch}\, a - \mathrm{ch}\, \beta)/\sqrt{(\mathrm{ch}\, a - \mathrm{ch}\, \beta)}$. This can be shown in fact to exist when

applied to smooth functions and is then defined by analytic continuation of

$$\frac{\theta(\operatorname{ch} a - \operatorname{ch} \beta)}{\Gamma(1 - \alpha)(\operatorname{ch} a - \operatorname{ch} \beta)^{\alpha}}.$$

Thus we are led to consider the integral

$$\frac{1}{8\pi^2}\left(\frac{2d}{da} - 1\right)\int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \int_0^a \operatorname{sh}\beta \, d\beta \, G(s_2 R_{\gamma, i\beta, \alpha} s_1, t) \frac{1}{\sqrt{2(\operatorname{ch} a - \operatorname{ch} \beta)}}\bigg|_{a = t/\theta}$$

$$= \frac{1}{4\pi}\sum_0^{\infty} P_l(s_2 \cdot s_1)\left[(l+1)e^{-(l+\frac{1}{2})\frac{t}{\theta}} + l e^{(l+\frac{1}{2})\frac{t}{\theta}}\right]e^{-l(l+1)\frac{t}{2\theta}}$$

$$= e^{\frac{t}{2\theta}}\frac{1}{4\pi}\sum_0^{\infty}\left[lP_{l-1}(s_2 \cdot s_1) + (l+1)P_{l+1}(s_2 \cdot s_1)\right]e^{-l(l+1)\frac{t}{2\theta}}.$$

Finally, making use of the known recursion relations for Legendre polynomials, it is seen that the series sums up to $e^{t/2\theta}(s_1 \cdot s_2) G(s_1 \cdot s_2, t)$. If we remark that $2(d/da) - 1$ can be written $2e^{a/2}(d/da)e^{-a/2}$ we see that we have established that

$$\frac{d}{da}\left\{e^{-a/2}\int_0^{2\pi}\frac{d\gamma}{2\pi}\int_0^a \operatorname{sh}\beta \, d\beta \frac{G(\cos\varphi \operatorname{ch}\beta + i\sin\varphi \operatorname{sh}\beta \cos\gamma, t)}{\sqrt{2(\operatorname{ch} a - \operatorname{ch}\beta)}}\bigg|_{a = t/\theta}\right. \qquad (15)$$

$$= \cos\varphi \, G(\cos\varphi, t).$$

We close this section by the last remark that the values of G at $\varphi = 0, \pi$ are related to the thermodynamics of rotation levels, by a suitable interpretation of t/θ as inversely proportional to the temperature.

4. Black Noise

In this section we find the probability distribution G in the case of strong correlations of the magnetic field, i.e., the function Q is a constant which, by a proper choice of θ, is taken equal to one. Our goal is still to compute g_l defined by (9). This we do directly by using Eqs. (4) and (5) giving expectation values of polynomials in $B_i(t)$. Calculations are greatly simplified by the fact that Q is a constant. Thus

$$\left\langle T\left\{\exp\int_0^t dt' \, \boldsymbol{B}(t') \cdot \vec{\tau}\right\}\right\rangle.$$

$$= 1 + \sum_1^{\infty}\int_{t \geq t_1 \geq \cdots \geq t_{2n} \geq 0}^{(2n)} dt_1 \ldots dt_{2n}\langle \boldsymbol{B}(t_1) \cdot \boldsymbol{L} \ldots \boldsymbol{B}(t_{2n}) \cdot \boldsymbol{L}\rangle$$

with

$$\langle \boldsymbol{B}(t_1) \cdot \boldsymbol{L} \ldots \boldsymbol{B}(t_{2n}) \cdot \boldsymbol{L}\rangle = \frac{1}{\theta^{2n}}\sum_{\substack{\text{all distinct} \\ \text{pairings}}}\delta_{\alpha, \alpha_2}\ldots\delta_{\alpha_{2n-1}\alpha_{2n}}L_{i_1}\ldots L_{i_{2n}}.$$

In this expression the indices $\alpha_1 \ldots \alpha_{2n}$ are a permutation of the indices $i_1 \ldots i_{2n}$. Next we notice the following integral over the unit three vector $\hat{p}$

$$\int \frac{d\Omega_{\hat{p}}}{4\pi} p_{i_1} \ldots p_{i_{2n}} = \frac{2^n n!}{(2n+1)!} \sum_{\substack{\text{all distinct} \\ \text{pairings}}} \delta_{\alpha_1 \alpha_2} \ldots \delta_{\alpha_{2n-1} \alpha_{2n}} .$$

The coefficient is found simply by taking, say, $i_1 = \ldots i_{2n} = 3$. Then the right-hand side reduces to $1/2n+1$ while the left-hand side is

$$\int_{-1}^{+1} \frac{d\cos\theta}{2} (\cos\theta)^{2n} = \frac{1}{2n+1} .$$

This means in the present case that

$$\langle \boldsymbol{B}(t_1) \cdot \boldsymbol{L} \ldots \boldsymbol{B}(t_{2n}) \cdot \boldsymbol{L} \rangle = \frac{(2n+1)!}{2^n n!} \int \frac{d\Omega_{\hat{p}}}{4\pi} (L\hat{p})^{2n} .$$

Furthermore

$$\frac{(2n+1)!}{2^n n!} = \frac{2^{n+1}}{\Gamma(1/2)} \int_0^\infty du\, e^{-u} u^{n+\frac{1}{2}} .$$

Putting everything together:

$$\left\langle T \left\{ \exp \int_0^t dt'\, \boldsymbol{B}(t') \cdot \boldsymbol{L} \right\} \right\rangle$$

$$= 1 + \sum_1^\infty \frac{1}{2n!} \left(\frac{t}{\theta} \right)^{2n} \frac{2^{n+1}}{\Gamma(1/2)} \int_0^\infty du\, e^{-u} u^{n+\frac{1}{2}} \int \frac{d\Omega_{\hat{p}}}{4\pi} (L \cdot \hat{p})^{2n} .$$

Since the integral

$$\int \frac{d\Omega_{\hat{p}}}{4\pi} (L \cdot \hat{p})^{2n+1}$$

vanishes, we can add the odd terms and we find

$$\left\langle T \left\{ \exp \int_0^t dt'\, \boldsymbol{B}(t') \cdot \boldsymbol{L} \right\} \right\rangle = \frac{2}{\sqrt{(1/2)}} \int_0^\infty du\, e^{-u} u^{\frac{1}{2}} \int \frac{d\Omega_{\hat{p}}}{4\pi} \exp\left(\frac{t\sqrt{2u}}{\theta} \boldsymbol{L} \cdot \hat{p} \right) .$$

Finally, we set $\alpha = (t/\theta)\sqrt{2u}$

$$\left\langle T \left\{ \exp \int_0^t dt'\, \boldsymbol{B}(t') \cdot \boldsymbol{L} \right\} \right\rangle = \frac{1}{\sqrt{2\pi}} \left(\frac{\theta}{t} \right)^3 \int_{-\infty}^{+\infty} d\alpha\, \alpha^2 e^{-\frac{\alpha^2 \theta^2}{2t^2}} \int \frac{d\Omega_{\hat{p}}}{4\pi} e^{\alpha \boldsymbol{L} \cdot \hat{p}} . \quad (16)$$

The integral over $\hat{p}$ yields, as already shown, an even function of α. This is why we have extended the α integral from $-\infty$ to $+\infty$. If $\alpha\hat{p}$ is interpreted as an effective magnetic field multiplied by time, we see indeed that we have to average over its direction. This is in agreement with the

heuristic arguments of Section 2. The time average that we were using for $t \to \infty$ is here replaced by the more exact α integration.

The coefficient $g_l(t)$ is now obtained by taking $1/2l+1$ times the trace of the above expression in the lth representation. Interchanging traces and integration, we see that

$$\sum_m (lm|e^{\alpha L \hat{p}}|lm) = \sum_{-l}^{l} e^{i\alpha m}$$

irrespectively of the direction of $\hat{p}$. Hence

$$g_l(t) = \frac{1}{2l+1} \sum_{-l}^{+l} \frac{1}{\sqrt{2\pi}} \left(\frac{\theta}{t}\right)^3 \int_{-\infty}^{+\infty} d\alpha\, \alpha^2\, e^{-\frac{\alpha^2}{2t^2}\theta^2 + im\alpha}$$

$$g_l(t) = \frac{1}{2l+1} \sum_{-l}^{+l} \Phi(mt) = \frac{1}{(2l+1)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\alpha\, e^{-\frac{\alpha^2}{2}} \alpha^2 \frac{\sin\frac{\alpha t}{\theta}(l+\frac{1}{2})}{\sin\frac{\alpha t}{2\theta}} \quad (17)$$

with

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\alpha\, \alpha^2\, e^{-\frac{\alpha^2}{2} + i\frac{t\alpha}{\theta}} = \left(1 - \frac{t^2}{\theta^2}\right) e^{-\frac{t^2}{2\theta^2}}.$$

And

$$G(s_2 \cdot s_1, t) = \frac{1}{4\pi} \sum_0^\infty P_l(s_2 \cdot s_1) \sum_{-l}^{+l} \Phi(mt)$$

$$G(s_2 \cdot s_1, t) = \frac{1}{4\pi} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\alpha\, e^{-\frac{\alpha^2}{2}} \alpha^2 \sum_0^\infty \frac{P_l(s_2 \cdot s_1) \sin\frac{\alpha t}{\theta}(l+\frac{1}{2})}{\sin\frac{\alpha t}{2\theta}}.$$

The series inside this integral sign can be summed and we find the two alternative expressions

$$G(s_2 \cdot s_1, t) = \frac{1}{4\pi} \sum_0^\infty P_l(s_2 \cdot s_1) \sum_{-l}^{+l} \left(1 - \frac{m^2 t^2}{\theta^2}\right) e^{-\frac{m^2 t^2}{2\theta^2}} \quad (18a)$$

$$G(s_2 \cdot s_1, t) = \frac{\theta}{4\pi} \frac{d}{dt} \sum_{n=-\infty}^{+\infty} \frac{1}{2\sqrt{\pi}} \int_{\varphi}^{2\pi-\varphi} d\alpha\, \frac{e^{-\frac{(\alpha-2n+1)^2\theta^2}{2t^2}}}{\sin\frac{\alpha}{2}} \frac{1}{\sqrt{\cos\varphi - \cos\alpha}}$$

$$(s_2 \cdot s_1 = \cos\varphi). \quad (18b)$$

If $\alpha \bmod 2\pi$ is thought as the angle of a rotation that brings s_1 over s_2 then clearly it can only run between φ and $2\pi - \varphi$. In fact this form could have been obtained more directly by a geometrical argument on the

integral (16) evaluated between s_2 and s_1. The series (18a) clearly exhibits the long time behaviour $t \gg \theta$ towards the equilibrium distribution

$$G(s_2 \cdot s_1, \infty) = \frac{1}{4\pi} \frac{1}{\sqrt{2(1 - s_2 \cdot s_1)}} = \frac{1}{4\pi |s_2 - s_1|}$$

as expected. The expansion (18b), on the other hand, is more suitable for short times. The dominant terms for $t \ll \theta$ are those with $n = 0, n = 1$ and they give equal results (changing α in $2\pi - \alpha$). Hence the leading behaviour is

$$G(\cos\varphi, t) \sim \theta \frac{d}{dt} \frac{1}{4\pi} \frac{1}{\sqrt{\pi}} \int_\varphi^{2\pi-\varphi} d\alpha \frac{e^{-\frac{\alpha^2 \theta^2}{2t^2}}}{\sin\frac{\alpha}{2}} \frac{1}{\sqrt{\cos\varphi - \cos\alpha}} \qquad t \to 0 .$$

For φ close to zero, we find

$$G(s_2 \cdot s_1, t) \sim \frac{1}{2\pi} \left(\frac{\theta}{t}\right)^2 e^{-|s_2 - s_1|^2 \frac{\theta^2}{2t^2}} \quad t \to 0 \quad s_1 \sim s_2$$

which could have been obtained by assimilating the sphere to its tangent plane and which shows that $|s_1 - s_2|$ is proportional to t/θ as if a velocity was defined; while for $s_1 = -s_2$, we have

$$G(-1, t) \sim \frac{1}{4} \left(\frac{\sqrt{\pi\theta}}{t}\right)^3 e^{-\frac{\pi^2\theta^2}{2t^2}} \qquad t \to 0 .$$

As was pointed out this "black noise" does not yield a Markovian process on the sphere. There is no semi-group law, no generator, hence no simple partial differential equation. In compact form we have

$$G(s_2, t_2 | s_1, t_1) = \int \frac{d\alpha}{\sqrt{2\pi}} \alpha^2 e^{-\frac{\alpha^2}{2}} \int \frac{d^2\hat{p}}{4\pi} (s_2 | e^{\frac{\alpha(t_2 - t_1)}{\theta} L \cdot \hat{p}} | s_1).$$

The calculation of higher probabilities, of the type $G_3(s_3, t_3; s_2, t_2 | s_1, t_1)$ although possible in principle, is not simple.

5. Behaviour in an External Field

In this last section, we examine the behaviour of the spin in the presence of a fixed external field B_0 added to the fluctuating one B. The motion is given by the equation:

$$\frac{ds}{dt} = (B_0 + B(t)) \cdot \vec{\tau} s .$$

It is both customary and practical to use a frame rotating around $\hat{B}_0$ at the frequency $|\boldsymbol{B}_0|$. By writing $s(t) = e^{\boldsymbol{B}_0 \cdot \vec{\tau} t} \tilde{s}(t)$ (at time zero the frames coincide)

$$\frac{d}{dt} \tilde{s}(t) = \boldsymbol{C}(t) \cdot \vec{\tau} \tilde{s}(t)$$

where

$$\boldsymbol{C}(t) \cdot \vec{\tau} = e^{-\boldsymbol{B}_0 \cdot \vec{\tau} t} \boldsymbol{B}(t) \cdot \vec{\tau} e^{\boldsymbol{B}_0 \cdot \vec{\tau} t}$$

is again a Gaussian random field but with a correlation matrix

$$\tilde{Q}_{ij}(t - t') = (e^{\boldsymbol{B}_0 \cdot \vec{\tau}(t - t')})_{ij} Q(t - t')$$

$$\tilde{Q}_{ij}(t - t') = \tilde{Q}_{ji}(t' - t).$$

If we deal initially with white noise $\tilde{Q} = Q$. Returning to the variable s this means that in the presence of B_0

$$G(s_2, t_2 | s_1, t_1)|_{B_0} = G(s_2 \cdot e^{\boldsymbol{B}_0 \cdot \vec{\tau}(t_2 - t_1)} s_1, t_2 - t_1)_{B_0 = 0}.$$

We interpret this fact in this particular case by saying that no renormalization effect arises. Apart from the blurring of the whole picture due to the random field, the spin precesses around B_0 with the same angular velocity it had in the absence of perturbing fluctuations. This is not in complete disagreement with the heuristic arguments of the introduction, since in this case the root mean square displacements envisioned do not exist. Nevertheless it shows that some care must be exercized in order to extract the correct answer. It is clear that, as soon as $Q \neq \delta$ function, $\tilde{Q} \neq Q$ and we have to do a real calculation. Since such a calculation to all orders is fairly difficult, we shall satisfy ourselves with a perturbative argument. We shall assume for that matter that we are in a position of having a correlation time t_c (i.e., a cut-off in frequencies $\sim 1/t_0$) and that the magnitude of fluctuations is so small, or $1/\theta$ so small, that $t_c \ll \theta$. We shall then look for what happens in the region $t_c \ll t \ll \theta$. We have seen that if $t_c = 0$ no renormalization occurs. Then it is natural to investigate them in power series in $1/\theta$ with a first term in $1/\theta^2$, in which case we approximate $\tilde{G}$ by

$$\tilde{G}(\tilde{s}_2, t_2; \tilde{s}_1, t_1)$$

$$\simeq \left(\tilde{s}_2 \left| I + \frac{1}{\theta^2} \int\limits_{t_2 \geq t \geq t' \geq t_1}^{(2)} dt \, dt' \, Q(t - t') \sum_{k=1}^{3} (e^{-\boldsymbol{B}_0 \cdot \boldsymbol{L} t} L_k e^{\boldsymbol{B}_0 \cdot \boldsymbol{L}(t - t')} \cdot L_k e^{\boldsymbol{B}_0 \cdot \boldsymbol{L} t'}) \cdot \right| \tilde{s}_1 \right).$$

$$\theta \gg t_2 - t_1 \gg t_2.$$

We assume further $\theta B_0 \ll 1$ since we are interested in static properties (i.e., the limit $B_0 \to 0$). In this case, the above simplifies further to

$$\tilde{G}(\tilde{s}_2, t_2; \tilde{s}_1, t_1)$$

$$\simeq \left(\tilde{s}_2 \middle| I + \frac{1}{\theta^2} \overset{(2)}{\underset{t_2 \geq t \geq t' \geq t_1}{\int}} dt \, dt' \, Q(t - t') \left(L^2 + (t - t') \, \boldsymbol{B}_0 \cdot \boldsymbol{L} \right) \middle| \tilde{s}_1 \right)$$

$$\simeq \left(\tilde{s}_2 \middle| \exp \frac{L^2}{\theta^2} \overset{(2)}{\underset{t_2 \geq t \geq t' \geq t_1}{\int}} dt \, dt' \, Q(t - t') \right.$$

$$\left. \cdot \exp \boldsymbol{B}_0 \cdot \boldsymbol{L} \, \frac{1}{\theta^2} \overset{(2)}{\underset{t_2 \geq t \geq t' \geq t_1}{\int}} dt \, dt' (t - t') \, Q(t - t') \middle| \tilde{s}_1 \right).$$

The interpretation of this formula is clear for (remembering that $[L_k, L^2] = 0$) it leads to

$$G(s_2, t_2 | s_1, t_1)_{B_0}$$

$$\simeq \left(s_2 \middle| \exp \left\{ \overset{(2)}{\underset{t_2 \geq t \geq t' \geq t_1}{\int}} dt \, dt' \, Q(t - t') \frac{L^2}{\theta^2} \right. \right.$$

$$\left. \left. + (t_2 - t_1) \, \boldsymbol{B}_0 \cdot \boldsymbol{L} \left(1 + \frac{1}{t_2 - t_1} \cdot \frac{1}{\theta^2} \overset{(2)}{\underset{t_2 \geq t \geq t' \geq 0}{\int}} dt \, dt' \, Q(t - t')(t - t') \right) \right\} \middle| s \right)$$

$$\theta \gg t_2 - t_1 \geq t_2, \, B_0 \to 0$$

and this is essentially the probability function in a frame rotating with frequency $|B_0| \, (1 + \delta \mu / \mu)$ where

$$\frac{\delta \mu}{\mu} = \lim_{T \to \infty} \frac{1}{T} \underset{T \geq t \geq t' \geq 0}{\int} dt \, dt' \, \frac{Q(t - t')}{\theta^2} \, (t - t') = \lim_{T \to \infty} \int_0^T dt \, \frac{t Q(t)}{\theta^2}$$

can be interpreted as a first order relative correction to the magnetic moment. At first sight this does not look like a negative quantity. An obvious counter-example is $(Q/\theta^2) \, (t - t') = (1/\theta^2) \, e^{-|t-t'|/t_c}$ which is indeed a positive kernel [with Fourier transform proportional to $1/(1 + \omega^2 t_c^2)$] and where $\delta \mu / \mu$ is obviously positive. Nevertheless, for the cases of physical interest, the Fourier transform of Q has the property that it is vanishing rather strongly for $\omega \to 0$ (like ω^3, say, for the standing modes in a large volume). It grows up to a cut-off frequency then decreases sharply. In this case we shall see that $\delta \mu / \mu$ is indeed negative. To compute the integral we write

$$\frac{Q(t - t')}{\theta^2} = \int_0^\infty dv(\omega) \cos \omega(t - t')$$

and find

$$\frac{\delta \mu}{\mu} = \lim_{T \to \infty} - \int_0^\infty \frac{dv(\omega)}{\omega^2} \left\{ 1 + \cos \omega T - 2 \frac{\sin \omega T}{\omega T} \right\}.$$

If $dv(\omega)/\omega^2$ is again a measure (this is clearly not the case with our previous counter-example) then indeed

$$\frac{\delta\mu}{\mu} = -\int_0^\infty \frac{dv(\omega)}{\omega^2} < 0 \,.$$

If we notice that [with $dv(\omega)$ even]

$$\frac{\delta\mu}{\mu} = \lim_{T\to\infty} \frac{1}{2} \int_{-\infty}^{+\infty} dv(\omega) \frac{d}{d\omega} \left\{ \frac{1 - \sin\omega T/\omega T}{\omega} \right\}$$

we see that in the case of white noise, $dv(\omega) = d\omega/\theta^2$, $\delta\mu/\mu$ vanishes indeed.

This discussion – a not very rigorous one admittedly – is, however, indicative that heuristic arguments can sometimes be fairly misleading. Since in the case of the electron anomalous magnetic moment, the correlation one would take is more or less of the type just discussed, simple non-relativistic arguments would lead to a negative correction which is totally wrong.

However, the whole theory of random motion on curved manifolds might find some other type of applications in theoretical physics. Some have been briefly indicated.

To conclude, it is tempting to make a conjecture that I was unable to prove rigorously but which is supported by the previous heuristic argument as well as the explicit examples. If $\int_{-\infty}^{+\infty} dt\, Q(t) < \infty$ then

$$\lim_{T\to\infty} G(s_2 \cdot s_1, t) = 1/4\pi, \text{ if } \int_{-\infty}^{+\infty} dt\, Q(t) = \infty, \text{ then } \lim G(s_2 \cdot s_2, t) = G_\infty(s_2 \cdot s_1)$$

$$= 1/(4\pi |s_2 - s_1|).$$

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