Commun. math. Phys. 35, 253—255 (1974)

An Application of the GHS Inequalities to Show the Absence of Phase Transition for Ising Spin Systems

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Received October 20, 1973

Abstract. We show that the GHS inequalities can be used instead of the Lee-Yang circle theorem to prove that there is no phase transition for the ν-dimensional Ising model in the presence of a (non-zero) external field.

It has been shown by Ruelle [3] that there is no phase transition for the ν-dimensional Ising model in the presence of a (non-zero) external field. A different proof of this result has been given by Lebowitz and Martin-Löf [2]. Both of these proofs use the Lee-Yang circle theorem and it is the object of this note to show that the result may be obtained by using the inequalities of Griffiths, Hurst and Sherman [1] rather than the circle theorem.

Let $\mathscr{C}$ denote the finite subsets of $\mathbb{Z}^\nu$ and let $\Phi : \mathscr{C} \to \mathbb{R}$ be a translation invariant, attractive pair potential, thus we have

$$\Phi(\emptyset) = 0 , \tag{1}$$

$$\Phi(A + x) = \Phi(A) \quad \text{for all} \quad A \in \mathscr{C},\ x \in \mathbb{Z}^\nu , \tag{2}$$

$$\Phi(A) = 0 \quad \text{if} \quad |A| \geqq 3, \quad (\text{where } |A| \text{ denotes the cardinality of } A) , \tag{3}$$

$$\Phi(A) \geqq 0 \quad \text{if} \quad |A| = 2 . \tag{4}$$

We will also assume that Φ satisfies:

$$\sum_{0 \neq x \in \mathbb{Z}^\nu} \Phi(\{0, x\}) < \infty , \tag{5}$$

$$\Phi(\{0\}) + \sum_{0 \neq x \in \mathbb{Z}^\nu} \Phi(\{0, x\}) = 0 . \tag{6}$$

Of course (6) just says that if we translate from "lattice gas" language to "spin" language then Φ corresponds to the Ising model in the absence of an external field.

For $\lambda \in \mathbb{R}$ let Φ_λ denote the potential got by adding an external field of size λ to Φ, thus

$$\Phi_\lambda(A) = \begin{cases} \Phi(A) + \lambda & \text{if} \quad |A| = 1, \\ \Phi(A) & \text{otherwise}. \end{cases}$$

Let $U_\lambda : \mathscr{C} \to \mathbb{R}$ be the energy corresponding to Φ_λ, thus

$$U_\lambda(A) = \sum_{B \subset A} \Phi_\lambda(B) = \sum_{B \subset A} \Phi(B) + \lambda |A|.$$

For $\Lambda \in \mathscr{C}$, $\lambda \in \mathbb{R}$ let

$$P_\Lambda(\lambda) = \frac{1}{|\Lambda|} \log \sum_{A \subset \Lambda} \exp U_\lambda(A);$$

then if $\Lambda \uparrow \mathbb{Z}^\nu$ (in the sense of van Hove) we have $P_\Lambda(\lambda)$ converges (for all $\lambda \in \mathbb{R}$) to the pressure $P(\lambda)$. Using the FKG inequalities and the convexity of P we have the following result of Lebowitz and Martin-Löf [2]:

Proposition 1. *If P is differentiable at $\lambda \in \mathbb{R}$ then phase transition does not occur for the interaction Φ_λ.*

(By the absence of phase transition we mean here that there exists only one infinite Gibbs state with potential Φ_λ.)

By the Lee-Yang circle theorem it follows that P is differentiable at λ if $\lambda \neq 0$. We will now show that this also follows from the GHS inequalities.

Lemma 1. *Let $I \subset \mathbb{R}$ be an open interval and for $n = 1, 2, \ldots$ let $f_n : I \to \mathbb{R}$ be convex and with $0 \leqq f_n \leqq 1$. Then there exists a subsequence $\{n_j\}$ such that $f_{n_j}(x)$ converges for all $x \in I$, (and if we denote the limit by $f(x)$ then of course $f : I \to \mathbb{R}$ is also convex).*

Proof. This is a well known result from real analysis.

Proposition 2. *P is differentiable at λ if $\lambda \neq 0$.*

Proof. Let $\tau : \{0, 1\}^{\mathbb{Z}^\nu} \to \{0, 1\}^{\mathbb{Z}^\nu}$ be the automorphism given by $\tau(A) = \mathbb{Z}^\nu - A$. Then τ induces an automorphism of the probability measures on $\{0, 1\}^{\mathbb{Z}^\nu}$ and it is well-known (and easily checked) that this automorphism maps Gibbs states with potential Φ_λ into Gibbs states with potential $\Phi_{-\lambda}$. Thus we need only consider the case $\lambda < 0$. For $\Lambda \in \mathscr{C}$ let $f_\Lambda = \dfrac{\partial P_\Lambda}{\partial \lambda}$; thus

$$f_\Lambda(\lambda) = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \varrho_{\Lambda, \lambda}(\{x\}),$$

where

$$\varrho_{\Lambda,\lambda}(\{x\}) = \frac{\sum_{x \in A \subset \Lambda} \exp U_\lambda(A)}{\sum_{A \subset \Lambda} \exp U_\lambda(A)}.$$

Now if $\lambda < 0$ then the inequalities of Griffiths, Hurst and Sherman [1] apply to Φ_λ and we thus get that $\varrho_{\Lambda,\lambda}(\{x\})$ is a convex function of λ on $(-\infty, 0)$ and hence by Lemma 1 we can find $\Lambda_n \uparrow \mathbb{Z}^\nu$ and a convex function $f : (-\infty, 0) \to \mathbb{R}$ such that for all $\lambda < 0$ we have both

$$f_{\Lambda_n}(\lambda) \to f(\lambda) \quad \text{and} \quad P_{\Lambda_n}(\lambda) \to P(\lambda) \quad \text{as} \quad n \to \infty .$$

Now let $\lambda < 0$ and $\lambda_0 < \lambda$; then

$$P(\lambda) - P(\lambda_0) = \lim_{n \to \infty} \int_{\lambda_0}^{\lambda} f_{\Lambda_n}(t)\, dt = \int_{\lambda_0}^{\lambda} f(t)\, dt$$

(where the last equality follows from the dominated convergence theorem, since $0 \leqq f_{\Lambda_n} \leqq 1$). But f is convex and thus in particular continuous, hence by the fundamental theorem of calculus we have that P is differentiable on $(-\infty, 0)$.

References

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Communicated by G. Gallavotti

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