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On the Connectedness Structure of the Coulomb S-Matrix*

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Abstract. The forward direction singularity of the non-relativistic Coulomb S-matrix is examined and discussed. The relativistic Coulomb S-matrix to order α is shown to have a similar singularity.

I. Introduction

It is well known that for *short range* forces, the S-matrix describing the scattering of a (spinless) particle from a potential can be usefully split up into two pieces,

$$S(\boldsymbol{k}_1, \boldsymbol{k}_2) = \delta(\boldsymbol{k}_1 - \boldsymbol{k}_2) + t(\boldsymbol{k}_1, \boldsymbol{k}_2). \tag{1}$$

This decomposition is useful and natural because after removal of an energy conserving delta function, $t(\boldsymbol{k}_1, \boldsymbol{k}_2)$ is a smooth (indeed, often analytic) function of its arguments. The "no scattering" part of S, $\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2)$, is called the "disconnected part" while $t(\boldsymbol{k}_1, \boldsymbol{k}_2)$ is the "connected part".

In Section II we calculate the explicit form of the Coulomb S-matrix, $S_c(\boldsymbol{k}_1, \boldsymbol{k}_2)$, and show that the decomposition (1) is far from natural. Indeed, in a sense to be defined more precisely, there is no delta-function component in S_c, and thus S_c is "totally connected". However, $S_c(\boldsymbol{k}_1, \boldsymbol{k}_2)$ does not have the structure of a connected part associated with a short range interaction. In fact as we will show, S_c is more singular than $\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2)$!

In Section III we discuss the one photon exchange diagram for relativistic Coulomb scattering and show that the S-matrix to order α has a similar singularity in the forward direction.

II. Forward Direction Singularity in the Coulomb Amplitude

Although the explicit form of the Coulomb scattering amplitude has long been known, it was only in 1964 that Dollard [1] gave the correct time dependent description of the scattering process. We briefly state his results:

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With

define[1]

$$H = H_0 + V(\boldsymbol{x}), \qquad H_0 = \boldsymbol{p}^2/2, \qquad V(\boldsymbol{x}) = \alpha/|\boldsymbol{x}| \tag{2}$$

$$H'_0(\boldsymbol{p}, t) = H_0 + V(\boldsymbol{p}t)\,\Theta(4H_0|t| - 1) \tag{3}$$

$$U_0(t) = \exp\left(-i\int_0^t ds\, H'_0(\boldsymbol{p}, s)\right). \tag{4}$$

Dollard proves the following:

(i) $\lim_{t\to\pm\infty} e^{iHt}U_0(t) = \Omega_\pm$ exist (in the sense of strong convergence).

(ii) If $\tilde{f}(\boldsymbol{x}) = \int e^{i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{k})\, d\boldsymbol{k}$, then

$$(\Omega_\pm \tilde{f})(\boldsymbol{x}) = \int \Psi_{\boldsymbol{k}}^\pm(\boldsymbol{x}) f(\boldsymbol{k})\, d\boldsymbol{k}. \tag{5}$$

Here the $\Psi_{\boldsymbol{k}}^\pm(\boldsymbol{x})$ are the usual stationary scattering eigenfunctions of H (see for example Schiff [2]).

Note that from (5) the S-operator

$$S_c = \Omega_+^* \Omega_- \tag{6}$$

can be calculated explicitly, for example from the expression

$$S_c(\boldsymbol{k}_1, \boldsymbol{k}_2) = \lim_{\varepsilon\to 0}\int e^{-\varepsilon|\boldsymbol{x}|}\,\overline{\psi}_{\boldsymbol{k}_1}^+(\boldsymbol{x})\,\psi_{\boldsymbol{k}_2}^-(\boldsymbol{x})\, d\boldsymbol{x} \tag{7}$$

which is valid in the sense of distributions. Since the integrals involved can be expressed in terms of known functions, it is reasonably straightforward to show from (7) that for $\boldsymbol{k}_1 \neq \boldsymbol{k}_2$

$$S_c(\boldsymbol{k}_1, \boldsymbol{k}_2) = (\gamma/2\pi i k_1)\, e^{2i\sigma(k_1)}\,\delta(k_1^2 - k_2^2)\left(\frac{1 - \hat{e}_1\cdot\hat{e}_2}{2}\right)^{-1-i\gamma} \tag{8}$$

where here

$$\gamma = \alpha/k_1, \qquad e^{2i\sigma_0(k_1)} = \Gamma(1 + i\gamma)/\Gamma(1 - i\gamma), \qquad \hat{e}_i = \boldsymbol{k}_i/k_i,$$

and thus we recover the usual Coulomb scattering amplitude. The result (8) has been derived by other authors using different techniques (see for example [3, 4] and references cited there). Note that the restriction to $\boldsymbol{k}_1 \neq \boldsymbol{k}_2$ is not trivial because the distribution $(1 - \hat{e}_1\cdot\hat{e}_2)^{-1-i\gamma}$ is undefined as it stands (it is not an integrable function). Furthermore, any extension is unique only up to a distribution with support at $\hat{e}_1 = \hat{e}_2$. Of course, Eq. (7) is sufficient to calculate S_c for all $\boldsymbol{k}_1, \boldsymbol{k}_2$ but we prefer another method which we feel is more instructive. It is based on the following proposition.

[1] While some sort of $t = 0$ cutoff is necessary in Eq. (4) to insure convergence, the particular choice $\Theta(4H_0|t| - 1)$ guarantees that the S-matrix will have the usual energy dependent phase and thus the standard singularity structure in the complex energy plane.

Proposition 1. *Suppose there exist two unitary operators, S_1 and S_2 which for each pair of C^∞ functions f and g with disjoint and compact support (in* **k** *space) satisfy*

$$(f, S_1 g) = (f, S_2 g) = (f, S_c g), \tag{9}$$

then $S_1 = S_2$. Stated more simply: there is at most one unitary extension of (8) *to all $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$.*

The proof of Proposition 1 is given in an appendix. We now simply write down *the* Coulomb S-operator. Its action on a continuously differentiable (and square integrable) function f is

$$(S_c f)(\boldsymbol{k}) = \lim_{\varepsilon\to 0}(\gamma/2\pi i k) e^{2i\sigma_0(k)} \int d\boldsymbol{k}' \delta(k^2 - k'^2)\left(\frac{1-\hat{e}\cdot\hat{e}'}{2}\right)^{-1+\varepsilon-i\gamma} f(\boldsymbol{k}'). \tag{10}$$

Note that such f are dense in $L_2(\mathbb{R}^3)$. We see that the correct extension of $(1-\hat{e}_1\cdot\hat{e}_2)^{-1-i\gamma}$ is just $\lim_{\varepsilon\to 0^+}(1-\hat{e}_1\cdot\hat{e}_2)^{-1+\varepsilon-i\gamma}$.

To show that S_c is unitary, let $f(\boldsymbol{k}) = Y_l^m(\hat{e})\, g(k)$. Making use of rotational invariance one easily derives

$$(S_c f)(\boldsymbol{k}) = c_l(k) f(\boldsymbol{k})$$

where

$$\begin{aligned} c_l(k) &= e^{2i\sigma_0}(\gamma/2i) \lim_{\varepsilon\to 0^+} \int_{-1}^{1} dx \left(\frac{1-x}{2}\right)^{-1-i\gamma+\varepsilon} P_l(x) \\ &= \Gamma(l+1+i\gamma)/\Gamma(l+1-i\gamma) \equiv e^{2i\sigma_l(k)}. \end{aligned} \tag{11}$$

That is, we have the expected result

$$(S_c f)(\boldsymbol{k}) = e^{2i\sigma_l(k)} f(\boldsymbol{k}) \tag{12}$$

proving that S_c is unitary. To arrive at Eq. (11) we have used a table of integrals [5] and some gamma-function identities.

We mention for future reference another representation of S_c which follows easily from Eq. (10):

$$\begin{aligned} (S_c f)(\boldsymbol{k}) = e^{2i\sigma_0(k)} \Bigg\{ & f(k) + (\gamma/2\pi i k)\int d\boldsymbol{k}' \\ & \delta(k^2 - k'^2)\left(\frac{1-\hat{e}\cdot\hat{e}'}{2}\right)^{-1-i\gamma} (f(\boldsymbol{k}') - f(\boldsymbol{k}))\Bigg\}. \end{aligned} \tag{13}$$

While at first glance Eq. (10) seems to imply $\lim_{\alpha\to 0}(f, S_c g) = 0$, we see at once from either Eq. (12) or Eq. (13) that as expected

$$\lim_{\alpha\to 0}(f, S_c g) = (f, g). \tag{14}$$

(The apparent paradox arises only if one interchanges the limits $\alpha \to 0$ and $\varepsilon \to 0$.)

We would now like to discuss the singularity structure of S_c at $\boldsymbol{k}_1 = \boldsymbol{k}_2$. If B is any bounded operator on $L_2(\mathbb{R}^3)$, there always exists a unique tempered distribution T on $\mathscr{S}(\mathbb{R}^6)$ such that $T(f \otimes g) = (\bar{f}, Bg)$ [6]. In particular since S_c is unitary

$$S_c(\boldsymbol{k}_1, \boldsymbol{k}_2) = \lim_{\varepsilon \to 0^+} (\gamma/2\pi i k_1) e^{2i\sigma_0(k_1)} \delta(k_1^2 - k_2^2) \left(\frac{1 - \hat{e}_1 \cdot \hat{e}_2}{2} \right)^{-1+\varepsilon - i\gamma} \tag{15}$$

is a tempered distribution, and it is as such that we will investigate its singularity structure.

As we mentioned in the introduction there are two different properties which are usually associated with a connected part: absence of delta functions and smoothness. Let us consider the first property first and ask whether $S_c(\boldsymbol{k}_1, \boldsymbol{k}_2)$ has any delta function component. Because, as it will turn out, S_c is a very singular object, this question is quite delicate and therefore we want to be precise. Thus we make the following definition:

Definition 1. A tempered distribution $T(\boldsymbol{k}_1, \boldsymbol{k}_2)$ is said to have "no component *concentrated* at $\boldsymbol{k}_1 = \boldsymbol{k}_2$" if for any h in $C_0^\infty(\mathbb{R}^3)$ (C^∞ functions of compact support) with $h(\boldsymbol{k}_1 - \boldsymbol{k}_2) = 1$ in a neighborhood of $\boldsymbol{k}_1 = \boldsymbol{k}_2$, the distributions $T_\lambda(\boldsymbol{k}_1, \boldsymbol{k}_2) = h(\lambda(\boldsymbol{k}_1 - \boldsymbol{k}_2))\, T(\boldsymbol{k}_1, \boldsymbol{k}_2)$ satisfy

$$\lim_{\lambda \to \infty} T_\lambda(f) = 0 \tag{16}$$

for each $f \in \mathscr{S}$.

We feel this to be a natural definition because $h_\lambda(\boldsymbol{k}_1 - \boldsymbol{k}_2) = h(\lambda(\boldsymbol{k}_1 - \boldsymbol{k}_2))$ is (for large λ) equal to one in a very small neighborhood of $\boldsymbol{k}_1 = \boldsymbol{k}_2$ and rapidly goes to zero elsewhere. If $T(\boldsymbol{k}_1, \boldsymbol{k}_2)$ is a sum of derivatives of $\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2)$ then of course $T_\lambda = T$ while if T is an integrable function $\lim_{\lambda \to \infty} T_\lambda = 0$ [2].

It is now a straightforward matter to verify that S_c has no component concentrated at $\boldsymbol{k}_1 = \boldsymbol{k}_2$. Rather than giving a direct proof of this statement we instead want to show how it follows from a more commonly used criterion, namely a spatial cluster property.

Proposition 2. *Let B be a bounded operator on $L_2(\mathbb{R}^3)$ and $T(\boldsymbol{a})$ the spatial translation operator $((T(\boldsymbol{a}) f(\boldsymbol{k}) = e^{-i\boldsymbol{k}\cdot\boldsymbol{a}} f(\boldsymbol{k}))$. Suppose for each $f, g \in L_2(\mathbb{R}^3)$*

$$\lim_{|\boldsymbol{a}| \to \infty} (T(\boldsymbol{a}) f, B\, T(\boldsymbol{a}) g) = 0\,. \tag{17}$$

[2] However, as the following example shows, given a distribution $T(x)$ this definition cannot be used to single out a *unique* component $T_0(x)$ with support at $x = 0$: If $T(x) = \text{P.V.}\,1/x$ then $\lim_{\lambda \to \infty} h(\lambda x)\, T(x) = \delta(x)\, T(h)$.

Then the tempered distribution $B(\boldsymbol{k}_1, \boldsymbol{k}_2)$ associated with B has no component concentrated at $\boldsymbol{k}_1 = \boldsymbol{k}_2$.

Proof. The statement (17) just means that the operators $B_{\boldsymbol{a}} = T(-\boldsymbol{a})\, B\, T(\boldsymbol{a})$ converge weakly to zero, or in terms of the corresponding tempered distributions $B_{\boldsymbol{a}}(f \otimes g) \to 0$ all $f, g \in \mathscr{S}$. But since $\|B_{\boldsymbol{a}}\| = \|B\|$, the tempered distributions $B_{\boldsymbol{a}}$ satisfy

$$|B_{\boldsymbol{a}}(f)| \leqq c |f|_n \quad \text{all} \quad \boldsymbol{a} \tag{18}$$

for some semi-norm $|\ |_n$, where c and n are independent of $\boldsymbol{a}$. From this and the fact that finite sums $\Sigma f_i \otimes g_i$ are dense in $\mathscr{S}$, it follows that

$$B_{\boldsymbol{a}}(f) \to 0 \quad \text{for each} \quad f \in \mathscr{S} . \tag{19}$$

Now define

$$g(\boldsymbol{a}) = B_{\boldsymbol{a}}(f) = B(e^{i(\boldsymbol{k}_1 - \boldsymbol{k}_2)\cdot \boldsymbol{a}} f) . \tag{20}$$

$g(\boldsymbol{a})$ is infinitely differentiable and $g(\boldsymbol{a}) \to 0$ as $|\boldsymbol{a}| \to \infty$. Thus, if $h \in C_0^\infty(\mathbb{R}^3)$, we have with $h_\lambda(\boldsymbol{k}) = h(\lambda \boldsymbol{k})$

$$\int g(\boldsymbol{a})\, \hat{h}_\lambda(\boldsymbol{a})\, d\boldsymbol{a} = B(h(\lambda(\boldsymbol{k}_1 - \boldsymbol{k}_2))\, f) = B_\lambda(f)$$

where $\hat{h}$ is the fourier transform of h. By a change of variable

$$B_\lambda(f) = \int g(\lambda \boldsymbol{a})\, \hat{h}(\boldsymbol{a})\, d\boldsymbol{a} \tag{21}$$

which has limit zero (as $\lambda \to \infty$) because of Lebesgue's dominated convergence theorem. This completes the proof.

To complete the discussion of the support properties of S_c we quote a result of Ross [7]: In the sense of weak operator convergence

$$T(-\boldsymbol{a})\, S_c\, T(\boldsymbol{a}) \to 0 \quad \text{as} \quad |\boldsymbol{a}| \to \infty . \tag{22}$$

Thus in the sense of our definition S_c has no component concentrated at $\boldsymbol{k}_1 = \boldsymbol{k}_2$. We remark that although the relation (22) may at first glance appear strange, it can be explained with reference to the classical theory. This is discussed elsewhere [8].

A word of caution is in order concerning the absence of a delta function in S_c. If instead of considering $S_c(\boldsymbol{k}_1, \boldsymbol{k}_2)$ as a distribution in two variables, we fix $\boldsymbol{k}_1 = \boldsymbol{k}_0$ and examine

$$S_c(\boldsymbol{k}_0, f) = (S_c f)(\boldsymbol{k}_0)$$

as a distribution in one variable we get very different results: Suppose h is as in Definition 1. Let

$$\begin{aligned} h_\lambda(\boldsymbol{k}_2) = h(\lambda(\boldsymbol{k}_0 - \boldsymbol{k}_2)) ; \quad \text{then} \quad \text{for} \quad \boldsymbol{k}_0 \neq 0 , \\ S_c(\boldsymbol{k}_0, h_\lambda f) \xrightarrow[\lambda \to \infty]{} e^{i\gamma \ln \lambda^2} f(\boldsymbol{k}_0)\, \mu . \end{aligned} \tag{23}$$

Here μ is a constant depending on $\boldsymbol{k}_0$ and the function h. Thus as a distribution in the variable $\boldsymbol{k}_2, S_c(\boldsymbol{k}_0, \boldsymbol{k}_2)$ is not without a component concentrated at $\boldsymbol{k}_2 = \boldsymbol{k}_0$. Note that the rapid oscillations in (23) are responsible for the fact that $S_c(h_\lambda f) \to 0$.

We now go on to consider the singularity structure of S_c. Because we are not interested in the behavior of $S_c(\boldsymbol{k}_1, \boldsymbol{k}_2)$ for large $\boldsymbol{k}_1, \boldsymbol{k}_2$ we restrict our test functions to have support in some fixed compact set Λ. Thus we consider S_c as a distribution on $\mathscr{D}(\Lambda)$, the set of C^∞ functions with support in Λ. We take for Λ the sphere $\{k \in \mathbb{R}^6 : k^2 \leqq a^2\}$.

Define the seminorms

$$|f|_n = \sup_{\substack{k \in \Lambda \\ |s| = n}} |D^s f(k)| \tag{24}$$

where $D^s = \partial^{|s|}/\partial k_1^{s_1} \dots \partial k_6^{s_6}$. The order of a distribution T on $\mathscr{D}(\Lambda)$, is then defined [9] as the smallest integer N for which

$$|T(f)| \leqq \sum_{n=0}^{N} C_n |f|_n \tag{25}$$

for some set of C_k and all f. We will use the order of a distribution as an index of its singularity.

Definition 2. A distribution T_2 (on $\mathscr{D}(\Lambda)$) is called "more singular" than a distribution T_1 (on $\mathscr{D}(\Lambda)$) if the order of T_2 is larger than the order of T_1.

We consider this definition reasonable because a distribution T of order N on $\mathscr{D}(\Lambda)$ can be uniquely extended to the larger class of functions $C^N(\Lambda)$, i.e. those functions with support in Λ which are only N times continuously differentiable, and T remains continuous on $C^N(\Lambda)$. Thus a distribution which is less singular than another is defined and continuous on a larger (and rougher) class of functions.

The next proposition shows that $S_c(\boldsymbol{k}_1, \boldsymbol{k}_2)$ is more singular than $\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2)$.

Proposition 3. *For any $\delta > 0$ there exists c_δ such that*

$$|S_c(f)| \leqq c_\delta |f|_0 + \delta |f|_1 . \tag{26}$$

The constant δ cannot be set equal to zero, and thus S_c has order 1.

Proof. The estimate (26) is proved simply after the integration region has been split up into the region $(1 - \hat{e}_1 \cdot \hat{e}_2) \leqq \lambda$ and its complement. We find that $|S_c(f)| \leqq C(\sqrt{\lambda}|f|_1 + (1 + 1/\lambda)|f|_0)$ and thus taking $\lambda = (\delta/C)^2$, (26) follows.

To show that δ cannot be taken equal to zero, let $1 \geqq \lambda > 0$ and

$$g_\lambda(\hat{e}_1, \hat{e}_2) = \left(\frac{1-\hat{e}_1 \cdot \hat{e}_2}{2}\right)^{i\gamma} \qquad \lambda \leqq \frac{1-\hat{e}_1 \cdot \hat{e}_2}{2} \leqq 1$$

$$= \lambda^{i\gamma} \qquad 0 \leqq \frac{1-\hat{e}_1 \cdot \hat{e}_2}{2} \leqq \lambda \,.$$

Then g_λ is a continuous function of $\hat{e}_1$ and $\hat{e}_2$ but

$$\lim_{\varepsilon \to 0^+} \int \frac{d\Omega_1}{4\pi} \frac{d\Omega_2}{4\pi} g_\lambda(\hat{e}_1, \hat{e}_2) \left(\frac{1-\hat{e}_1 \cdot \hat{e}_2}{2}\right)^{-1-i\gamma+\varepsilon} \tag{27}$$

$$= i/\gamma - \ln \lambda \,.$$

Thus if for example $f_\lambda(\boldsymbol{k}_1, \boldsymbol{k}_2) = g_\lambda(\hat{e}_1, \hat{e}_2)\, e^{-2i\sigma_0(k_1)}\, h\left(\frac{k_1^2+k_2^2}{2}\right)$ with $h \in C_0^\infty(\mathbb{R})$ and $\operatorname{supp} h \subseteqq [a^2/4, a^2/2]$, then $f \in C^0(\Lambda)$ and

$$S_c(f) = 4\pi \int dk\, k^2 (1 + i\gamma \ln \lambda)\, h(k^2) \,. \tag{28}$$

Because $|f_\lambda|_0 = \sup_x |h(x)|$ is independent of λ, if $\int_0^\infty dk^2\, h(k^2) \neq 0$ then for small enough λ

$$|S_c(f_\lambda)| \geqq C \ln \lambda^{-1} |f_\lambda|_0 \,. \tag{29}$$

Since λ can be made as small as desired, the proof is complete.

To summarize the results of this section, we have shown that S_c has no delta function component although it is in fact more singular than a delta function. Although S_c does not satisfy the smoothness criterion usually satisfied by a connected part arising from a short range interaction, we feel that it nevertheless deserves the adjective "connected".

III. Relativistic Coulomb Scattering to Order

The purpose of this section is to clarify an apparent discrepancy between the non-relativistic and the relativistic S-matrix for Coulomb scattering, the latter being given by the usual Feynman-Dyson expansion. To simplify matters we consider the scattering of 2 different spinless charged particles of equal mass. We consider the S-matrix as a limit of a massive photon theory where the photon propagator is replaced by

$$g_{\mu\nu}/k^2 - \lambda^2 + i\varepsilon$$

and $\lambda \to 0$. Then to first order in α we have the two Feynman diagrams in Fig. 1

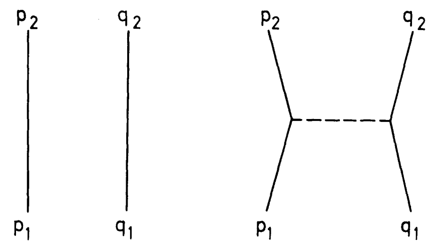

Fig. 1

which give

$$S_\lambda(\boldsymbol{p}_2, \boldsymbol{q}_2; \boldsymbol{p}_1, \boldsymbol{q}_1) \tag{30}$$
$$= \delta^3(\boldsymbol{p}_2 - \boldsymbol{p}_1)\delta^3(\boldsymbol{q}_2 - \boldsymbol{q}_1) + \frac{i\alpha}{4\pi} \frac{\delta^4(p_2 + q_2 - p_1 - q_1)}{\sqrt{\omega_{q_1}\omega_{q_2}\omega_{p_1}\omega_{p_2}}} \frac{(p_1 + p_2)\cdot(q_1 + q_2)}{(p_1 - p_2)^2 - \lambda^2}.$$

With $\lambda \neq 0$, this distribution has of course the structure of a short range interaction S-matrix, but we should expect that with $\lambda \to 0$ we will obtain something more like the non-relativistic result for Coulomb scattering. (This statement should *not* be true to higher orders in α where one is *forced* to include the effects of soft photon radiation [3].) The discrepancy we are talking about is the apparent presence of an "identity piece" (the first diagram in Fig. 1) even when $\lambda \to 0$. In what follows we first take the limit $\lambda \to 0$ in Eq. (3) and remove an infinite "Coulomb phase". We then show that the result (in the non-relativistic limit) agrees with Eq. (13) for S_c up to a phase (again of course up to order α).

Thus consider the limiting form of

$$(S_\lambda f)(\boldsymbol{p}_2, \boldsymbol{q}_2) \equiv \int d\boldsymbol{p}_1 d\boldsymbol{q}_1 S(\boldsymbol{p}_2, \boldsymbol{q}_2; \boldsymbol{p}_1, \boldsymbol{q}_1) f(\boldsymbol{p}_1, \boldsymbol{q}_1) \tag{31}$$

when $\lambda \to 0$. (Since it is not necessary to smear out in $(\boldsymbol{p}_2, \boldsymbol{q}_2)$ we do not do so.) With

$$s = (p_2 + q_2)^2, \quad \beta^2 = \lambda^2/s - 4m^2 \tag{32}$$

it is straightforward to show that if f is continuously differentiable

$$(S_\lambda f)(\boldsymbol{p}_2, \boldsymbol{q}_2) = f(\boldsymbol{p}_2, \boldsymbol{q}_2)\left(1 + i\alpha \frac{p_2 \cdot q_2 \ln \beta^2}{\sqrt{(p_2 \cdot q_2)^2 - m^4}}\right) + \frac{i\alpha}{4\pi}(Df)(\boldsymbol{p}_2, \boldsymbol{q}_2) + \mathcal{O}(\beta^2 \ln \beta) \tag{33}$$

[3] See, however, Zwanziger [10] where a redefinition of the S-matrix in Q.E.D. allows consideration of "Coulomb scattering" alone. Zwanziger makes plausible the statement that the full amplitude contains only a connected part.

where

$$(Df)(\boldsymbol{p}_2, \boldsymbol{q}_2) = \int \frac{d\boldsymbol{p}_1 \, d\boldsymbol{q}_1}{\sqrt{\omega_{p_1}\omega_{q_1}\omega_{p_2}\omega_{q_2}}} \frac{\delta^4(p_2+q_2-p_1-q_1)}{(p_1-p_2)^2} \left\{(p_1+p_2)\cdot(q_1+q_2) f(\boldsymbol{p}_1, \boldsymbol{q}_1) - 4p_2\cdot q_2 \bigg/ \sqrt{\frac{\omega_{p_2}\omega_{q_2}}{\omega_{p_1}\omega_{q_1}}} f(\boldsymbol{p}_2, \boldsymbol{q}_2)\right\}. \tag{34}$$

Thus to *first order in* α

$$S_\lambda \xrightarrow[\lambda\to 0]{} \exp\left[\frac{i\alpha}{v(p_1, q_1)} \ln \beta\right] S \exp\left[\frac{i\alpha}{v(p_2, q_2)} \ln \beta\right] \tag{35}$$

where $v(p, q) = (1 - m^4/(p \cdot q)^2)^{\frac{1}{2}}$ and

$$S = \delta^3(\boldsymbol{p}_2 - \boldsymbol{p}_1)\, \delta^3(\boldsymbol{q}_2 - \boldsymbol{q}_1) + \frac{i\alpha}{4\pi} D(\boldsymbol{p}_2, \boldsymbol{q}_2; \boldsymbol{p}_1, \boldsymbol{q}_1). \tag{36}$$

Eq. (35) is to be interpreted in the following way. When both sides are applied to smooth wavefunctions and the result expanded to first order in α, their difference tends to zero. The connisseur will recognize the phase in Eq. (35) as the Coulomb phase [11, 12], which we have dropped to get the infrared divergence free S-matrix of Eq. (36).

We now take the non-relativistic limit of (36) and go to "relative" coordinates in order to compare our result with potential scattering. We skip the details and just give the result: The operator S goes over to an operator $S_r(\boldsymbol{k}, \boldsymbol{k}')$ where

$$(S_r f)(\boldsymbol{k}) = f(\boldsymbol{k}) + (\gamma/2\pi i k) \int d\boldsymbol{k}' \, \delta(k^2 - k'^2) \left(\frac{1 - \hat{e}\cdot\hat{e}'}{2}\right)^{-1} (f(\boldsymbol{k}') - f(\boldsymbol{k})). \tag{37}$$

Eq. (37) is to be compared with Eq. (13). After removal of $e^{2i\sigma_0(k)}$ they are identical to first order in α. We remark that one should expect agreement of Eqs. (37) and (13) only up to a phase because the "Coulomb phase" is ambiguous up to anything which is finite. This is the reason why the factor $e^{2i\sigma_0}$ must be removed before (37) and (13) agree.

To conclude our discussion we remark that it is impossible to identify a component of S_r with support at $\boldsymbol{k}_1 = \boldsymbol{k}_2$. That is the limit of $h(\lambda(\boldsymbol{k}_1 - \boldsymbol{k}_2)) \cdot S_r(\boldsymbol{k}_1, \boldsymbol{k}_2)$ as $\lambda \to \infty$ does not exist and thus it is meaningless to talk about whether or not S_r contains a delta function.

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Appendix: Proof of Proposition I

We first show that $B = S_1 - S_2$ is given by

$$B(\boldsymbol{k}_1, \boldsymbol{k}_2) = \delta(\boldsymbol{k}_1 - \boldsymbol{k}_2)\, b(\boldsymbol{k}_2) \tag{A 1}$$

with b an L^∞ function. (Here we use the same letter to denote both the operator B and the associated tempered distribution.)

Thus let $D = \{(\boldsymbol{k}_1, \boldsymbol{k}_2) : \boldsymbol{k}_1 = \boldsymbol{k}_2\}$ and suppose

$$f \in \mathscr{D}(\mathbb{R}^6),\ \operatorname{supp} f \cap D = \phi\,. \tag{A 2}$$

We want to show that the condition (A 2) implies $B(f) = 0$. By constructing a suitable partition of unity it follows that we need only show this for those f with $\operatorname{supp} f$ contained in a cube E which does not intersect D. But such f can be approximated (in the topology of $\mathscr{S}$) by finite sums of functions of the form $g(\boldsymbol{k}_1)\, h(\boldsymbol{k}_2)$ with $\operatorname{supp} g$, $\operatorname{supp} h$ compact and $\operatorname{supp} g \cap \operatorname{supp} h = \phi$, from which $B(f) = 0$ follows.

Since B therefore has support in D it is a finite sum [13]

$$B(\boldsymbol{k}_1, \boldsymbol{k}_2) = \sum_s (D^s \delta)\, (\boldsymbol{k}_1 - \boldsymbol{k}_2) \otimes T_s\left(\frac{\boldsymbol{k}_1 + \boldsymbol{k}_2}{2}\right) \tag{A 3}$$

where $T_s \in \mathscr{S}'(\mathbb{R}^3)$. The fact that $s = 0$ alone occurs follows from Eq. (18)

$$|B(e^{i(\boldsymbol{k}_1 - \boldsymbol{k}_2)\cdot \boldsymbol{a}} f)| \leqq c |f|_n\,. \tag{A 4}$$

Finally, since B is a bounded operator $T_0 = b \in L^\infty$.

Now by assumption S_1 and S_2 have the additional property

$$(f, S_i g) = \int d\boldsymbol{k}_1, d\boldsymbol{k}_2\, \bar{f}(\boldsymbol{k}_1)\, S_c(\boldsymbol{k}_1, \boldsymbol{k}_2)\, g(\boldsymbol{k}_2) \tag{A 5}$$

for all f, g in C^∞ with disjoint compact supports. Unitarity implies

$$(S_2 + B)^*(S_2 + B) = 1 + S_2^* B + B^* S_2 + B^* B = 1 \tag{A 6}$$

or for $\boldsymbol{k}_1 \neq \boldsymbol{k}_2$

$$\bar{S}_c(\boldsymbol{k}_2, \boldsymbol{k}_1)\, b(\boldsymbol{k}_2) + \bar{b}(\boldsymbol{k}_1)\, S_c(\boldsymbol{k}_1, \boldsymbol{k}_2) = 0\,. \tag{A 7}$$

After removal of the energy conserving delta functions we have for $\hat{e}_1 \neq \hat{e}_2$

$$b(k\hat{e}_2)\, (1 - \hat{e}_1 \cdot \hat{e}_2)^{i\gamma} + \bar{b}(k\hat{e}_1)\, (1 - \hat{e}_1 \cdot \hat{e}_2)^{-i\gamma} = 0\,. \tag{A 8}$$

If R is a rotation around the $\hat{e}_1$ axis, (A 8) implies $b(kR\hat{e}_2) = b(k\hat{e}_2)$ and since $\hat{e}_1$ is essentially arbitrary $b(k\hat{e}) = c(k)$. But since $(1 - \hat{e}_1 \cdot \hat{e}_2)^{i\gamma}$ and its complex conjugate are linearly independent functions of $\hat{e}_1 \cdot \hat{e}_2$, $c(k) = 0$. Thus $S_1 = S_2$ and the proof is complete.

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