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Dilation-Analyticity and Decay Properties of Interactions

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Abstract. Let $H = H_0 + V$ be a Schrödinger operator on $L^2(\mathbb{R}^n)$. We show that the more dilation analytic V is, the slower it must decay at infinity.

1. Introduction

In the theory of the Schrödinger operator $H = H_0 + V$, various assumptions are made about the interaction V in order to be able to prove useful theorems about the spectral and scattering properties of the operator. Two assumptions which are often made are dilation analyticity assumptions (see [1] and [2]) and decay assumptions (see, for example, [4]). These usually have not occurred together (at least explicitly). It is the purpose of this paper to explore the interrelations between these two assumptions. In particular we will show that *the more dilation analytic V is, the slower it must decay at infinity.*

Our proof is based on a certain complex variable result (Lemma 3.2) which gives a sufficient condition for an analytic function defined in an angular sector to be 0. This is a consequence of the Phragmen – Lindelöf theorem and a theorem of Carlson.

2. The Main Theorem

We will denote by $\mathscr{H}$, the Hilbert space $L^2(R^n)$ of complex square integrable functions on $\mathbb{R}^n$. As usual, the inner product is defined by:

$$(\psi_1, \psi_2) = \int_{\mathbb{R}^n} \overline{\psi_1(x)}\, \psi_2(x)\, dx\,.$$

Also $\|\psi\|^2 \equiv (\psi, \psi)$. $\mathscr{H}_+$ will denote the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|\psi\|_+ \equiv \|H_0 \psi\| + \|\psi\|$ where H_0 is the usual self-adjoint

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operator on $\mathscr{H}$ corresponding to the negative Laplacian

$$-\Delta = -\sum_{j=1}^{n}\left(\frac{\partial}{\partial x^j}\right)^2 .$$

Note $\mathscr{H}_+ \subsetneqq \mathscr{H}$, $\mathscr{B}(\mathscr{H}_+, \mathscr{H})$ will denote the space of bounded linear mappings from $\mathscr{H}_+$ to $\mathscr{H}$. $|||\cdot|||$ will denote the usual norm in $\mathscr{B}(\mathscr{H}_+, \mathscr{H})$. *Most interactions of physical interest in nonrelativistic quantum mechanics are elements of* $\mathscr{B}(\mathscr{H}_+, \mathscr{H})$ *for* $n=3$.

Let $\mathbb{R}^+$ denote the *multiplicative group* of positive real numbers. Let $U(\varrho)$ be the unitary representation of R^+ on $\mathscr{H}$ defined by

$$(U(p)\,\varphi)\,(x) \equiv \varrho^{n/2}\,\varphi(\varrho x)$$

for $\varrho \in \mathbb{R}^+$, $\varphi \in \mathscr{H}$. $\{U(\varrho) : \varrho \in \mathbb{R}^+\}$ is the *dilation group on* $\mathscr{H}$. For later use, we will need the following lemma.

Lemma 2.1. *Let* $\psi \in C_0^\infty(\mathbb{R}^n)$ *and* $\varrho \in \mathbb{R}^+$. *Then*

$$\|U(\varrho^{-1})\,\psi\|_{+1} = \frac{1}{\varrho^2}\,\|H_0\psi\| + \|\psi\| . \tag{2.1}$$

In particular,

$$\|U(\varrho^{-1})\,\psi\|_{+1} \leqq \|\psi\|_{+1} \quad \textit{for} \quad \varrho \geqq 1 . \tag{2.2}$$

Proof. A direct computation shows

$$U(\varrho)\,H_0\,U(\varrho^{-1})\,\psi = \varrho^{-2} H_0\,\psi .$$

This implies

$$\|H_0\,U(\varrho^{-1})\,\varphi\| = \varrho^{-2}\,\|H_0\,\psi\|$$

which, together with the unitarity of $U(\varrho^{-1})$ on $\mathscr{H}$, implies (2.1). □

Let $\mathscr{R}$ be the Riemann surface for $\ln z$. Thus $\mathscr{R} = \{(\varrho, \varphi) | \varrho > 0, \varphi \in \mathbb{R}\}$ with the analytic structure defined by the function $\Psi(\varrho, \varphi) = \varrho e^{i\varphi}$. Let $0 < \theta \leqq \infty$ and let $S_\theta = \{(\varrho, \varphi) \in \mathscr{R}: -\theta < \varphi < -\theta\}$. S_θ is called the θ-sector in $\mathscr{R}$. When $\theta < \pi$, we identify S_θ with the obvious sector in $\mathbb{C}$.

Definition. Set $V \in \mathscr{B}(\mathscr{H}_{+1}, \mathscr{H})$ and let $V(\varrho) = U(\varrho)\,V\,U(\varrho^{-1})$. [It follows from Lemma 2.1 that $V(\varrho) \in \mathscr{B}(\mathscr{H}_+, \mathscr{H})$.] We say V is S_θ *dilation analytic* if $V(\varrho)$ can be continued to an $\mathscr{B}(\mathscr{H}_+, \mathscr{H})$-valued analytic function in S_θ. The potential V is said to be exactly S_θ-analytic if V is S_θ-analytic but not $S_{\theta+\varepsilon}$-analytic for any $\varepsilon > 0$.

Examples. A function V with compact support, viewed as a multiplication operator, is not dilation analytic at all. A Yukawa potential $V = \frac{e^{-kr}}{r}$ $(n=3)$ is exactly $S_{\pi/2}$ analytic. A Coulomb potential $V = \frac{1}{r}$ $(n=3)$ is S_∞-analytic.

Remark 2.2. A key property of an S_θ dilation analytic V is the obvious

$$V(\varrho e^{i\varphi}) = U(\varrho)\, V(e^{i\varphi})\, U(\varrho^{-1}) \tag{2.3}$$

for $-\theta < \varphi < \theta$. [We have abused notation by writing $V(\varrho e^{i\varphi})$ rather than $V(\varrho, \varphi)$. No confusion should arise.]

We can now state our main theorem.

Theorem 2.3. *Set $V \in \mathscr{B}(\mathscr{H}_+, \mathscr{H})$ and let $\alpha > 0$ be such that*

1. *V is S_θ-dilation analytic for some $\theta > \dfrac{\pi}{2\alpha}$.*
2. *$e^{k|x|^\alpha} V \in \mathscr{B}(\mathscr{H}_+, \mathscr{H})$ for some $k > 0$.*

Then $V = 0$.

The theorem says essentially that if V is S_θ dilation analytic for some $\theta > \dfrac{\pi}{2\alpha}$, then V must "decay more slowly" than $e^{-k|x|^\alpha}$ for any $k > 0$. The theorem is in some sense sharp since for $n \geqq 2$, $V = \dfrac{e^{-k|x|^\alpha}}{r} \in \mathscr{B}(\mathscr{H}_+, \mathscr{H})$ $k > 0$, $\alpha > 0$ and $e^{k|x|^\alpha} V \in \mathscr{B}(\mathscr{H}_+, \mathscr{H})$ and V is S_θ-dilation analytic for all $\theta < \dfrac{\pi}{2\alpha}$.

It seems difficult, however, to formulate a necessary and sufficient condition for a potential to be exactly dilation – analytic in the angle $-\dfrac{\pi}{2\alpha} < \varphi < \dfrac{\pi}{2\alpha}$.

For example, the function $e^{-r}(1 + \sin r^2)$ decays as e^{-r}, but is dilation – analytic in the angle $-\dfrac{\pi}{4} < \varphi < \dfrac{\pi}{4}$.

3. An Application of the Phragmen - Lindelöf Theorem

In this section we prove a result from complex function theory on which the proof of our main theorem is based.

Let us first state the following corollary of the Phragmen – Lindelöf theorem.

Lemma 3.1. *Suppose that the function $f(z)$ is analytic in an angular sector*

$$\{z = \varrho e^{i\varphi} \mid \varphi_1 \leqq \varphi \leqq \varphi_2,\ 1 \leqq \varrho < \infty\}$$

where $\varphi_2 - \varphi_1 < \pi$, and that $f(z)$ satisfies the estimates for some $K, b, c > 0$,

(i) $|f(\varrho e^{i\varphi_i})| \leqq K e^{-b\varrho}$ *for* $1 \leqq \varrho < \infty$, $i = 1, 2$,

(ii) $|f(\varrho e^{i\varphi})| \leqq K e^{c\varrho}$, *for* $\varphi_1 < \varphi < \varphi_2$, $1 \leqq \varrho < \infty$.

Then

$$(f(\varrho e^{i\varphi})| \leqq K_1 e^{-b\varrho} \quad \textit{for} \quad \varphi_1 \leqq \varphi \leqq \varphi_2, \quad 1 < \varrho < \infty, \quad \textit{and some} \quad K_1 > 0.$$

Proof. We can assume that $\varphi_1 = -\varphi_0, \varphi_2 = \varphi_0, 0 < \varphi_0 < \frac{\pi}{2}$. Consider the function

$$g(z) = e^{\frac{b}{\cos\varphi_0} z} f(z).$$

This function satisfies the estimates

(i) $|g(\varrho e^{i\varphi_i})| \leqq K$, for $1 \leqq \varrho < \infty$, $i = 1, 2$.

(ii) $|g(\varrho e^{i\varphi})| \leqq K e^{\left(b\frac{\cos\varphi}{\cos\varphi_0} + c\right)\varrho}$ for $-\varphi_0 < \varphi < \varphi_0$, $1 \leqq \varrho < \infty$.

By the Phragmen – Lindelöf theorem ([6], p. 177),

$$|g(\varrho e^{i\varphi})| \leqq K_1 \quad \text{for} \quad -\varphi_0 \leqq \varphi \leqq \varphi_0, \quad 1 \leqq \varrho < \infty$$

where $K_1 = \max\{K, K_2\}, K_2 = \max\limits_{-\varphi_0 \leqq \varphi \leqq \varphi_0} |g(e^{i\varphi})|$.

This implies

$$|f(\varrho e^{i\varphi})| \leqq K_1^{-b\frac{\cos\varphi}{\cos\varphi_0}\varrho} \leqq K_1 e^{-b\varrho} \quad \text{for} \quad -\varphi_0 \leqq \varphi \leqq \varphi_0, \quad 1 \leqq \varrho < \infty. \quad \square$$

By means of Lemma 3.1 we can now prove

Lemma 3.2. *Suppose that the function $h(\varrho e^{i\varphi})$ is analytic in the angular sector $\{\varrho e^{i\varphi} | -\varphi_0 \leqq \varphi \leqq \varphi_0, 0 < \varrho < \infty\}$ where $\frac{\pi}{2} < \varphi_0 < \pi$, and that $h(\varrho e^{i\varphi})$ satisfies the estimates*

(i) $|h(\varrho)| \leqq K e^{-a_1\varrho}$ *for* $1 < \varrho < \infty$,

(ii) $|h(\varrho e^{i\varphi})| \leqq K e^{a_2\varrho}$ *for* $0 < |\varphi| < \varphi_0$, $1 \leqq \varrho < \infty$,

where $0 \leqq a_2 < a_1 |\cos\varphi_0|$.

Then $h(\varrho e^{i\varphi}) \equiv 0$.

Proof. Choose ε and $\varphi_1, 0 < \varepsilon < a_1, \frac{\pi}{2} < \varphi_1 < \varphi_0$, such that

$$-a = (a_1 - \varepsilon)\cos\varphi_1 + a_2 < 0.$$

Consider the function

$$g(z) = e^{(a_1 - \varepsilon)z} h(z).$$

The function $g(z)$ satisfies the estimates

(i′) $|g(\varrho) \leqq K e^{-\varepsilon\varrho}$ for $1 \leqq \varrho < \infty$, (ii′) $|g(\varrho e^{i\varphi})| \leqq K e^{[(a_1-\varepsilon)\cos\varphi + a_2]\varrho}$ for $0 < |\varphi| \leqq \varphi_1$, $1 \leqq \varrho < \infty$,

in particular

(iii′) $|g(\varrho e^{-+i\varphi_0})| \leqq K e^{-a\varrho}$.

Applying Lemma 3.1 to the function $g(\varrho e^{i\varphi})$ on the sector

$$\{\varrho e^{i\varphi} | 0 \leqq \varphi \leqq \varphi_1, 1 \leqq \varrho < \infty\} \quad \text{and} \quad \{\varrho e^{i\varphi} | -\varphi_1 \leqq \varphi \leqq 0, 1 \leqq \varrho < \infty\}$$

we obtain

$$|g(\varrho e^{i\varphi})| \leqq K e^{-b\varrho} \quad \text{for} \quad -\varphi_1 \leqq \varphi \leqq \varphi_1, \quad 1 \leqq \varrho < \infty,$$

where $b = \min\{\varepsilon, a\}$.

An application of Carlson's theorem ([5], p. 185) or [3], Lemma 1.2[1], now implies $g(\varrho e^{i\varphi}) \equiv 0$ and hence $h(\varrho e^{i\varphi}) \equiv 0$. □

4. Proof of the Main Theorem

Proof of the Theorem for $\alpha = 1$. Let $\hat{\mathscr{D}}$ be the set of functions $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\psi \subset \{x \in R^n | |x| \geqq \varepsilon\}$ for some $\varepsilon = \varepsilon_\psi > 0$. $\hat{\mathscr{D}}$ is dense in $\mathscr{H}$. Thus it is sufficient to show that the analytic function

$$F_{\psi,\xi}(z) \equiv (\psi, V(z)\,\xi) \qquad z = \varrho e^{i\varphi}, \qquad z \in S_\theta,$$

is identically equal to 0 for all $\psi \in \hat{\mathscr{D}}$, $\xi \in C_0^\infty(\mathbb{R}^n)$. We will show that $F_{\psi,\xi}(\varrho e^{i\varphi})$ satisfies the hypothesis of Lemma 3.2 for $K_3 = 0$.

Let φ_0 be such that $\pi/2 < \varphi_0 < \theta$, $\varphi_0 < \pi$, and let

$$\operatorname{supp}(\psi) \subseteqq \{x \in R^n \,|\, |x| \geqq \varepsilon\}, \qquad \varepsilon > 0.$$

Then, for $\varrho \geqq 1$, $-\varphi_0 \leqq \varphi \leqq \varphi_0$, we have:

$$\begin{aligned} |F_{\psi,\xi}(\varrho e^{i\varphi})| &= |(\psi, U(\varrho)\, V(e^{i\varphi})\, U(\varrho^{-1})\,\xi)| \\ &= |(U(\varrho^{-1})\,\psi, V(e^{i\varphi})\, U(\varrho^{-1})\,\xi)| \qquad (4.1) \\ &\leqq \|\psi\|_2 \|\xi\|_+ \sup_{-\varphi_0 \leqq \varphi \leqq \varphi_0} |||V(e^{i\varphi})||| = K < \infty\,. \end{aligned}$$

We have used (2.2), (2.3), the unitarity of $U(\varrho^{-1})$ on $\mathscr{H}$, and the fact that $\varphi \rightsquigarrow V(e^{i\varphi})$ is continuous from $[-\varphi_0, \varphi_0]$ to $\mathscr{B}(\mathscr{H}_+, \mathscr{H})$. This shows that (3.2) is satisfied for $K_3 = 0$ and $K = K_1$.

To see the exponential decay along the positive real axis we need the following lemma.

[1] Titchmarsh's statement of Carlson's theorem assumes that $f(Z)$ is regular for $\operatorname{Re} z \geqq 0$. However, his proof only requires that f be regular for $\operatorname{Re} z \geqq 0$, $|z| \geqq r_0$ for some $r_0 > 0$ (see [5], p. 182). Similarly the proof of [3], Lemma 1.2, is obviously valid if $\{z \,|\, |z| < r_0\}$ is delated from the domain of the function.

Lemma 4.1. *Let* $\psi \in \hat{\mathscr{D}}$ *be such that* $\operatorname{supp}\psi \subset \{x \mid |x| \geqq \varepsilon\}$. *Let* $k, \alpha > 0$. *Then*

$$\|e^{-k|x|^\alpha} U(\varrho^{-1})\psi\| \leqq e^{-k\varepsilon^\alpha \varrho^\alpha} \|\psi\|, \qquad \varrho \geqq 1. \tag{4.2}$$

Proof. First of all, note that $\operatorname{supp}(U(\varrho^{-1})\psi) \subseteqq \{x \in R^n \mid |x| \geqq \varrho\varepsilon\}$. Then

$$\begin{aligned}
&\|e^{-k|x|^\alpha} U(\varrho^{-1})\psi\|^2 \\
&\quad = \int_{\mathbb{R}^n} e^{-2k|x|^\alpha} |U(\varrho^{-1})\psi|^2 \, dx \\
&\quad = \int_{|x| \geqq \varrho\varepsilon} e^{-2k|x|^\alpha} |U(\varrho^{-1})\psi|^2 \, dx \\
&\quad \leqq e^{-2k\varrho\varepsilon^\alpha \varrho^\alpha} \int |U(\varrho^{-1})\psi|^2 \, dx \\
&\quad \leqq e^{-2k\varepsilon^\alpha \varrho^\alpha} \|U(\varrho^{-1})\psi\|^2 \\
&\quad = e^{-2k\varepsilon^\alpha \varrho^\alpha} \|\psi\|^2
\end{aligned}$$

which establishes (4.2). □

Applying this lemma to $F_{\psi,\xi}(z)$ we have:

$$\begin{aligned}
|F_{\psi,\xi}(\varrho)| &= |(\psi, U(\varrho) V U(\varrho^{-1})\xi)| \\
&= |(e^{-k|x|} U(\varrho^{-1})\psi, e^{i|x|} V U(\varrho^{-1})\xi)| \\
&\leqq (|||e^{k|x|} V||| \, \|\psi\| \, \|\xi\|_+) \, e^{-k\varepsilon\varrho}.
\end{aligned}$$

Thus letting $k\varepsilon = K_2$, $H_3 = 0$, $K_1 = \max(K, |||e^{k|x|} V||| \, \|\psi\| \, \|\xi\|_+)$, and using (4.1) and (4.3) we see the hypotheses of Theorem 3.1 are satisfied and therefore $F_{\psi,\xi} \equiv 0$ on S_θ.

Proof of the Theorem for $\alpha > 0$.

As before, it is sufficient to show that $F_{\psi,\xi}(z) \equiv (\psi, V(z)\xi)$ is identically 0 for $\psi \in \hat{\mathscr{D}}$, $\xi \in C_0^\infty(\mathbb{R}^n)$, $z \in S_\theta$. First of all we can show in exactly the same way as above, that there exists $K_1, K_2 > 0$ such that

$$|F_{\psi,\xi}(\varrho e^{i\varphi})| \leqq K_1, \qquad \varrho \geqq 1, \qquad -\theta < \varphi < \theta \tag{4.4}$$

and

$$|F_{\psi,\xi}(\varrho)| \leqq K_1 e^{-K_2 \varrho^\alpha}, \qquad \varrho \geqq 1. \tag{4.5}$$

Consider the mapping $f: S_{\alpha\theta} \to S_\theta : (\varrho, \varphi) \rightsquigarrow (\varrho^{\alpha^{-1}}, \alpha^{-1}\varphi)$. This is clearly an analytic bijection. Let $G = F_{\psi,\xi} \circ f$. This is an analytic function on $S_{\alpha\theta}$ and note $\alpha\theta > \pi/2$. Moreover (4.4) and (4.5) imply

$$|G(\varrho e^{i\varphi})| \leqq K_1, \qquad \varrho \geqq 1, \qquad -\alpha\theta < \varphi < \alpha\theta \tag{4.6}$$

and

$$|G(\varrho)| \leqq K_1 e^{-K_2 \varrho}, \qquad \varrho \geqq 1. \tag{4.7}$$

Applying Theorem 3.1 with $\pi/2 < \varphi_0 < \alpha\theta$, $\varphi_0 < \pi$ K_1, K_2 as in (4.6) and (4.7) and, $K_3 = 0$, we see $G \equiv 0$ or $S_{\alpha\theta}$. This implies $G \circ f^{-1} = F_{\psi,\xi} \equiv 0$ on S_θ. □

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