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A Proposition-State Structure

I. The Superposition Principle

Vittorio Berzi and Antonio Zecca*
Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Italy,
Istituto di Scienze Fisiche dell'Università, Milano, Italy

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Abstract. A generalization of the superposition principle of quantum mechanics is proposed introducing the concept of maximal state of a logic.

1. Introduction and Basic Axioms

It is generally admitted that the superposition principle (in the sense of Dirac's book) is the point where the departure of quantum theory from the classical physics is more evident. In order to have a precise mathematical formulation of this principle, it is particularly convenient the language of lattice theory where the difference between classical and Quantum theories can be made very transparent. There exist at present, many expositions of the lattice approach [1, 2]. In view of the formulation of the superposition Principle, the Varadarajan's framework is of particular interest [2]. There it is given a notion of superposition of states which includes both the concept of classical mixture of states as well as the concept of quantum superposition of states.

In this way it is open the possibility of the existence of (non trivial) pure superpositions of pure states. This situation comes out, of course, specializing the lattice structure, to get, via Piron's theorem [1a], the standard logic. In view of the special nature of the assumptions leading to a standard logic, it is of some interest the question if the above mentioned situation occurs in a more general context. In this paper we attempt to give a positive answer to that question.

We associate to the physical system a pair (L, S): the logic L represents the set of all the classes (propositions) of equivalent yes-no experiments and the set S represents the set of all the preparing procedures pertaining to the physical system. The mathematical assumptions on (L, S) are the following. The set L has the structure of complete, orthocomplemented

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and atomic lattice. The elements $\bigwedge_\alpha a_\alpha \equiv \wedge \{a_\alpha\}$, $\bigvee_\alpha a_\alpha \equiv \vee \{a_\alpha\}$ denote respectively the greatest lower bound and the least upper bound for every family $\{a_\alpha\} \subset L$. The term a' denotes the complement of $a \in L$. The elements $\mathbb{1} = \vee L$ and $\emptyset = \wedge L$ are the greatest and the least elements of L. An element $s \in S$ is a map from L in $[0, 1]$. The number $s(a)$ $(a \in L, s \in S)$ is interpreted to give the probability of the outcome "yes" for a test of the class a when the physical system has been prepared with the practical procedures pertaining s. Denoting $S_1(a) = \{s \in S : s(a) = 1\}$ and $S_0(a) = \{s \in S : s(a) = 0\}$ $(a \in L)$, the set S is a subset of $[0, 1]^L$ satisfying the following conditions.

A 1. $a, b \in L$ $a \leqq b \Leftrightarrow S_1(a) \subset S_1(b)$ ($\leqq$ is the order relation in L).
A 2. $S_1(a) = S_0(a') \, \forall a \in L$.
A 3. $S_1(\mathbb{1}) = S$.
A 4. $S_1\left(\bigwedge_\alpha a_\alpha\right) = \bigcap_\alpha S_1(a_\alpha)$ for every family $\{a_\alpha\} \subset L$.
A 5. S is a σ-convex set: if $\{s_i\} \subset S$ and $\{\alpha_i\} \subset (0, 1)$, with $\sum_i \alpha_i = 1$, are countable sets, then there exists $s \in S$ such that $s(a) = \sum_i \alpha_i s_i(a) \; \forall a \in L$, namely $s = \Sigma \alpha_i s_i$.

Definition 1. A pair (L, S) with L a complete, orthocomplemented, atomic lattice and S satisfying the assumptions A 1–A 5, is said to be a proposition-state structure.

It is immediate that a proposition-state structure has the properties: i) $S_1(a) = S_1(b)$ iff $a = b$; ii) $S_0(a) = S_1(a') \, \forall a \in L$; iii) $S_1\left(\bigvee_\alpha a_\alpha\right) \supset \bigcup_\alpha S_1(a_\alpha)$ for every family $\{a_\alpha\} \subset L$; iv) $S_1(\emptyset) = S_0(\mathbb{1}) = \Phi$ (the empty set of states) and $S_0(\emptyset) = S$; v) $\forall a \in L$, $S_1(a)$ and $S_0(a)$ are σ-convex subsets of S; vi) $\emptyset \neq a \in L$ implies $S_1(a) \neq \Phi$.

Remark 1. If the logic L is the complete, orthocomplemented, weakly modular and atomic lattice of all the closed subspaces of a separable complex Hilbert space and S is the set of all the trace class states, then one can easily verify that (L, S) is a proposition-state structure.

We do not require the σ-additivity for the states as done in Ref. [2].

In the next sections, we use the notions of ideal and dual ideal of a lattice, to formulate the concept of superposition of states. We also introduce the concept of characteristic state which coincides with the concept of pure state in the Hilbert model.

If we call maximal state a state contained in $S_1(e)$ when e is an atom, the main result (Proposition 2) of the paper is the existence of maximal (non trivial) superpositions of maximal states.

In this sense we give an indication for structures more general than the standard ones admitting a quantum-like superposition Principle.

Finally we propose a condition (A 6) which ensures the maximality of the pure states and which implies a strong result in the case of completely distributive lattice (Proposition 1).

2. Ideals and Superposition of States

The states of S can be used to introduce ideals in L. Indeed the subsets of L, $L(s)=\{a\in L: s\in S_1(a)\}$ and $O(s)=\{a\in L: s\in S_0(a)\}$ have the following properties:

a) $L(s)=\{a'\in L: a\in O(s)\}\equiv O(s)' \quad (O(s)\equiv L(s)')$

b) $\begin{cases} \emptyset\notin L(s),\ \mathbb{1}\in L(s)\ \forall s\in S \\ a\in L(s),\ b\in L \ \text{ and } \ a\leqq b\Rightarrow b\in L(s) \\ a,b\in L(s)\Rightarrow a\wedge b\in L(s) \\ a\in L(s),\ b\in L\Rightarrow a\vee b\in L(s) \end{cases}$

c) $O(s)$ has the dual properties which hold for $L(s)$ in b).

A subset $I\subset L$ is a dual ideal in L if and only if there exists $\emptyset\neq a\in L$ such that $I=\{x\in L: x\geqq a\}$: then $a=\wedge I\equiv \bigwedge_{x\in I} x$ [1 f]. In a dual way I is an ideal in L if and only if there exists $\mathbb{1}\neq b\in L$ such that $I=\{x\in L: x\leqq b\}$: then $b=\vee I\equiv\bigvee_{x\in I} x$. By A4 we have $\wedge L(s)\in L(s)$ and $\vee O(s)\in O(s)$. Hence, taking into account b) and c), $O(s)$ and $L(s)$ are respectively an ideal and a dual ideal in L completely determined by the elements $\vee O(s)$ and $\wedge L(s)$. The element $\wedge L(s)$ is the least proposition in L which is certainly true on s and $(\wedge L(s))'\equiv\vee O(s)$ is the greatest proposition in L which is impossible on s. An atom of L is an element $e\in L(e\neq\emptyset)$ such that if $\emptyset\neq b\in L$ and $b\leqq e$ implies $b=e$. An element $m\in L(m\neq\mathbb{1})$ such that if $\mathbb{1}\neq b\in L$ and $b\geqq m$ implies $b=m$, is called a maximal element in L. If the term A denotes the set of all the atoms of L, then A' will be the set of all the maximal element in L, L being orthocomplemented.

It must be noticed that a dual ideal $I\subset L$ is maximal if and only if $\wedge I\in A$. Hence there is a bijection between A and the set of all the maximal dual ideals in L and between A' and the set of the maximal ideals in L.

Ideals can be introduced in L in a more general way. Indeed, for every $D\subset S$, defining

$$L(D)=\{a\in L: s\in S_1(a)\ \forall s\in D\} \quad \text{and} \quad O(D)=\{a\in L: s\in S_0(a)\ \forall s\in D\}$$

we get:

$$L(D)=\bigcap_{s\in D} L(s) \quad \text{and} \quad O(D)=\bigcap_{s\in D} O(s).$$

Lemma 1. *If* $D \subset S$ *then* $\wedge\, L(D) = \bigvee_{s \in D} (\wedge\, L(s))$.

Proof. From $L(D) \subset L(s)\ \forall s \in D$, it follows $\wedge\, L(D) \geqq \bigvee_{s \in D} (\wedge\, L(s))$. On the other hand it holds $\bigvee_{s \in D} (\wedge\, L(s)) \geqq \wedge\, L(D)$ because $\bigvee_{s \in D} (\wedge\, L(s))$ is an element of $L(D)$.

In a dual way it can be shown that $\vee\, O(D) = \bigwedge_{s \in D} (\vee\, O(s))$.

Definition 2. The state $s \in S$ is said to be a superposition of the states in $D \subset S$ if $L(s) \supset L(D)$.

That is the same as saying that s is a superposition of the states in D if and only if $O(s) \supset O(D)$. This means that s is a superposition of the states in D if and only if $a \in L$ and $a \in O(\bar{s})\ \forall \bar{s} \in D \Rightarrow a \in O(s)$ which is taken as a definition in the book of Varadarajan [3].

If $L(s) = L(D)$ we say that s is a classical superposition of the states in D.

Lemma 2. *If* $s = \sum_i \alpha_i s_i$ *as in A5, then* $L(s) = \bigcap_i L(s_i)$ *and* $O(s) = \bigcap_i O(s_i)$.

Proof. The relations follow directly considering that $s(a) = 1$ $(s(a) = 0)$ if and only if $s_i(a) = 1\, (s_i(a) = 0)$ for every index i.

From the very definition of $L(s)$ and $O(s)$ we also get $\left(\bigcap_i L(s_i)\right)' = \bigcap_i L(s_i)'$.

3. Characteristic and Pure States

Definition 3. A state $s \in S$ is said to be characteristic if $s' \in S$ and $L(s) = L(s') \Rightarrow s = s'$. The set of characteristic states will be denoted with S_C.

Lemma 3. *If* $s \in S_C$ *then* $L(s)$ *and* $O(s)$ *are maximal ideals.*

Proof. Let I be a proper dual ideal such that $I \supset L(s)$ strictly. Then $s \in S_1(\wedge\, L(s))$ and $s \notin S_1(\wedge\, I) \neq S_1(\emptyset)$. Hence $s' \in S_1(\wedge\, I) \subset S_1(\wedge\, L(s))$ exists such that $s \neq s'$ and $L(s') \supset L(s)$. Considering $\bar{s} = \alpha s + (1-\alpha) s'$ $(\alpha \in (0,1))$ then $\bar{s} \neq s$ and $L(\bar{s}) = L(s)$ by Lemma 2. This is a contradiction, s being characteristic. Hence $L(s)$ is maximal, $\wedge\, L(s)$ is an atom of L and $S_1(\wedge\, L(s)) \equiv \{s\}$. If now an ideal I exists such that $I \supset O(s)$ strictly, then $I' \supset O(s)' = L(s)$ and this is not possible, I' being a dual ideal and $L(s)$ a maximal ideal.

Definition 4. An element $s \in S$ such that $s = \alpha s_1 + (1-\alpha) s_2$ $(s_1, s_2 \in S, \alpha \in (0,1))$ implies $s = s_1 = s_2$ is said to be a pure state. The set of pure states will be denoted with S_P.

Lemma 4. $S_C \subset S_P$.

Proof. Suppose $s \in S_C$ and $s = \alpha s_1 + (1-\alpha) s_2$, $\alpha \in (0,1)$. By Lemma 2, $L(s)$ being maximal, it must be $L(s) = L(s_1) = L(s_2)$ and hence $s = s_1 = s_2$, s being characteristic.

Remark 2. If L is assumed to be a discrete direct union of standard logics each of which is the lattice of all closed subspaces of a complex separable Hilbert space and S is the set of all positive, σ-additive functions s on L such that $s(\mathbb{1})=1$, then $S_C \equiv S_P$ [1 f]. In that model there also is a bijection between atoms and pure states and between pure states and maximal ideals.

If (L, S) is the pair of Definition 1, we have, as a special case of iii), $S_1(a) \supset \bigcup_{e \in A(a)} S_1(e)$ where $A(a)=\{e \in A : e \leqq a\}$ $(a \in L, a \neq \emptyset)$. However in general, there is no explicit connection between the set of atoms and the set of pure states.

We propose a condition which is easily seen to hold in the case of Remark 1, to get the counter part of Lemma 3 in what concerns the pure states.

A6. $S_1(a)=\left\{\sum_i \alpha_i s_i : \{s_i\} \subset \bigcup_{e \in A(a)} S_1(e);\ \{\alpha_i\} \subset (0,1);\ \sum_i \alpha_i = 1\right\}$, $\forall a \in L$.

Indeed, defining $a = \wedge L(s)$ for $s \in S$ we have $s=\sum_i \alpha_i s_i$ as in A6. If $s \in S_P$ it follows $s = s_i\ \forall i$.

Hence there exists $e \in A(a)$ such that $s \in S_1(e)$, $L(s)$ is maximal and $e = \wedge L(s) \in A$.

So the condition A6 implies $\{\wedge L(s) : s \in S_C\} \subset \{\wedge L(s) : s \in S_P\} \subset A$, but it does not imply a bijection between A and S_P.

Proposition 1. *Let (L, S) be a proposition-state structure satisfying A6 and such that L is a completely distributive lattice. Then*

i) *if $s \in S$ and $L(s)$ is maximal, then $s \in S_C$ and $s(a)=0$ or $s(a)=1$ $\forall a \in L$,*

ii) *if $\bar{s} \in S_P, D \subset S_P$ and $L(\bar{s}) \supset L(D) \Rightarrow \bar{s} \in D$.*

Proof. i) Let $s \in S$ and $L(s)$ be a maximal dual ideal. Then $\wedge L(s)=e$ is an atom and $\vee O(s)=e'$ is a maximal element in L. If $a \in L$ we have $a=(a \wedge e) \vee (a \wedge e')$ so that either $a \geqq e$ or $a \leqq e'$. This means $L(s) \cup O(s)=L$ and hence $s(a)=1$ or $s(a)=0$ $\forall a \in L$. From this it immediately follows $S_1(\wedge L(s))=\{s\}$ and hence $s \in S_C$.

ii) The point i) and A6 imply $\bar{s} \in S_C$ and $D \subset S_C$. The elements $\wedge L(\bar{s})=e$, $\wedge L(s)=e(s)\ \forall s \in D$ are atoms. From the assumptions and Lemma 1, it holds $e \leqq \bigvee_{s \in D} e(s)$. From the complete distributivity it follows $e = e \wedge \left(\bigvee_{s \in D} e(s)\right) = \bigvee_{s \in D} (e \wedge e(s))$. Hence there exists $s' \in D$ such that $e=e(s')$. This implies $\bar{s}=s' \in D$.

Remark 3. As a consequence of Proposition 1 i), we get $S_C \equiv S_P$. In this case the atomicity of L and the assumptions A1–A6 on S imply a bijection between atoms and pure states: $A \equiv \{\wedge L(s) : s \in S_P\}$. Moreover, for every $D \subset S_P$, defining $\bar{D}$ to be the set of all the pure states which are superposition of the states in D and M the class of all $D \subset S_P$ such that $D = \bar{D}$, by Proposition 1 ii), M is infact the set of all the subsets of S_P [3]. The map $a \to A(a)\,(a \in L)$ is easily seen to be an isomorphism (L being completely distributive and atomic) between L and the set of all subsets of A. Hence there also is an isomorphism between L and M.

The next Lemma is useful for the general case.

Lemma 5. *Let L be a complete, orthocomplemented, atomic lattice satisfying the following property:*

$$B \subset A, \; \bar{e} \in A \quad \text{and} \quad \bar{e} \leqq \bigvee_{e \in B} e \Rightarrow \bar{e} \in B .$$

Then L is a completely distributive lattice.

Proof. Let $P(A)$ be the Boolean algebra of all subsets of A. For every $B \in P(A)$ consider the map $B \to \varphi(B) \equiv \bigvee_{e \in B} e$ from $P(A)$ in L. If $\varphi(B) = \varphi(B')$ for some $B, B' \in P(A)$, $B \neq B'$, then an atom $\bar{e}$ would exist such that $\bar{e} \in B'$, $\bar{e} \notin B$ (or vice-versa) and $\bar{e} \leqq \varphi(B)$ which contradicts the assumption. Hence φ is a bijection of $P(A)$ onto L. Let now $\{A_\alpha\}$ be a family of elements of $P(A)$. From the definition of φ one gets $\varphi\left(\bigcap_\alpha A_\alpha\right) = \bigwedge_\alpha \varphi(A_\alpha)$. Further $\varphi(B') = \varphi(B)'$ for every $B \in P(A)$ with $B' = A - B$, so that $\varphi\left(\bigcup_\alpha A_\alpha\right) = \bigvee_\alpha \varphi(A_\alpha)$. Then φ is a completely distributive isomorphism of $P(A)$ onto L. This completes the proof[1].

Proposition 2. *Let (L, S) be a proposition-state structure and suppose L not to be a Boolean lattice. Then $\bar{s} \in S$, $D \subset S$, $\bar{s} \notin D$ exist such that $L(\bar{s})$ and $L(s)\,\forall s \in D$ are maximal dual ideals and $\bar{s}$ is a superposition of the states in D which is not a classical superposition.*

Proof. From the assumptions L is not completely distributive. By Lemma 5, $\bar{e} \in A$, $B \subset A$ exist such that $\bar{e} \leqq \bigvee_{e \in B} e$ with $\bar{e} \notin B$. Choose now a state $\bar{s} \in S_1(\bar{e})$ and a state $s(e) \in S_1(e)$ for every $e \in B$. Then $L(\bar{s})$ and $L(s(e))\,\forall e \in B$ are maximal dual ideals such that $L(\bar{s}) \supset \bigcap_{e \in B} L(s(e))$. The theorem is proved pointing out that $\bar{e} \in L(\bar{s})$ but $\bar{e} \notin \bigcap_{e \in B} L(s(e))$ and setting $D = \{s(e) : e \in B\}$.

Remark 4. If (L, S) is a proposition-state structure satisfying A 6 and e is an atom, then, a priori, the following possibilities cannot be excluded:

1) $S_1(e)$ does not contain pure states. Then $S_1(e)$ contains at least two states.

[1] For an alternative proof of Lemma 5, compare Ref. [4] Chapter X, Section 16, Theorem 16.

2) $S_1(e) \subset S_P$. If $S_1(e) \equiv \{s\}$ then s is a pure state which is characteristic. If $S_1(e)$ does contain more then one state, then all the states in $S_1(e)$ are pure but not characteristic.

3) $S_1(e) \cap S_P \neq \Phi$ and $S_1(e) \cap S_P \neq S_1(e)$. None of the elements of $S_1(e)$ is characteristic.

It may be noticed that if we assume $A = \{\wedge\ L(s) : s \in S_P\}$ for the pair (L, S) of Definition 1, we have immediately the maximality of $L(s)$ for $s \in S_P$ without assuming A6. Furthermore the standard property of the pure superpositions of pure states of quantum mechanics is easily seen to hold choosing, in the proof of Proposition 2, $\bar{s}$ and the states of D to be pure. Anyway, also in that case, the situations 2) and 3) might, a priori, arise.

In a forthcoming paper the Definition 2 will be used to describe the reversible dynamical processes directly in the abstract (L, S) scheme.

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A. Zecca
Istituto di Scienze Fisiche
dell'Università di Milano
Via Celoria, 16
I-20133 Milano, Italy