On the Equivalence
of KMS and Gibbs Conditions for States
# of Quantum Lattice Systems

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Abstract. The equivalence of a Gibbsian equilibrium condition and the KMS condition is proven for one-dimensional quantum lattice systems with a finite range interaction at arbitrary temperature, and for quantum lattice systems of arbitrary dimension, with a finite body interaction, at high temperature.

§ 1. Introduction

There are a large number of conditions which characterize equilibrium states of infinitely extended lattice systems (see [5] for example). Most of them are known to be mutually equivalent for classical and quantum lattice systems. One exception has been the equivalence of the Gibbsian equilibrium condition and the KMS condition for quantum lattice systems. We shall show this equivalence for one-dimensional quantum lattice systems at all temperatures and for a general class of quantum lattice systems at high temperature.

In Section 2, we shall fix our notation and terminology. In Section 3, we collect some technical estimates. In Section 4, we formulate a Gibbs condition, which becomes equivalent to equilibrium conditions of Lanford and Ruelle, and Dobrushin when the interaction is classical. In Section 5, we derive the equivalence of the Gibbs condition and the KMS condition. In Section 6, we derive the equivalence of the Gibbs condition and the variational principle when the state is invariant under lattice translations.

§ 2. The KMS Condition

We consider a ν-dimensional quantum lattice system. For a finite subset I of $\mathbb{Z}^\nu$, $N(I)$ denotes the number of points in I and $d(I)$ denotes the largest distance between 2 points in I.

The C^*-algebra $\mathfrak{A}$ (of quasi-local observables) contains a sub-C^*-algebra $\mathfrak{A}(I)$ for each subset I of $\mathbb{Z}$ satisfying the following conditions:

(1) $\mathfrak{A}(I)$ for each finite subset I is a full matrix algebra of some finite dimension. (2) If I and I' are disjoint, $\mathfrak{A}(I)$ and $\mathfrak{A}(I')$ commute element-wise. (3) If I is the union of I_n, then $\mathfrak{A}(I)$ is generated (as a C^*-algebra) by the union of $\mathfrak{A}(I_n)$. For each $Q \in \mathfrak{A}$, we call the infimum (in the set inclusion order) of sets I satisfying $Q \in \mathfrak{A}(I)$ the support of Q and denote it by $supp\,Q$. We denote $N(Q) \equiv N(supp\,Q)$, $d(Q) \equiv d(supp\,Q)$. $\|Q\|_{N \leq m}$ denotes the infimum of $\sum_i \|Q_i\|$ such that $Q = \sum_i Q_i$ and $N(Q) \leq m$. If such $\{Q_i\}$ does not exist, we define $\|Q\|_{N \leq m} = \infty$. $\|Q\|_{d \leq m}$ denotes the infimum of $\sum_i \|Q_i\|$ such that $Q = \sum_i Q_i$ and $d(Q_i) \leq m$. If such $\{Q_i\}$ does not exist, we define $\|Q\|_{d \leq m} = \infty$.

A representation τ of the translation group $\mathbb{Z}^\nu$ by $*$-automorphisms of $\mathfrak{A}$ is supposed to exist such that $\tau(x)\,\mathfrak{A}(I) = \mathfrak{A}(I + x)$ for every $x \in \mathbb{Z}^\nu$, where $I + x = \{x + a; a \in I\}$.

For each finite subset I of $\mathbb{Z}$, we are given a self-adjoint element $\Phi(I) \in \mathfrak{A}(I)$, called an interaction potential, satisfying the following conditions: (1) Φ (empty set) $= 0$, (2) $\tau(x)\,\Phi(I) = \Phi(I + x)$, (3) $\|\Phi\| \equiv \sum_{I \ni 0} \|\Phi(I)\|/N(I) < \infty$. Let $N(\Phi)$ denote the supremum of $N(I)$ such that $\Phi(I) \neq 0$. In the following we shall be interested in Φ with a finite $N(\Phi)$ (namely the case where n-body interaction is absent for large n). Let $d(\Phi)$ denote the supremum of $d(I)$ such that $\Phi(I) \neq 0$. A finite range interaction is characterized by $d(\Phi) < \infty$.

The energy $U^\Phi(I)$ in a finite region I is defined by $U^\Phi(I) = \sum_{I' \subset I} \Phi(I')$. We have $\|U^\Phi(I)\| \leq \|\Phi\|\, N(I)$. The time translation automorphisms σ_t for $-\infty < t < \infty$ are defined by

$$\sigma_t Q = \lim_{I \to \infty} e^{iU^\Phi(I)t} Q\, e^{-iU^\Phi(I)t}, \qquad Q \in \mathfrak{A} \tag{2.1}$$

where the limit is known to exist if $N(\Phi) < \infty$. ([13] or [15]. For a weaker condition on Φ, see [14], p. 193.)

A state φ of $\mathfrak{A}$ is defined to be a KMS state with an inverse temperature β (in the units where the Boltzmann constant is 1) if φ satisfies the following KMS condition: For any elements Q_1 and Q_2 in $\mathfrak{A}$, there exists a function $F(z)$ of z in the complex plane which is continuous for $\mathrm{Im}\,z \in [0, \beta]$, holomorphic for $\mathrm{Im}\,z \in (0, \beta)$ and satisfies $F(t) = \varphi(Q_2 \sigma_t Q_1)$, $F(t + i\beta) = \varphi(\sigma_t(Q_1) Q_2)$.

For each state φ, there exists a Hilbert space $\mathscr{H}_\varphi$, a representation π_φ of $\mathfrak{A}$ on $\mathscr{H}_\varphi$ and a cyclic vector $\Omega_\varphi \in \mathscr{H}$ such that $\varphi(Q) = (\Omega_\varphi, \pi_\varphi(Q)\Omega_\varphi)$. (Here an inner product (x, y) is linear in y according to the physicists' convention.) If the state is invariant under time translation (namely $\varphi(\sigma_t(Q)) = \varphi(Q)$ for any $Q \in \mathfrak{A}$, $t \in \mathbb{R}$), then there exists a strongly continuous unitary representation $U_\varphi(t)$ of $t \in \mathbb{R}$ such that $U_\varphi(t)\,\pi_\varphi(Q)\Omega_\varphi = \pi_\varphi(\sigma_t Q)\Omega_\varphi$.

If φ is a KMS state, φ is invariant under σ_t. Furthermore the vector Ω_φ is cyclic and separating for the von Neumann algebra $\mathfrak{M} = \pi_\varphi(\mathfrak{A})''$. (The cyclicity by definition, and the separating property by the KMS condition.) It should be added that $\mathfrak{A}$ is simple, π_φ is faithful and hence φ is faithful.

By Stone's theorem, there exists a self-adjoint operator H_φ such that $U_\varphi(t) = e^{iH_\varphi t}$. The positive self-adjoint operator

$$\Delta = e^{-\beta H_\varphi}(= \Delta_{\Omega_\varphi}) \tag{2.6}$$

is called the modular operator for the cyclic and separating vector Ω_φ of $\mathfrak{M}$. The automorphism $\sigma_t^\varphi = \sigma_{-\beta t}$ is called the modular automorphism.

§ 3. Technical Lemmas
1. Time Translation

First we consider the case of arbitrary dimension ν and a finite-body interaction $(N(\Phi) < \infty)$. Let $\delta_A Q$ denote the commutator $[A, Q]$ and $\delta_{U^\Phi(I)} Q \equiv \sum_{I' \subset I} [\Phi(I'), Q]$ for finite or infinite I. The sum is norm convergent if $\|Q\|_{N \leq m} < \infty$ for some m.

Lemma 1. *If $N(\Phi) < \infty$ and $\|Q\|_{N \leq m} < \infty$ for some m, then*

$$\|(\delta_{U^\Phi(I)})^n Q\| \leq \{2(N(\Phi) - 1) \|\Phi\|'\}^n (n-1)! \, an^a \|Q\|_{N \leq m} \tag{3.1}$$

where $\|\Phi\|' = \sum_{I \ni 0} \|\Phi(I)\|$ and $a = m/(N(\Phi) - 1)$. The power series

$$\sigma_t^I Q \equiv \sum_{n=0}^\infty n!^{-1}(it)^n (\delta_{U^\Phi(I)})^n Q \tag{3.2}$$

is absolutely convergent for

$$|t| < r(\Phi) \equiv \{2(N(\Phi) - 1) \|\Phi\|'\}^{-1}$$

if $\|Q\|_{N \leq m} < \infty$ for some m. The convergence is uniform for $|t| \leq T$ and $\|Q\|_{N \leq m} \leq M$ if $T < r(\Phi)$. For such t and Q,

$$\|\sigma_t^I Q\| \leq a(T; m) \|Q\|_{N \leq m} \tag{3.3}$$

for some constant $a(T; m)$.

The proof is essentially in [13]. We denote $\sigma_t^{\mathbb{Z}^\nu}$ by σ_t.

The corresponding Lemma for the case $\nu = 1$ is the following:

Lemma 2. *If $\nu = 1$, $d(\Phi) < \infty$ and $\|Q\|_{d \leq m} < \infty$, then the power series (3.2) converges absolutely for any t in the complex plane. The convergence is uniform for $|t| \leq T$ for any T and (3.3) holds for some constant $a(T; m)$.*

The proof is in [2].

The absolute convergence in the above Lemmas is established for the series (3.2) after the defining summation for $(\delta_{U^\Phi(I)})^n Q$ is substituted. From this it immediately follows that under the conditions of Lemmas 1 and 2

$$\lim_{I \to \mathbb{Z}^\nu} \sigma_t^I Q = \sigma_t Q \quad \text{(norm convergence)}. \tag{3.4}$$

If I is finite, then

$$\sigma_t^I Q = e^{iU^\Phi(I)t} Q e^{-iU^\Phi(I)t}. \tag{3.5}$$

If t is real in addition, σ_t^I for finite I obviously extends to a one-parameter group of $*$-automorphisms of $\mathfrak{A}$. Hence $\|\sigma_t^I\| = 1$ for real t. By taking limits, we obtain the same properties for general I and in particular for σ_t; so then (3.4) holds for extended σ_t^I and σ_t.

For real s and t, we have $\sigma_{t+s}^I Q = \sigma_t^I(\sigma_s^I Q)$ for any I and $Q \in \mathfrak{A}$. By an analytic continuation, the same equation holds for complex s provided that the two sides are definable by (3.2). Hence we may define σ_{t+is}^I for real t and s satisfying $|s| < r(\Phi)$ by

$$\sigma_{t+is}^I Q = \sigma_t^I(\sigma_{is}^I Q). \tag{3.6}$$

2. Surface Energy

Let $I'\backslash I$ denote the intersection of I' and the complement I^c of I. Suppose $I' \supset I$ and let

$$W_{I,I'} = \sum_{\emptyset \neq I_1 \subset I, \, \emptyset \neq I_2 \subset I'\backslash I} \Phi(I_1 \cup I_2), \tag{3.7}$$

$$W_I = W_{I,\mathbb{Z}^\nu}. \tag{3.8}$$

W_I is the sum of the interaction potential across the boundary of I and its complement.

Lemma 3. *As a finite subset I' tends to $\mathbb{Z}^\nu$, $\|W_{I,I'} - W_I\|$ tends to 0 if $N(I) < \infty$.*

Proof. This follows from the absolute convergence of the sum (3.7) for $I' = \mathbb{Z}^\nu$ and finite I. The latter follows from

$$\sum \|\Phi(I_1 \cup I_2)\| \leqq \sum_{p \in I} \sum_{I'' \ni p} \|\Phi(I'')\| = N(I) \|\Phi\|'. \qquad \text{Q.E.D.}$$

Let $h_a(I)$ denote the ratio of the maximum number of mutually disjoint translates of a cube of side a which are contained in I, and the minimum number of translates of a cube of side a the union of which can cover I. When we say that I tends to $\mathbb{Z}^\nu$ in the sense of van Hove, we consider a net of finite subsets I_α such that I_α tends to $\mathbb{Z}^\nu$ in set inclusion order and $h_a(I_\alpha) \to 1$ for every a.

Lemma 4. *If I tends to $\mathbb{Z}^\nu$ in the sense of van Hove, then* $\|W_I\|_{N \leq N(\Phi)}/N(I)$ *tends to* 0, *provided that* $N(\Phi) < \infty$.

Proof. Let $C(a)$ denote the cube of side $2a$ and center at the origin. $N(C(a)) = (2a + 1)^\nu$. Since $\|\Phi\|' < \infty$, there exists an a for any given $0 < \varepsilon < 1$ such that $\|\Phi\|' - \sum\limits_{C(a) \supset I \ni 0} \|\Phi(I)\| < \varepsilon$. Let the integer n satisfy $1 - (1 - n^{-1})^\nu < \varepsilon$. Let $b = 2an$. For any finite set I, let $I_\varepsilon^{\mathrm{in}}$ be the set of points p such that $p + C(a)$ is in I. If $h_b(I) > 1 - \varepsilon$, then there exist mutually disjoint translates J_k of $C(b)$ contained in I such that

$$\sum_k N(J_k) \geqq h_b(I) \, N(I) > (1 - \varepsilon) \, N(I). \tag{3.9}$$

On the other hand

$$N((J_k)_\varepsilon^{\mathrm{in}}) = [\{(n - 1)2a + 1\}/(2an + 1)]^\nu \, N(J_k) > (1 - \varepsilon) \, N(J_k). \tag{3.10}$$

Since the point sets $(J_k)_\varepsilon^{\mathrm{in}}$ are mutually disjoint and contained in $I_\varepsilon^{\mathrm{in}}$, we have

$$N(I_\varepsilon^{\mathrm{in}}) \geqq \sum_k N((J_k)_\varepsilon^{\mathrm{in}}) > (1 - \varepsilon)^2 \, N(I). \tag{3.11}$$

Let $I \backslash I_\varepsilon^{\mathrm{in}} = I_\varepsilon^{\mathrm{sur}}$. Inequality (3.11) implies

$$N(I_\varepsilon^{\mathrm{sur}})/N(I) < \varepsilon(2 - \varepsilon). \tag{3.12}$$

Let

$$W_1 = \sum_{I_1 \subset I, \, I_1 \cap I_\varepsilon^{\mathrm{in}} \neq \emptyset} \; \sum_{I_2 \subset I^c} \Phi(I_1 \cup I_2),$$

$$W_2 = \sum_{I_1 \subset I_\varepsilon^{\mathrm{sur}}} \; \sum_{I_2 \subset I^c} \Phi(I_1 \cup I_2).$$

Then

$$\|W_1\|_{N \leq N(\Phi)} \leqq \sum_{p \in I_\varepsilon^{\mathrm{in}}} \; \sum_{I' \ni p, \, I' \not\subset p + C(a)} \|\Phi(I')\| \leqq N(I_\varepsilon^{\mathrm{in}}) \varepsilon < N(I)\varepsilon,$$

$$\|W_2\|_{N \leq N(\Phi)} \leqq \sum_{p \in I_\varepsilon^{\mathrm{sur}}} \; \sum_{I' \ni p} \|\Phi(I')\| \leqq N(I_\varepsilon^{\mathrm{sur}}) \|\Phi\|'.$$

Since $W_I = W_1 + W_2$, we have

$$\|W_I\|_{N \leq N(\Phi)}/N(I) < \varepsilon + \varepsilon(2 - \varepsilon) \|\Phi\|'$$

which tends to 0 as $\varepsilon \to 0$. Q.E.D.

Lemma 5. *If I tends to $\mathbb{Z}^\nu$ in the sense of Van Hove, then* $\|W_I\|_{d \leq d(\Phi)}/N(I)$ *tends to* 0, *provided that* $d(\Phi) < \infty$.

The proof is similar to that of Lemma 4.

3. The Intertwiner Λ_I^t

Let

$$\Lambda_{I,I'}^t = \mathrm{Exp}_l\left(\int_0^1; \ \sigma_{ts}^{I'}(it\,W_{I,I'})\,ds\right) \tag{3.13}$$

where the expansional is defined in [3]; (3.13) is well-defined by Lemma 1 if $|\mathrm{Im}\,t| < r(\Phi)$. By Propositions 3 and 4 of [3], it has an inverse:

$$(\Lambda_{I,I'}^t)^{-1} = \mathrm{Exp}_r\left(\int_0^1; \ -\sigma_{ts}^{I'}(it\,W_{I,I'})\,ds\right). \tag{3.14}$$

For $I' = \mathbb{Z}^\nu$, we write Λ_I^t instead of $\Lambda_{I,\mathbb{Z}^\nu}^t$. For finite I', we have by Theorem 3 of [3]

$$\Lambda_{I,I'}^t = e^{itu^\Phi(I')}e^{-it(U^\Phi(I') - W_{I,I'})}. \tag{3.15}$$

Since $U^\Phi(I') - W_{I,I'} = U^\Phi(I) + U^\Phi(I'\backslash I)$, we have for $Q \in \mathfrak{A}(I)$

$$\Lambda_{I,I'}^t\, e^{itU^\Phi(I)}Q = \sigma_t^{I'}(Q)\,\Lambda_{I,I'}^t\, e^{itU^\Phi(I)}. \tag{3.16}$$

When t is real, this equality for an arbitrary I' is obtained from the same for finite I' by a limit argument. (Note that $\sigma_{ts}^{I'}$ for real ts is a $*$-automorphism.) Hence by analytic continuation, (3.16) holds for general I' for $Q \in \mathfrak{A}(I)$, finite I and $|\mathrm{Im}\,t| < r(\Phi)$ if $N(\Phi) < \infty$, and for $Q \in \mathfrak{A}(I)$, finite I and any t if $\nu = 1$ and $d(\Phi) < \infty$. In particular

$$\Lambda_I^t\, e^{itU^\Phi(I)}Q = \sigma_t(Q)\,\Lambda_I^t\, e^{itU^\Phi(I)}, \tag{3.17}$$

$$e^{-itU^\Phi(I)}(\Lambda_I^t)^{-1}\,\sigma_t(Q) = Q\,e^{-itU^\Phi(I)}(\Lambda_I^t)^{-1}. \tag{3.18}$$

§ 4. Gibbs Condition

Let φ_0^I denote the central state (i.e. tracial state) of $\mathfrak{A}(I)$. Since $\mathfrak{A}(I)$ is a full matrix algebra of finite dimension if I is finite, we may view $\mathfrak{A}$ as the tensor product $\mathfrak{A}(I) \otimes \mathfrak{A}(I^c)$ [12]. For a state φ of $\mathfrak{A}$, we now introduce the following:

Gibbs Condition. *For each finite $I \subset \mathbb{Z}^\nu$, there exists a positive linear functional φ^{I^c} of $A(I^c)$ such that*

$$\varphi(Q) = (\varphi_0^I \otimes \varphi^{I^c})\,(e^{-\beta U^\Phi(I)/2}\,\Lambda_I^*\,Q\,\Lambda_I\,e^{-\beta U^\Phi(I)/2}), \tag{4.1}$$

where $\Lambda_I \equiv \Lambda_I^{i\beta/2}$.

We shall first show that this condition is equivalent to conditions of Lanford and Ruelle [11] and Dobrushin [7, 8] when the interaction is classical, namely when each $\Phi(I)$ belongs to an abelian subalgebra $\mathfrak{M}$ of $\mathfrak{A}$ and φ is restricted to $\mathfrak{M}$; $\mathfrak{M}$ is the algebra generated by number operators at each lattice site. If a state φ satisfies (4.1) and the interaction

is classical then the state is classical too, in the usual sense [6] that the vanishing of the conditional expectation of an operator Q with respect to $\mathfrak{M}$, $\varphi_0(Q|\mathfrak{M}) = 0$, implies $\varphi(Q) = 0$. ($\hat{Q} \equiv \varphi_0(Q|\mathfrak{M})$ is defined by the requirements $\hat{Q} \in \mathfrak{M}$ and $\varphi_0(QQ_1) = \varphi_0(\hat{Q}Q_1)$ for all $Q_1 \in \mathfrak{M}$.) This follows from (4.1) for $Q \in \mathfrak{A}(I)$ for any finite I and hence for all $Q \in \mathfrak{A}$. We then restrict φ to $\mathfrak{M}$. Since $\sigma_t(Q) = Q$ for $Q \in \mathfrak{M}$, we have $\Lambda_I = e^{\beta W_I/2}$. So we have as Gibbs condition

$$\varphi(Q) = (\varphi_0^I \otimes \varphi^{I^c})(e^{-\beta(U^\Phi(I) + W_I)}Q), \qquad Q \in \mathfrak{M}. \tag{4.2}$$

For $Q_1 \in \mathfrak{A}(I) \cap \mathfrak{M}$, let us consider the conditional expectation

$$f_I(Q_1) = \varphi_0^I(e^{-\beta(U^\Phi(I) + W_I)}Q_1) \in \mathfrak{A}(I^c) \cap \mathfrak{M}. \tag{4.3}$$

Then

$$\varphi(Q_1 \otimes Q_2) = \varphi^{I^c}(f_I(Q_1)Q_2) \tag{4.4}$$

for $Q_1 \in \mathfrak{A}(I) \cap \mathfrak{M}$, $Q_2 \in \mathfrak{A}(I^c) \cap \mathfrak{M}$. Lanford and Ruelle consider the situation where there is a minimal projection P_0^I in $\mathfrak{A}(I) \cap \mathfrak{M}$ (the projection onto the no-particle state in I) such that $e^{-\beta(U^\Phi(I) + W_I)}P_0^I = P_0^I$. Then $f_I(P_0^I) = \varphi_0^I(P_0^I)$ and (4.4) implies $\varphi^{I^c}(Q_2) = \varphi(P_0^I)^{-1}\varphi(P_0^I \otimes Q_2)$. Hence

$$\varphi(Q_1 \otimes Q_2) = \varphi(P_0^I \otimes \{f_I(Q_1)Q_2\})/\varphi(P_0^I). \tag{4.5}$$

The Eq. (4.5) when written in terms of measures on the spectrum of $\mathfrak{M}$, is the equilibrium condition of Lanford and Ruelle.

To be more explicit, for a classical lattice gas the spectrum of $\mathfrak{M}$ may be seen as the space $\mathcal{P}(\mathbb{Z}^\nu)$ of all subsets of the lattice, and φ gives rise to a measure μ there. The isomorphism of $\mathfrak{M}$ with $L^\infty(\mathcal{P}(\mathbb{Z}^\nu), \mu)$ may be written, for Q in $\mathfrak{A}(I)$, as

$$Q = \sum_{X \subset I} \tilde{Q}(X) \, n(X) \, \bar{n}(I \setminus X)$$

where $n(X) = \bigotimes_{x \in X} n(x)$, $n(x) = a^*(x)a(x)$ is the number operator at the site x, and similarly $\bar{n}(X) = \bigotimes_{x \in X} a(x)a^*(x)$. Thus the function $\tilde{Q}$ corresponds with the operator Q, so

$$\varphi(Q) = \int_{\mathcal{P}(\mathbb{Z}^\nu)} \tilde{Q}(X) \, \mu(dX)$$

$$= \sum_{X \subset I} \int_{\mathcal{P}(I^c)} \tilde{Q}(X \cup Y) \, \varrho_I(X, dY),$$

where the latter equality defines $\varrho_I(X, dY)$ ([11], Eq. (3.2)). Then we have

$$\varphi^{I^c}(Q_2) = \varphi(P_0^I \otimes Q_2) \, \varphi_0^I(P_0^I)^{-1}$$

$$= \varphi_0^I(P_0^I)^{-1} \sum_{X \subset I} \int_{\mathcal{P}(I^c)} \tilde{P_0^I}(X) \tilde{Q_2}(Y) \, \varrho_I(X, dY)$$

$$= \varphi_0^I(P_0^I)^{-1} \int_{\mathcal{P}(I^c)} \tilde{Q_2}(Y) \, \varrho_I(\emptyset, dY).$$

Now also

$$f_I(Q_1) = \varphi_0^I(P_0^I) \sum_{X \subset I} \exp\left(-\sum_{S \subset X \cup Y : S \cap X \neq \emptyset} \tilde{\Phi}(S)\right) \tilde{Q}_1(X \cup Y).$$

[The weight coming from φ_0^I in the summation over $X \subset I$ is a constant, which is equal, for example, to the weight for $X = \emptyset$ and the latter is $\varphi_0^I(P_0^I)$.] So, if Q_1 is in $\mathfrak{A}(I)$, Eq. (4.5) becomes

$$\sum_{X \subset I} \int_{\mathscr{P}(I^c)} \tilde{Q}_1(X) \tilde{Q}_2(Y) \varrho_I(X, dy)$$

$$= \int_{\mathscr{P}(I^c)} f_I(Q_1)\tilde{\,}(Y) \tilde{Q}_2(Y) \varrho_I(\emptyset, dY)/\varphi(P_0^I)$$

$$= \sum_{X \subset I} \int_{\mathscr{P}(I^c)} \exp\left(-\sum_{S \subset X \cup Y : S \cap X \neq \emptyset} \tilde{\Phi}(S)\right)$$

$$\cdot \tilde{Q}_1(X) \tilde{Q}_2(Y) \varrho_I(\emptyset, dY)$$

expressing exactly the condition in (3.10) of [11].

Conversely a classical state $\tilde{\varphi}$ has a unique extension φ to the whole algebra $\mathfrak{A}$ under the requirement that for every Q, $\varphi(Q) = \tilde{\varphi}(\varphi_0(Q|\mathfrak{M}))$. Then if (4.5) holds we have (4.2) for $Q = Q_1 \otimes Q_2$, $Q_1 \in \mathfrak{A}(I) \cap \mathfrak{M}$ and $Q_2 \in \mathfrak{A}(I^c) \cap \mathfrak{M}$, where $\varphi^{I^c}(Q_2) = \varphi(P_0^I \otimes Q_2)$. Since such Q are total in $\mathfrak{M}$ in the setting of Lanford and Ruelle, (4.2) holds for $Q \in \mathfrak{M}$ and hence (4.1) holds for $Q \in \mathfrak{A}$. This shows the desired equivalence.

The condition given by Dobrushin is known to be equivalent to that given by Ruelle and Lanford.

§ 5. The Equivalence of Gibbs and KMS Conditions

Theorem 1. *When $N(\Phi) < \infty$ and $\beta/2 < r(\Phi)$, or when $v = 1$, $d(\Phi) < \infty$ and β is arbitrary, a state φ satisfies the Gibbs condition if and only if it satisfies the KMS condition.*

Proof. First assume the Gibbs condition (4.1) for φ. [Our Gibbs condition is formulated only for $\beta/2 < r(\Phi)$ if v is general and $N(\Phi) < \infty$.] Let $Q = Q_1 \sigma_{i\beta/2}(Q_2)$, $Q_1 \in A$, $Q_2 \in \mathfrak{A}(I)$. Then by (3.17) with $t = i\beta/2$,

$$\sigma_{i\beta/2}(Q_2) \Lambda_I e^{-\beta U^\Phi(I)/2} = \Lambda_I e^{-\beta U^\Phi(I)/2} Q_2. \tag{5.1}$$

By taking adjoints and replacing Q_2^* by Q_2, we have

$$Q_2 e^{-\beta U^\Phi(I)/2} \Lambda_I^* = e^{-\beta U^\Phi(I)/2} \Lambda_I^* \sigma_{-i\beta/2}(Q_2). \tag{5.2}$$

Using (5.1), the tracial property of φ_0^I and (5.2), we obtain

$$\varphi(Q_1 \sigma_{i\beta/2}(Q_2)) = \varphi(\sigma_{-i\beta/2}(Q_2)Q_1) \tag{5.3}$$

whenever $Q_2 \in \mathfrak{A}(I)$, $Q_1 \in \mathfrak{A}$. By taking the limit, we obtain (5.3) for any Q_2 with $\|Q_2\|_{N \leq m} < \infty$. If $Q_2 \in \mathfrak{A}(I)$, then $\|(\delta_{U^\Phi(\mathbb{Z}^v)})^n Q_2\|_{N \leq m} < \infty$ for

$m = n(N(\Phi) - 1) + N(I)$. Hence (5.3) holds if Q_2 is replaced by $(d/dz)^n \sigma_z(Q_2)|_{z=0}$.

For $Q_2 \in \mathfrak{A}(I)$, I finite, consider the function

$$f(z) = \varphi(\sigma_z(Q_2)).$$

It is well defined, uniformly bounded and holomorphic for z satisfying $|\mathrm{Im}\, z| \leqq \beta/2$ $(< r(\Phi))$. By (5.3) with $Q_1 = 1$, we have $f^{(n)}(i\beta/2) = f^{(n)}(-i\beta/2)$ for $n = 0, 1, 2, \ldots$. Hence

$$f(z + i\beta/2) = f(z - i\beta/2) \tag{5.4}$$

holds for z in some neighbourhood of 0. By analytic continuation, (5.4) holds in a neighbourhood of real z. Hence $f(z)$ can be extended to a periodic entire function. A periodic entire function can be uniformly bounded only when it is a constant. Hence we have $f(z) = f(0)$. Namely φ is σ_t-invariant.

We now consider $Q_1(f)$, where $Q_1 \in A$, $\tilde{f} \in \mathscr{D}(\mathbb{R})$, the space of infinitely differentiable functions of compact support,

$$f(t) = (2\pi)^{-1} \int \tilde{f}(p) e^{ipt}\, dp, \quad \text{and} \quad Q_1(f) = \int \sigma_t(Q_1) f(t)\, dt.$$

We have $\sigma_t(Q_1(f)) = Q_1(f_t)$ with $f_t(s) = f(s - t)$, which holds obviously for real t and has an analytic continuation for arbitrary t. By the σ_t-invariance, we have

$$\varphi(Q_1(f_t)\, \sigma_s(Q_2)) = \varphi(\sigma_{-s}(Q(f_t))Q_2) = \varphi(Q_1(f_{t-s})Q_2)$$

which holds for real s and hence by analytic continuation for all s we are considering $[|\mathrm{Im}\, s| < r(\Phi)$ for general v and $N(\Phi) < \infty$, arbitrary s for $v = 1$, $d(\Phi) < \infty]$.

Substituting $Q_1(f_{t-i\beta/2})$ for Q_1 in (5.3), we obtain

$$\varphi(\sigma_{t-i\beta}(Q_1(f))Q_2) = \varphi(Q_2 \sigma_t Q_1(f)). \tag{5.5}$$

By taking limits, this equation holds for arbitrary $Q_2 \in \mathfrak{A}$. Taking $F(z)$ to be $\varphi(Q_2 \sigma_z[Q_1(f)])$, we have $F(t) = (Q_2 \sigma_t(Q_1))$ and $F(t + i\beta) = (\sigma_t[Q_1(f)]Q_2)$, which is the KMS condition for operators $Q_1(f)$ and Q_2.

It is known that this implies the KMS condition for any Q_1 and Q_2 in $\mathfrak{A}$ (see Lemma 2.2 of [1], for example).

We now prove the converse. Assume the KMS condition for φ. We consider arbitrary β when $v = 1$ and $d(\Phi) < \infty$, and β satisfying $\beta/2 < r(\Phi)$ when v is general and $N(\Phi) < \infty$. Let

$$\hat{\phi}_I(Q) = \varphi((\Lambda_I^*)^{-1} e^{\beta U^\Phi(I)/2} Q\, e^{\beta U^\Phi(I)/2} \Lambda_I^{-1}). \tag{5.6}$$

By the KMS condition one has

$$\varphi(B \sigma_{i\beta/2}(Q_1)) = \varphi(\sigma_{-i\beta/2}(Q_1) B)$$

for $Q_1 \in \mathfrak{A}(I)$ and $B \in \mathfrak{A}$, where I is finite. By (3.18), we have

$$Q_1 e^{\beta U^{\Phi}(I)/2} \Lambda_I^{-1} = e^{\beta U^{\Phi}(I)/2} \Lambda_I^{-1} \sigma_{i\beta/2}(Q_1).$$

By taking the adjoint and replacing Q_1^* by Q_1, we also have

$$\sigma_{-i\beta/2}(Q_1)(\Lambda_I^*)^{-1} e^{\beta U^{\Phi}(I)/2} = (\Lambda_I^*)^{-1} e^{\beta U^{\Phi}(I)/2} Q_1.$$

Hence we have

$$\hat{\varphi}_I(Q_2 Q_1) = \hat{\varphi}_I(Q_1 Q_2) \tag{5.7}$$

for any $Q_2 \in \mathfrak{A}$, $Q_1 \in \mathfrak{A}(I)$, which implies $\hat{\varphi}_I = \varphi_0^I \otimes \varphi^{I^c}$ where φ^{I^c} is the restriction of $\hat{\varphi}_I$ to $\mathfrak{A}(I^c)$. Solving (5.6) for φ, we obtain the Gibbs condition.   Q.E.D.

§ 6. Equivalence of the Gibbs Condition with the Variational Principle for Translationally Invariant States

Theorem 2. *A state φ, invariant under the lattice translations $\tau(a)$, satisfies the Gibbs condition if and only if it satisfies the variational principle:*

$$s(\varphi) - \beta \varphi(A_{\Phi}) \geqq s(\psi) - \beta \psi(A_{\Phi}) \tag{6.1}$$

for all translationally invariant states ψ, where

$$A_{\Phi} = \sum_{I \ni 0} \Phi(I)/N(I) \tag{6.2}$$

and $s(\varphi)$ is the mean entropy of φ([9]). β is arbitrary when $v = 1$, $d(\Phi) < \infty$ and β is restricted to be in $[0, 2r(\Phi))$ when v is arbitrary and $N(\Phi) < \infty$.

Proof. It is known that the limit

$$P(\Phi) = \lim N(I)^{-1} \log \varphi_0^I(e^{-\beta U^{\Phi}(I)}) \tag{6.3}$$

as I tends to $\mathbb{Z}^v$ in the sense of van Hove satisfies

$$P(\Phi) = \sup(s(\psi) - \beta \psi(A_{\Phi})) \tag{6.4}$$

where the supremum is over all translationally invariant states ψ.

We now assume the KMS condition for a state φ and show that $s(\varphi) - \beta \varphi(A_{\Phi}) = P(\Phi)$. Let

$$\varphi_I(Q) \equiv \varphi((\Lambda_I^*)^{-1} Q \Lambda_I^{-1}).$$

By the Gibbs condition, we have

$$\varphi_I(Q) = c \varphi_0^I(e^{-\beta U^{\Phi}(I)} Q) \tag{6.5}$$

for any $Q \in \mathfrak{A}(I)$, where the constant c can be determined as

$$c = \varphi_0^I(e^{-\beta U^{\Phi}(I)})^{-1} \varphi_I(1).$$

Let $\varrho_I(\psi) \in \mathfrak{A}(I)^+$ be defined by

$$\psi(Q) = \varphi_0^I(\varrho_I(\psi)Q), \qquad Q \in \mathfrak{A}(I).$$

The mean entropy is defined by

$$s(\psi) = -\lim N(I)^{-1} \psi(\log \varrho_I(\psi)).$$

[This definition differs by a constant from the definition using the matrix trace instead of the normalized trace φ_0^I. However our definition (6.3) of $P(\Phi)$ is also in terms of φ_0^I and hence (6.4) holds.] From (6.5) we have

$$\varrho_I(\varphi_I) = \varphi_0^I (e^{-\beta U^\Phi(I)})^{-1} e^{-\beta U^\Phi(I)} \varphi_I(1).$$

Let $\mathscr{H}_\varphi, \pi_\varphi$ and Ω_φ be associated with φ by the GNS construction. Then φ_I is the vector state whose vector is

$$\Phi \equiv \pi_\varphi(\Lambda_I^{-1})\Omega_\varphi = \mathrm{Exp}_r\left(\int_0^{1/2}; \sigma_{-is}^\varphi(\beta W_I)\, ds\right)\Omega_\varphi = \Omega_\varphi(\beta W_I)$$

where $\Omega_\varphi(h)$ for a relative Hamiltonian h is defined in [5]. By Proposition 4.1 of [5], $\Omega_\varphi(\beta W_I) \in V_{\Omega_\varphi}$. Hence by Theorem 12 of [4] we have

$$\varphi_I \leqq \|\Lambda_I^{-1}\|^2 \varphi, \qquad \varphi \leqq \|\Lambda_I\|^2 \varphi_I.$$

[The latter follows from $\Omega_\varphi = \pi_\varphi(\Lambda_I)\Phi$ and $V_\Phi = V_{\Omega_\varphi}$.]

The above inequalities when restricted to $\mathfrak{A}(I)$ yield

$$\|\Lambda_I^{-1}\|^{-2} \varrho_I(\varphi_I) \leqq \varrho_I(\varphi) \leqq \|\Lambda_I\|^2 \varrho_I(\varphi_I).$$

Since $\log \varrho$ is operator monotone for positive ϱ,

$$-2\log\|\Lambda_I^{-1}\| \leqq \log \varrho_I(\varphi) - \log \varrho_I(\varphi_I) \leqq 2\log\|\Lambda_I\|.$$

Since

$$\log \varrho_I(\varphi_I) = -\beta U^\Phi(I) - \log \varphi_0^I(e^{-\beta U^\Phi(I)}) + \log \varphi_I(1),$$

$$\|\Lambda_I\|^{-2} \leqq \varphi_I(1) \leqq \|\Lambda_I^{-1}\|^2 \quad (\text{due to } \varphi(1) = 1),$$

and

$$\lim N(I)^{-1} \varphi(U^\Phi(I)) = \varphi(A_\Phi)$$

for translationally invariant φ, we obtain

$$s(\varphi) - \beta\varphi(A_\Phi) - P(\Phi) = 0$$

if we prove

$$\lim N(I)^{-1} \log\|\Lambda_I^{-1}\| = 0, \tag{6.6}$$

$$\lim N(I)^{-1} \log\|\Lambda_I\| = 0. \tag{6.7}$$

By Lemma 1 for $N(\Phi) < \infty$, $|s| \leqq T < r(\Phi)$, and by Lemma 2 for $v = 1$, $d(\Phi) < \infty$, $|s| \leqq T$, we have

$$\|\sigma_{is}(W_I)\| \leqq a(T, k(\Phi)) \|W_I\|_{k \leqq k(\Phi)},$$

where $k = N$ for general v, and $k = d$ for $v = 1$. Hence

$$\log \|\Lambda_I\| \leqq (\beta/2) \, a(\beta/2, k(\Phi)) \, \|W_I\|_{k \leqq k(\Phi)} \, ,$$

$$\log \|\Lambda_I^{-1}\| \leqq (\beta/2) \, a(\beta/2, k(\Phi)) \, \|W_I\|_{k < k(\Phi)} \, .$$

Since $\|\Lambda_I^{-1}\|^{-1} \leqq \|\Lambda_I\|$, we have

$$\left| \log \|\Lambda_I\| \right| \leqq (\beta/2) \, a(\beta/2, k(\Phi)) \, \|W_I\|_{k \leqq k(\Phi)} \, ,$$

$$\left| \log \|\Lambda_I^{-1}\| \right| \leqq (\beta/2) \, a(\beta/2, k(\Phi)) \, \|W_I\|_{k \leqq k(\Phi)} \, .$$

By Lemmas 4 and 5, we have (6.6) and (6.7).

It is known that a state satisfying the variational principle satisfies the KMS condition [10]. Q.E.D.

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