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Markov Processes, Bernoulli Schemes, and Ising Model

Francesco di Liberto, Giovanni Gallavotti, and Lucio Russo
Istituto di Fisica Teorica dell'Università, Napoli, Italia

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Abstract. We give conditions for the Bernoullicity of the ν-dimensional Markov processes.

1. Symbols and Definitions

Z^ν is the ν-dimensional lattice of the points with integral coordinates and $K = I^{Z^\nu} = \prod_{\xi \in Z^\nu} I$, $I = \{0, 1\}$, is the space of sequences of 0's and 1's labelled with the points $\xi \in Z^\nu$.

The space K is compact if endowed with the topology obtained as product of the discrete topologies on the factors I.

Similarly if $\Theta \subset Z^\nu$ we define the compact space $K_\Theta = I^\Theta = \prod_{\xi \in \Theta} I$.

We shall identify the elements $X \in K_\Theta$ as subsets of Θ: so that $X = (x_1, x_2 \ldots x_p) \in K_\Theta$ means the sequence $X \in K_\Theta$ with values 1 in $x_1, x_2, \ldots, x_p$ and 0 in $\Theta \setminus X$.

If $X \in K$ and $\xi \in Z^\nu$ we put $\tau_\xi X = X + \xi = (x_1 + \xi, x_2 + \xi, \ldots)$ if $X = (x_1, x_2 \ldots)$. The transformations $\tau_\xi : K \to K$ form a ν-dimensional group which we denote with the symbol τ; τ transforms Borel sets into Borel sets.

If μ is a Borel probability measure on K which is τ-invariant and $\Lambda \subset Z^\nu$ is a finite set (i.e. $|\Lambda| < \infty$), then we can define Borel measures

$$\mu_\Lambda(X, E), \qquad Q_\Lambda(E) \quad \text{on} \quad K_{Z^\nu \setminus \Lambda}$$

as

$$\mu_\Lambda(X, E) = \mu(\{Y \mid Y \in K;\ Y \cap \Lambda = X;\ Y \cap (Z^\nu \setminus \Lambda) \in E\}) \ E \subset K_{Z^\nu \setminus \Lambda}, \tag{1.1}$$

$$Q_\Lambda(E) = \sum_{X \subset \Lambda} \mu_\Lambda(X, E) = \mu(\{Y \mid Y \in K,\ Y \cap (Z^\nu \setminus \Lambda) \in E\}). \tag{1.2}$$

The Radon-Nikodym derivative, defined for $X \subset \Lambda$ and $Y \subset Z^\nu \setminus \Lambda$

$$\frac{\mu_\Lambda(X, dY)}{Q_\Lambda(dY)} = f_\Lambda(X \mid Y) \tag{1.3}$$

is the conditional probability "for finding X in Λ given that Y is realized outside Λ".

In general $\tau', \tau'', \theta', \theta'', \ldots$ will be ν-dimensional groups of transformations on Lebesgue measure spaces (K', μ'), $(K'', \mu''), \ldots$ which preserve the measures and are isomorphic to the group Z^ν.

A τ-invariant measure μ is called a *non-singular Markov process* if, calling $\partial_1 \Lambda = \{\xi \mid \xi \in Z^\nu, \xi \notin \Lambda$ and distance of ξ from Λ equals $1\}$,

$$\text{i)} \qquad f_\Lambda(X \mid Y) > 0 \qquad Q_\Lambda - \text{a.e.} \tag{1.4}$$

$$\text{ii)} \qquad f_\Lambda(X \mid Y) = f_\Lambda(X \mid Y') \quad \text{if} \quad Y \cap \partial_1 \Lambda = Y' \cap \partial_1 \Lambda \tag{1.5}$$

the last equation being understood $Q_\Lambda \times Q_\Lambda -$ a.e.

Define, next,

$$\begin{aligned} |X| &= \text{number of points in } X \\ [X] &= \text{number of nearest neighbours in } X \\ i(X \mid Y) &= \text{number of couples of nearest neighbours} \\ &\qquad (\xi, \eta) \text{ such that } \xi \in X, \eta \in Y \end{aligned} \tag{1.6}$$

then the following very remarkable theorem holds [1]:

Theorem 1. *A τ-invariant probability measure on K is a non singular Markov process if and only if there are two real parameters $z \geqq 0$, β such that $\forall X \subset \Lambda \ \forall Y \subset Z^\nu \setminus \Lambda$ ($Q_\Lambda -$ a.e.)*

$$f_\Lambda(X \mid Y) = \frac{z^{|X|} e^{4\beta i(X \mid Y)} e^{4\beta [X]}}{\sum\limits_{X' \subset \Lambda} z^{|X'|} e^{4\beta i(X' \mid Y)} e^{4\beta [X']}}\,; \tag{1.7}$$

because of this theorem we shall refer to a Markov process as to a (z, β)-Markov process.

There is a natural two set partition $\mathscr{P}$ of the space K on which the above Markov processes act:

$$\begin{aligned} \mathscr{P} &= (P_0, P_1)\,, \\ P_0 &= \{X \mid X \in K,\ 0 \notin X\}\,, \\ P_1 &= \{X \mid X \in K,\ 0 \in X\}\,. \end{aligned} \tag{1.8}$$

If $\Lambda \subset Z^\nu$ is a finite region the $2^{|\Lambda|}$ atoms of the partition $\mathscr{P}_\Lambda = \bigvee_{\xi \in \Lambda} \tau_\xi \mathscr{P}$ are of the form $A_\Lambda(X) = \{Y \mid Y \in K, Y \cap \Lambda = X\}$, and their measure will be denoted

$$f_\Lambda(X) = \mu(A_\Lambda(X)) = \mu(\{Y \mid Y \in K, Y \cap \Lambda = X\})\,. \tag{1.9}$$

2. Description of the Results

It has been recently shown that the non-singular Markov chains (i.e. 1-dimensional non-singular Markov processes) are uniquely determined by their conditional probabilities [2] and are Bernoulli schemes for all values of (z, β) [3].

In two or more dimensions the same questions are more difficult. It happens that the conditional probabilities do not necessarily determine the process which generates them [4]. It might even happen that a measure μ with conditional probabilities (1.7) is not necessarily τ-invariant [5].

It is, therefore, particularly interesting to ask whether a τ-invariant Markov process (z, β) is a Bernoulli scheme.

In this paper we consider two extreme situations and show that the corresponding Markov processes are actually of Bernoulli type. The two situations correspond to the cases:

$$\begin{aligned} &\text{i) } \beta \text{ fixed and } z \ll 1; \\ &\text{ii) } z = e^{-8\beta}, \beta \gg 1. \end{aligned} \tag{2.1}$$

These two cases are extreme in the sense that in case i) the conditional probabilities uniquely determine a measure μ which is, furthermore, known to be τ-invariant, ergodic and, better, a K-system [6]; in case ii) the conditional probabilities do not determine μ [4] and it is known that the corresponding τ-invariant ergodic measures are just two [7] (and furthermore they are both mixing).

The proof will consist in showing that the partition $\mathscr{P}$ is "finitely determinate" (see next section) in a Markov process (z, β) verifying i), ii). It is known that this fact together with the fact that $\mathscr{P}$ is a τ-generator for (z, β) implies that (z, β) is a Bernoulli scheme [8].

The finite determinability of $\mathscr{P}$ relative to (z, β) is deduced from the strong cluster property

$$\sum_{X_1 \subset \Lambda_1} \sum_{X_2 \subset \Lambda_2} |f_{\Lambda_1 \cup \Lambda_2}(X_1 \cup X_2) - f_{\Lambda_1}(X_1)\, f_{\Lambda_2}(X_2)| \leqq \eta(\Lambda_1, \Lambda_2), \tag{2.2}$$

valid for $|\Lambda_1|, |\Lambda_2| < \infty$ and where η is defined in terms of two suitable non negative functions $A(z, \beta)$, $\varkappa(z, \beta)$ as well as in terms of the geometric objects $d(\Lambda_1, \Lambda_2) =$ (distance of Λ_1 from Λ_2) and $|\partial_1 \Lambda_i| =$ (number of elements neighbouring Λ_i) as:

$$\eta(\Lambda_1, \Lambda_2) = \left(\exp A e^{-\varkappa d(\Lambda_1, \Lambda_2)} \left(\min_{i=1,2} |\partial_1 \Lambda_i|\right) - 1\right); \tag{2.3}$$

this result is proven in Section 4, 5 for cases i) or ii) respectively.

A second type of results will concern the ergodic properties of the one dimensional dynamical system associated with a Markov process μ on K and a one dimensional subgroup τ_j of the group τ: we shall show that this dynamical system is a Bernoulli scheme with infinite entropy: many factors, with finite entropy, of this scheme are exhibited and, in terms of them, we discuss some conjectures.

This paper contains the proof that (2.2), (2.3) imply that $\mathscr{P}$ is finitely determined with respect to (z, β) (Section 3). Section 4, 5 contain the proof of (2.3). Section 6 contains some concluding remarks and the study of some factors of the one dimensional dynamical systems associated with (z, β).

The proof in Section 3 is very similar to that in Ref. [3] and we give here only the necessary changes and new definitions: we shall also freely use, here, the definitions conventions and lemmas of § 2 of Ref. [3].

After completing this work a paper by Dobruschin [17] has appeared in which an inequality slightly weaker than (2.2) is proven. This inequality would be as good as (2.2) for the proof of the isomorphism result. The technique of Ref. [17] is rather different from ours which allows to prove, besides inequality (2.2), the strong inequality (4.8).

The results in case i) have been obtained also by Ornstein (private communication).

In the paper by Dobruschin [17] an inequality slightly weaker than (2.2) is proven also for the case β small and z arbitrary; so using the results of Section 3 it follows that the process (z, β) is a Bernoulli scheme also in this case.

3. Finite Determinability of $\mathscr{P}$

We assume, from now on, that the Markov process (z, β) on K, denoted by μ, verifies (2.2), (2.3) (hence is mixing). For simplicity we shall also fix $v = 2$.

More generally if (K', μ') is a Lebesgue measure space and τ' is a group of measure preserving transformations of K' and if $\mathscr{P}'$ is a partition of K' we shall call the couple $(\mathscr{P}', \tau')$ a process on (K', μ'). Thus a Markov process could be regarded as a process $(\mathscr{P}, \tau)$ on (K, μ).

Definition. A process $(\mathscr{P}, \tau)$ will be called a weak Bernoulli process of exponential type (wbe-process) if there is a function $F(\alpha): R^+ \to R^+$ such that $\lim_{\alpha \to 0^+} F(\alpha) = 0$ and, for any two disjoint regions $\Lambda_1, \Lambda_2 \subset Z^2$ the two partitions

$$\mathscr{Q}_{\Lambda_1} = \bigvee_{\xi \in \Lambda_1} \tau_\xi \mathscr{P}, \qquad \mathscr{Q}_{\Lambda_2} = \bigvee_{\xi \in \Lambda_2} \tau_\xi \mathscr{P}$$

are such that

$$\sum_{q_1 \in \mathscr{Q}_{\Lambda_1}} \sum_{q_2 \in \mathscr{Q}_{\Lambda_2}} |\mu(q_1 \cap q_2) - \mu(q_1)\,\mu(q_2)| < F(\alpha_{\Lambda_1 \Lambda_2}),$$

where

$$\alpha_{\Lambda_1 \Lambda_2} = \left(\min_{i=1,2} |\partial_1 \Lambda_i|\right) e^{-\varkappa d(\Lambda_1 \Lambda_2)},$$

$$d(\Lambda_1, \Lambda_2) = \text{distance of } \Lambda_1 \text{ from } \Lambda_2,$$

$$|\partial_1 \Lambda| = \text{number of points } \xi \text{ in } Z^2 \text{ neighbouring } \Lambda \text{ and } \xi \notin \Lambda.$$

If $(\mathscr{P}', \tau'), (\mathscr{P}'', \tau'')$ are processes on $(K', \mu'), (K'', \mu'')$ respectively we shall consider couples Φ, Ψ of isomorphisms of (K', μ') and (K'', μ'') into the unit interval with Lebesgue measure (X, m) and then define

$$d((\mathscr{P}', \tau'), (\mathscr{P}'', \tau'')) = \sup_{\Lambda} \inf_{\Phi, \Psi} \frac{1}{|\Lambda|} \sum_{\xi \in \Lambda} D(\Phi(\tau'_\xi \mathscr{P}'), \Psi(\tau''_\xi \mathscr{P}'')),$$

where the sup is taken over the finite squares Λ centered at the origin [9] and, if $\mathscr{P}$ and $\mathscr{Q}$ are two partitions, each with n sets, of the same measure space (K, μ), $D(\mathscr{P}, \mathscr{Q}) = \sum_{i=1}^{n} \mu(P_i \triangle Q_i)$.

Let us define a useful family of subsets of Z^2:

a) Λ = finite square $= \{\xi \mid \xi \in Z^2\ a_1 \leqq \xi_1 \leqq b_1, a_2 \leqq \xi_2 \leqq b_2\}$ with a_i, b_i integers; $i = 1, 2$,

b) $\Lambda_n^0 = \{\xi \mid \xi \in Z^2, |\xi_i| \leqq n, i = 1, 2\}$,

c) $\Lambda_n = \{\xi \mid \xi \in Z^2, O \leqq \xi_i \leqq n-1,\ i = 1, 2\}$,

d) if Λ is the set in a) above we put

$$\Lambda^- = \{\xi \mid \xi \in Z^2 \text{ either } \xi_1 < a_1 \text{ or } \xi_1 \leqq b_1 \text{ and } \xi_2 < a_2\},$$

e) $\tilde{\Lambda}_n = \Lambda_n^0 \cap \{0\}^-$,

f) if $\boldsymbol{x} \in Z^2$ and Λ is as in a) above we put

$$\boldsymbol{x} + \Lambda = \{\xi \mid \xi \in Z^2,\ x_i + a_i \leqq \xi_i \leqq x_i + b_i\ \ i = 1, 2\}.$$

If $\mathscr{P}, \mathscr{Q}$ are partitions of the same measure space and a is a set we define:

$$E(\mathscr{P}) = -\sum_{p \in \mathscr{P}} \mu(p) \log \mu(p),$$

$$E(\mathscr{P}/a) = -\sum_{p \in \mathscr{P}} \frac{\mu(p \cap a)}{\mu(a)} \log \frac{\mu(p \cap a)}{\mu(a)},$$

$$E(\mathscr{P}/\mathscr{Q}) = E(\mathscr{P} \vee \mathscr{Q}) - E(\mathscr{Q}) = -\sum_{q \in \mathscr{Q}} \sum_{p \in \mathscr{P}} \mu(p \cap q) \log \frac{\mu(p \cap q)}{\mu(q)}$$

$$= \sum_{q \in \mathscr{Q}} \mu(q)\, E(\mathscr{P}/q).$$

Definition. If $(\mathscr{P}, \tau)$ is a process on (K, μ) we define the entropy $E(\mathscr{P}, \tau)$ as

$$E(\mathscr{P}, \tau) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} E\Big(\bigvee_{\xi\in\Lambda_n} \tau_\xi \mathscr{P}\Big) = \lim_{n\to\infty} E\Big(\mathscr{P} / \bigvee_{\xi\in\tilde{\Lambda}_n} \tau_\xi \mathscr{P}\Big).$$

The limit in the r.h.s. is obtained monotonically (by decreasing). We can now give the following important definition:

Definition. A process $(\mathscr{P}', \tau')$ on (K', μ') is finitely determined if, given $\varepsilon > 0$, one can find $\delta_\varepsilon > 0$ and a finite square Λ^ε such that if a process $(\tilde{\mathscr{P}}, \tilde{\tau})$ on $(\tilde{K}, \tilde{\mu})$ has the properties

i) $$d\Big(\bigvee_{\xi\in\Lambda^\varepsilon} \tau'_\xi \mathscr{P}', \bigvee_{\xi\in\Lambda^\varepsilon} \tilde{\tau}_\xi \tilde{\mathscr{P}}\Big) < \delta_\varepsilon,$$

ii) $$|E(\mathscr{P}', \tau') - E(\tilde{\mathscr{P}}, \tilde{\tau})| < \delta_\varepsilon,$$

then

$$d((\mathscr{P}, \tau), (\mathscr{P}', \tau')) < \varepsilon.$$

We shall prove the following theorem:

Theorem 2. *A (w.b.e.)-process is finitely determined.*

We assume the reader familiar with the paper [3]: from this paper we take the Lemmas 1, 2, 3, 4 and use them here: We only remark that these lemmas are simple general consequences of the definition of ε-independence.

Lemma 5. *Let $\mathscr{P}, \mathscr{Q}$ be partitions of a measure space (K, μ). Given $\varepsilon > 0$ there is a $\delta(\varepsilon)$ such that $E(\mathscr{P}/\mathscr{Q}) > E(\mathscr{P}) - \delta(\varepsilon)$ implies $\mathscr{P} \perp^\varepsilon \mathscr{Q}$.*

Proof. See [11].

Lemma 6. *Let $n > 0$; if $(\mathscr{P}', \tau')$ is a process on (K', μ')*

$$n^2 E(\mathscr{P}', \tau') = \lim_{L\to\infty} E\Big(\bigvee_{\xi\in\Lambda_n} \tau_\xi \mathscr{P}' / \bigvee_{\xi\in\Lambda_n^- \cap \Lambda_L^0} \tau'_\xi \mathscr{P}'\Big) \tag{3.1}$$

and the limit is approached monotonically (by decreasing).

Proof. Call $\mathscr{P}'_n = \bigvee_{\xi\in\Lambda_n} \tau'_\xi \mathscr{P}'$ and observe that, if L is a multiple of n:

$$E(\mathscr{P}', \tau') = \lim_{L\to\infty} \frac{1}{L^2} E\Big(\bigvee_{\xi\in\Delta_L} \tau'_\xi \mathscr{P}'_n\Big),$$

where Δ_L is the set of points in Λ_L with coordinates divisible by n. We order lexicographically the points $\xi \in \Delta_L$ and we can say that

$$E\Big(\bigvee_{\xi\in\Delta_L} \tau'_\xi \mathscr{P}'_n\Big) = \sum_{K=1}^{L^2/n^2}\left(E\Big(\bigvee_{i=1}^{K}\tau'_i\mathscr{P}'_n\Big) - E\Big(\bigvee_{i=1}^{K-1}\tau'_i\mathscr{P}'_n\Big)\right) + E(\mathscr{P}'_n)\,;$$

dividing by L^2 and observing that the bulk terms in the sum tend as $L\to\infty$ to the limit in the r.h.s. of (3.1) we obtain the desired result.

Lemma 7. *Let $\mathscr{P}$ be a partition and let* $\mathscr{P}_1 = \bigvee_{\xi\in\Lambda_L} \tau_\xi\mathscr{P}$,

$$\mathscr{Q} = \bigvee_{\xi\in\Lambda_n^0\cap\Lambda_{\bar L}} \tau_\xi\mathscr{P}\,,\qquad \mathscr{R} = \bigvee_{\xi\in\Lambda_{\bar L}\cap(\Lambda_{n+m}^0-\Lambda_n^0)} \tau_\xi\mathscr{P}\,.$$

Let $1>\varepsilon>0$, *and let* $\delta(\varepsilon)$ *be as in Lemma 5. Suppose*

$$E(\mathscr{P}_1/\mathscr{Q}) \leqq L^2 E(\mathscr{P},\tau) + \varepsilon\delta(\varepsilon)\,. \tag{3.2}$$

Then there exist $\mathscr{Q}_1 \subseteqq \mathscr{Q}$ *such that*

i) $$\mu\Big(\bigcup_{q\in\mathscr{Q}_1} q\Big) > 1-\varepsilon\,,$$

ii) $$\mathscr{P}_1/q \perp^\varepsilon \mathscr{R}/q \quad \textit{if} \quad q\in\mathscr{Q}_1\,.$$

Proof. Lemma 6 implies that

$$L^2 E(\mathscr{P},\tau) \leqq E(\mathscr{P}_1/\mathscr{Q}\vee\mathscr{R}) \leqq E(\mathscr{P}_1/\mathscr{Q})\,,$$

Lemma 5, implies that either

α) $$\mathscr{P}_1/q \perp^\varepsilon \mathscr{R}/q$$

or

β) $$E(\mathscr{P}_1/q/\mathscr{R}/q) < E(\mathscr{P}_1/q) - \delta(\varepsilon)\,.$$

If β holds for $q\in\mathscr{Q}_2$ and $\mu\Big(\bigcup_{q\in\mathscr{Q}_2} q\Big) \geqq \varepsilon$, then

$$\begin{aligned}
&E(\mathscr{P}_1/q) - E(\mathscr{P}_1/\mathscr{Q}\vee\mathscr{R})\\
&\quad= \sum_{q\in\mathscr{Q}} \mu(q)\left(E(\mathscr{P}_1/q) - \sum_{r\in\mathscr{R}} \frac{\mu(q\cap r)}{\mu(q)} E(\mathscr{P}_1/q\cap r)\right)\\
&\quad= \sum_{q\in\mathscr{Q}} \mu(q)\,(E(\mathscr{P}_1|q) - E(\mathscr{P}_1|q|\mathscr{R}|q))\\
&\quad\geqq \sum_{q\in\mathscr{Q}_2} \mu(q)\,\delta(\varepsilon) \geqq \varepsilon\delta(\varepsilon)\,,
\end{aligned}$$

which contradicts the assumption.

Define the function $K(\varepsilon)$ as

$$F\left(16e^{-\frac{\varkappa}{2}K(\varepsilon)}\right)=\varepsilon^2\,. \tag{3.3}$$

Then:

Lemma 8. *Assume $(\mathscr{P},\tau)$ is a w.b.e.-process on (K,μ) and let $1>\varepsilon>0$. Let $H\leqq e^{\frac{\varkappa}{2}K}$ with $K=K(\frac{1}{9}\varepsilon^2)$ (see (3.3)). There exist $n^*,\eta>0$ such that if $(\mathscr{P}',\tau')$ is a process on (K',μ') and*

i) $$d\left(\bigvee_{\xi\in\Lambda_{n^*+H+K}}\tau_\xi\mathscr{P},\ \bigvee_{\xi\in\Lambda_{n^*+H+K}}\tau'_\xi\mathscr{P}'\right)<\eta\,, \tag{3.4}$$

ii) $$|E(\mathscr{P},\tau)-E(\mathscr{P}',\tau')|<\eta\,, \tag{3.5}$$

then, if $\boldsymbol{k}=(K,K)\in Z^2$ and $\boldsymbol{e}=(1,1)\in Z^2$,

$$\bigvee_{\xi\in\boldsymbol{k}+\Lambda_H}\tau'_\xi\mathscr{P}'\perp^{\varepsilon}\bigvee_{\xi\in\Delta}\tau'_\xi\mathscr{P}' \tag{3.6}$$

for all finite subsets $\Delta\subset(\Lambda_{H+K}+\boldsymbol{e})^-$.

Identical results and estimates hold if the role of the axis 1 is interchanged with the role of axis 2.

Proof. Call

$$\mathscr{P}_1=\bigvee_{\xi\in\boldsymbol{e}+\Lambda_{H+K}}\tau_\xi\mathscr{P}\,,\qquad \mathscr{P}_2=\bigvee_{\xi\in(\boldsymbol{k}+\Lambda_H)}\tau_\xi\mathscr{P}\,,$$

$$\mathscr{Q}=\bigvee_{\xi\in\Lambda_n^0\cap(\boldsymbol{e}+\Lambda_{H+K})^-}\tau_\xi\mathscr{P}\,,\qquad \mathscr{R}=\bigvee_{\xi\in(\Lambda_{n+m}^0-\Lambda_n^0)\cap(\boldsymbol{e}+\Lambda_{H+K})^-}\tau_\xi\mathscr{P}\,.$$

The choice of K,H implies that $\mathscr{P}_2\perp^{1/9\,\varepsilon^2}\mathscr{Q}$. Fix $n=n^*$ so large that

$$E(\mathscr{P}_1/\mathscr{Q})\leqq(K+H)^2E(\mathscr{P},\tau)+\varepsilon^2/9\ \ \delta(\varepsilon^2/9)\,. \tag{3.7}$$

By (3.7) and (3.4) we can also guarantee, for η small:

$$E(\mathscr{P}_1'/\mathscr{Q}')<(K+H)^2E(\mathscr{P},\tau)+\varepsilon^2/9\ \ \delta(\varepsilon^2/9)\,,$$

and this, together with (3.5), implies for η small enough:

$$E(\mathscr{P}_1'/\mathscr{Q}')<(K+H)^2E(\mathscr{P}',\tau')+\varepsilon^2/9\ \ \delta(\varepsilon^2/9)\,. \tag{3.8}$$

Choose η so small that i) and $\mathscr{P}_2\perp^{1/9\,\varepsilon^2}\mathscr{Q}$ imply $\mathscr{P}_2'\perp^{1/9\,\varepsilon^2}\mathscr{Q}'$; Lemma 7 and (3.8) imply the existence of $\mathscr{Q}_1'\subseteqq\mathscr{Q}'$ such that

a) $$\mu'\left(\bigcup_{q\in\mathscr{Q}_1'}q\right)>1-\tfrac{1}{9}\varepsilon^2\,,$$

b) $$\mathscr{P}'/q\perp^{\varepsilon^2/9}\mathscr{R}'/q\qquad\forall\,q\in\mathscr{Q}_1'\,.$$

Therefore Lemma 4 applied to $\mathcal{Q}', \mathcal{R}', \mathcal{P}'_2$ implies

$$\mathcal{P}'_2 \perp^{\varepsilon^2/3} \mathcal{Q}' \vee \mathcal{R}',$$

and this holds for all $m \geqq 0$.

If $\mathcal{S}'$ is a partition refined by $\mathcal{Q}' \vee \mathcal{R}'$ we find [1]

$$\sum_{p \in \mathcal{P}'_2} \sum_{s \in \mathcal{S}'} |\mu'(p \cap s) - \mu'(p)\,\mu'(s)| \leqq \sum_{p \in \mathcal{P}'_2} \sum_{r \in \mathcal{Q}' \vee \mathcal{R}'} |\mu'(p \cap r) - \mu'(p)\,\mu'(r)| \leqq \varepsilon^2 .$$

Hence using again the just quoted proposition:

$$\mathcal{P}'_2 \perp^{\varepsilon} \mathcal{S}',$$

and this proves the lemma.

Lemma 9. *Assume $(\mathcal{P}, \tau)$ is a (w.b.e.)-process on (K, μ) and let $1 > \varepsilon > 0$. There exist $n_1 > 0$, $\varepsilon/20 > \eta > 0$ so that if $(\mathcal{P}', \tau')$ is a process on (K', μ') with the properties:*

$$\text{i)} \qquad d\Big(\bigvee_{\xi \in \Lambda_{n1}} \tau_\xi \mathcal{P}, \bigvee_{\xi \in \Lambda_{n1}} \tau'_\xi \mathcal{P}'\Big) < \eta, \tag{3.9}$$

$$\text{ii)} \qquad |E(\mathcal{P}, \tau) - E(\mathcal{P}', \tau')| < \eta, \tag{3.10}$$

then there exist sequences $\{\mathcal{P}'_\xi\}$, $\{\mathcal{P}_\xi\}$, $\xi \in Z^2_+$ of partitions of (X, m) and a positive integer $n_2 > 0$ such that

$$d\Big(\bigvee_{\xi \in \Lambda} \tau_\xi \mathcal{P}\Big) = d\Big(\bigvee_{\xi \in \Lambda} \mathcal{P}_\xi\Big), \qquad \forall\, \Lambda \subset Z^2_+ \qquad \Lambda \supset \Lambda_{n_2}, \tag{3.11}$$

$$d\Big(\bigvee_{\xi \in \Lambda} \tau'_\xi \mathcal{P}'\Big) = d\Big(\bigvee_{\xi \in \Lambda} \mathcal{P}'_\xi\Big), \qquad \forall\, \Lambda \subset Z^2_+, \qquad \Lambda \supset \Lambda_{n_2}, \tag{3.12}$$

$$\sum_{\xi \in \Lambda_n} D(\mathcal{P}_\xi, \mathcal{P}'_\xi) < \varepsilon n^2, \qquad n \geqq n_2 . \tag{3.13}$$

Proof. Let $H = e^{\frac{\alpha}{2} K}$ and let $K \geqq K(1/9(\varepsilon/20)^2)$ (c.f.r. (3.3)) so large that $\frac{K}{K+H} < \frac{\varepsilon}{64}$. Let η, n^* be the numbers provided by Lemma 8 corresponding to $\varepsilon/20$ instead of ε. Let $n_1 = n_* + H + K$.

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[1] We use here the following simple consequence of the definition of ε-independence: If $\mathcal{P}$ and $\mathcal{Q}$ are partitions of (K, μ) and if $\mathcal{P} \perp^{\varepsilon} \mathcal{Q}$, then $\sum_{p \in \mathcal{P}} \sum_{q \in \mathcal{Q}} |\mu(p \cap q) - \mu(p)\,\mu(q)| < 3\varepsilon$. Conversely if $\sum_{p \in \mathcal{P}} \sum_{q \in \mathcal{Q}} |\mu(p \cap q) - \mu(p)\,\mu(q)| < \varepsilon^2$ it follows that $\mathcal{P} \perp^{\varepsilon} \mathcal{Q}$.

The assumptions on $(\mathscr{P},\tau)$ imply $\bigvee_{\xi\in \boldsymbol{k}+\Lambda_H} \tau_\xi \mathscr{P} \perp^{\varepsilon/20} \mathscr{P}$ $\Big($actually they imply $\frac{1}{9}\left(\frac{\varepsilon}{20}\right)^2$-independence rather than $\varepsilon/20$-independence, but this is not needed$\Big)$. Furthermore Lemma 8 implies

$$\bigvee_{\xi\in \boldsymbol{k}+\Lambda_H} \tau'_\xi \mathscr{P}' \perp^{\varepsilon/20} \mathscr{P}' .$$

Let $\mathscr{P}_0, \mathscr{P}'_0$ be partitions of (X, m) such that

$$d(\mathscr{P}_0)=d(\mathscr{P}), \qquad d(\mathscr{P}'_0)=d(\mathscr{P}') .$$

Because of Lemma 3 we can find partitions $\mathscr{P}'_\xi, \mathscr{P}_\xi$ of (X, m) with $\xi\in\Lambda_H+\boldsymbol{k}$ such that

$$d\Big(\mathscr{P}_0 \vee \bigvee_{\xi\in\Lambda_H+\boldsymbol{k}} \mathscr{P}_\xi\Big)=d\Big(\mathscr{P}\vee \bigvee_{\xi\in \boldsymbol{k}+\Lambda_H} \tau_\xi\mathscr{P}\Big),$$
$$d\Big(\mathscr{P}'_0 \vee \bigvee_{\xi\in \boldsymbol{k}+\Lambda_H} \mathscr{P}'_\xi\Big)=d\Big(\mathscr{P}'\vee \bigvee_{\xi\in \boldsymbol{k}+\Lambda_H} \tau'_\xi\mathscr{P}'\Big),$$

and

$$\sum_{\xi\in \boldsymbol{k}+\Lambda_H} D(\mathscr{P}_\xi, \mathscr{P}'_\xi) < 5H^2\frac{\varepsilon}{20}=\frac{\varepsilon}{4}H^2 . \tag{3.14}$$

Define $\mathscr{P}_\xi, \mathscr{P}'_\xi$ for $\xi\in\Lambda_{H+K}$, $\xi\neq 0$ $\xi\notin \boldsymbol{k}+\Lambda_H$ so that (3.11), (3.12) hold for $\Lambda=\Lambda_{H+K}$. Formule (3.14) implies:

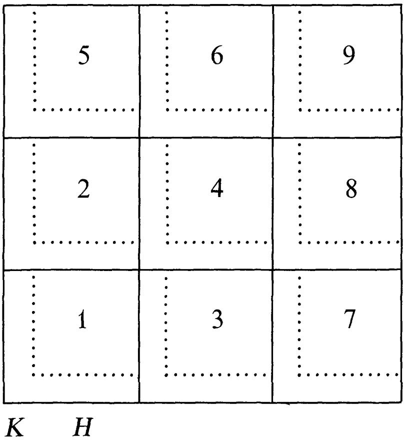
$$\begin{aligned}\sum_{\xi\in\Lambda_{H+K}} D(\mathscr{P}_\xi, \mathscr{P}'_\xi) &\leqq 4K(K+H)+\frac{\varepsilon}{4}H^2\\ &=(K+H)^2\left(\frac{\varepsilon}{4}\,\frac{H^2}{(H+K)^2}+\frac{4K}{K+H}\right)\\ &\leqq \varepsilon(K+H)^2 .\end{aligned} \tag{3.15}$$

Let $n_2=(K+H)$. Cover the set Z^2_+ with a sequence $\{\Lambda(j)\}$ of squares which are translates of Λ_{H+K} and label them as in the picture.

The corridor of width K in $\Lambda(j)$ will be denoted as $\Gamma(j)$. $\Gamma(j)$ is an appropriate translate of $\Lambda_{H+K}-(\Lambda_H+\boldsymbol{k})$. We call $\Delta(j)=\Lambda(j)-\Gamma(j)$, $\Delta(1)=\boldsymbol{k}+\Lambda_H$. We now assume, inductively, that $\mathscr{P}_\xi, \mathscr{P}'_\xi$ have been constructed for $\xi\in\bigcup_{K=1}^{N}\Lambda(K)$ so that (3.11), (3.12) hold for $\Lambda=\bigcup_{K=1}^{N}\Lambda(K)$ and also

$$\sum_{\xi\in\Delta(j)} D(\mathscr{P}_\xi, \mathscr{P}'_\xi) < H^2\ \varepsilon/4 \qquad j=1,2,\dots,N . \tag{3.16}$$

The definition of (w.b.e.)-process and the choice of K imply that



$$\bigvee_{\xi\in\Delta(N+1)} \tau_\xi \mathscr{P} \perp^{\varepsilon/20} \bigvee_{\xi\in\bigcup_{K=1}^{N}\Delta(K)} \tau_\xi \mathscr{P} \tag{3.17}$$

$\left(\text{here too the definitions imply } \frac{1}{9}\left(\frac{\varepsilon}{20}\right)^2\text{-independence}\right)$. Lemma 8 and translation invariance implies

$$\bigvee_{\xi\in\Delta(N+1)} \tau'_\xi \mathscr{P}' \perp^{\varepsilon/20} \bigvee_{\xi\in\bigcup_{K=1}^{N}\Delta(K)} \tau'_\xi \mathscr{P}'. \tag{3.18}$$

Furthermore (3.9) and translation invariance imply

$$d\Bigg(\bigvee_{\xi\in\Delta(N+1)} \tau_\xi \mathscr{P}, \bigvee_{\xi\in\Delta(N+1)} \tau'_\xi \mathscr{P}'\Bigg) < \varepsilon/20. \tag{3.19}$$

We now use Lemma 3 by choosing $\mathscr{P}_0$ and $\mathscr{P}'_0$ as

$$\mathscr{P}_0 = \bigvee_{\xi\in\bigcup_{j=1}^{N}\Delta(j)} \mathscr{P}_\xi, \quad \mathscr{P}'_0 = \bigvee_{\xi\in\bigcup_{j=1}^{N}\Delta(j)} \mathscr{P}'_\xi. \tag{3.20}$$

Then Lemma 3 implies the existence of partitions $\mathscr{P}_\xi, \mathscr{P}'_\xi$ $\xi\in\Delta(N+1)$ such that denoting $(\tilde{\mathscr{P}}, \tilde{\tau})$ either $(\mathscr{P}, \tau)$ or $(\mathscr{P}', \tau')$:

$$d\Bigg(\bigvee_{\xi\in\bigcup_{j=1}^{N}\Delta(j)} \tilde{\mathscr{P}}_\xi \vee \bigvee_{\xi\in\Delta(N+1)} \tilde{\mathscr{P}}_\xi\Bigg) = d\Bigg(\bigvee_{\xi\in\bigcup_{j=1}^{N}\Delta(j)} \tilde{\tau}_\xi \tilde{\mathscr{P}} \vee \bigvee_{\xi\in\Delta(N+1)} \tilde{\tau}_\xi \tilde{\mathscr{P}}_\xi\Bigg), \tag{3.21}$$

and

$$\sum_{\xi \in \Delta(N+1)} D(\mathscr{P}_\xi, \mathscr{P}'_\xi) < H^2 \frac{\varepsilon}{4}. \tag{3.22}$$

Hence (3.22) implies that (3.16) holds for $N+1$.

Furthermore we define $\mathscr{P}_\xi, \mathscr{P}'_\xi$ for $\xi \in \Gamma(N+1)$ so that (3.11), (3.12) hold for $\Lambda = \bigcup_{j=1}^{N+1} \Lambda(j)$.

Finally we check (3.13): let $j(K+H) \leqq n < (j+1)(K+H)$, $j \geqq 1$; then

$$\sum_{\xi \in \Lambda_n} D(\mathscr{P}_\xi, \mathscr{P}'_\xi) = \sum_{s:\Delta(s) \subset \Lambda_n} \sum_{\xi \in \Delta(s)} D(\mathscr{P}_\xi, \mathscr{P}'_\xi) + \sum_{\xi \in \mathscr{B}} (\mathscr{P}_\xi, \mathscr{P}'_\xi),$$

where $\mathscr{B}$ is a boundary strip ($\mathscr{B} = \emptyset$ if n is divided by $(K+H)$). The sum over $\xi \in \mathscr{B}$ can be split in two parts: the part coming from the $\xi \in \Delta(r)$ for some r and the part coming from the $\xi \in \Gamma(r)$ for some r. There are, at most, $4Kn$ ξ's of the second type while the ξ's of the first type can be collected into groups of ξ's belonging to the same $\Delta(r)$: there are at most $\frac{2n}{K+H}$ such groups. Hence since (3.22) implies

$$\sum_{\xi \in \Lambda(s)} D(\mathscr{P}_\xi, \mathscr{P}'_\xi) \leqq 4K(K+H) + \frac{\varepsilon}{4} H^2,$$

$$\sum_{\xi \in \Delta(r)} D(\mathscr{P}_\xi, \mathscr{P}'_\xi) \leqq \frac{\varepsilon}{4} H^2;$$

it follows, remembering that

$$j(H+K) \leqq n < (j+1)(H+K), \quad j \geqq 1,$$

$$\begin{aligned}
\sum_{\xi \in \Lambda_n} D(\mathscr{P}_\xi, \mathscr{P}'_\xi) &\leqq \left(4K(K+H) + \varepsilon \frac{H^2}{4}\right) j^2 + 8Kn + \frac{\varepsilon}{4} H^2 \frac{2n}{K+H} \\
&= j^2 (K+H)^2 \left(\frac{4K}{K+H} + \frac{\varepsilon}{4} \frac{H^2}{(K+H)^2}\right) + \frac{8K}{n} n^2 + \frac{\varepsilon}{4} \frac{2H^2}{n(K+H)} n^2 \\
&\leqq n^2 \left(\frac{4K}{K+H} + \frac{\varepsilon}{4} + \frac{8K}{K+H} + \frac{\varepsilon}{4} \frac{2H^2}{(K+H)^2}\right) \\
&\leqq n^2 \left(\frac{4\varepsilon}{64} + \frac{\varepsilon}{4} + \frac{8\varepsilon}{64} + \frac{\varepsilon}{2}\right) < \varepsilon n^2.
\end{aligned}$$

A corollary of Lemma 9 is Theorem 2 from which one can deduce the fact that $(\mathscr{P}, \tau)$ is a Bernoulli scheme (see [8]): one has to suitably adapt the rest of the proof of [3] using the generalized versions of the one dimensional theorems of Rohlin and McMillan [8].

4. Proof of (2.2), (2.3) when $z \ll 1$

In this section we shall consider the space $\mathscr{I}$ of the functions defined on the set $\hat{Z}_0^\nu$ of the sequences of integers $X = \{n_\xi\}$, $\xi \in Z^\nu$ $n_\xi = 0, 1$ such that $|X| = \sum_\xi n_\xi < \infty$; we think of these sequences as finite subsets of Z^ν with multiply occupied sites. We put $X! = \prod_{\xi \in \tilde{X}} n_\xi!$ and $\tilde{X}$ = set of $\xi \in Z^\nu$ such that $n_\xi > 0$.

If $\lambda(\xi)$ is a function on Z^ν we set

$$\lambda(X) = \prod_{\xi \in \tilde{X}} \lambda(\xi)^{n_\xi}, \qquad X \in \hat{Z}_0^\nu. \tag{4.1}$$

The following Theorems 3 and 4 are known to hold [12]:

Theorem 3. *There exists a function $\varphi^T \in \mathscr{I}$ such that for all finite Λ's in Z^ν*

$$\sum_{X \subset \Lambda} e^{4\beta[X]} z^{|X|} \lambda(X) = \exp \sum_{\tilde{X} \subset \Lambda} \frac{\varphi^T(X)}{X!} z^{|X|} \lambda(X), \tag{4.2}$$

provided $\max_\xi |z\lambda(\xi)| \leqq z_0(\beta)$, where $z_0(\beta)$ is a suitable function of β. Furthermore the function φ^T has the following properties:

i) $\varphi^T(X) = \varphi^T(\xi + X) \quad \forall \xi \in Z^\nu$, *(translation invariance)*,

ii) $\varphi^T(X)$ *is z and λ independent*,

iii)
$$\sum_{\substack{\tilde{X} \cap \Lambda_1 \neq \emptyset \\ \tilde{X} \cap \Lambda_2 \neq \emptyset}} \frac{|\varphi^T(X)|}{X!} z_0^{|X|} \leqq A(\beta)\, e^{-\varkappa(\beta) d(\Lambda_1 \Lambda_2)}. \tag{4.3}$$

A proof of this theorem can be found in [12] (in this paper, however, there is a combinatorial mistake so that several factors of the form $X!$ are missing; for convenience of the reader we give in appendix a short proof of Theorem 3).

Theorem 4. *If $z < z_0(\beta)\, e^{-16|\beta|}$ the functions (1.7) uniquely determine the measure associated with the process (z, β). Furthermore the measures $f_\Lambda(X)$ of the atoms of $\mathscr{P}_\Lambda$ (see (1.9)) are given by*

$$f_\Lambda(X) = \lim_{M \to \infty} \frac{\sum_{T \subset M \setminus \Lambda} e^{4\beta[X \cup T]} z^{|X \cup T|}}{\sum_{Y \subset M} e^{4\beta[Y]} z^{|Y|}}, \qquad X \subset \Lambda, \tag{4.4}$$

where M is a square, concentric with Λ, and with side tending to infinity.

This theorem is a standard result in statistical Mechanics, see [2]. To prove the inequality (2.2) we write

$$f_\Lambda^{(M)}(X) = \frac{\sum\limits_{T \subset M \setminus \Lambda} e^{4\beta[T]} e^{4\beta[X]} z^{|X|} z^{|T|} e^{4\beta i(X|T)}}{\sum\limits_{Y \subset M} e^{4\beta[Y]} z^{|Y|}}; \tag{4.5}$$

hence, introducing the function

$$\lambda_X(\xi) = e^{4\beta i(X|\xi)} \leqq e^{16|\beta|}, \tag{4.6}$$

we can apply Theorem 3 to the regions $M \setminus \Lambda$ and M and, remarking that $\lambda_X(\xi) = 1$, if $\xi \notin \partial_1 \Lambda$ (i.e. if ξ is not adjacent to Λ), we find

$$f_\Lambda^{(M)}(X) = z^{|X|} e^{+4\beta[X]} \cdot \exp\left(- \sum_{\substack{Y \cap \Lambda \neq \emptyset \\ \tilde{Y} \subset M}} \frac{\varphi^T(Y)}{Y!} z^{|Y|} + \sum_{\substack{Y \cap \partial_1 \Lambda \neq \emptyset \\ \tilde{Y} \subset M \setminus \Lambda}} \frac{\varphi^T(Y)}{Y!} z^{|Y|} (\lambda_X(Y) - 1) \right),$$

and, if $z \leqq z_0(\beta) e^{-16|\beta|}$, in view of (4.3), the limit as $M \to \infty$ in this expression does not offer difficulties and one finds:

$$f_\Lambda(X) = z^{|X|} e^{+4\beta[X]} \cdot \exp\left(- \sum_{Y \cap \Lambda \neq \emptyset} \frac{\varphi^T(Y)}{Y!} z^{|Y|} + \sum_{Y \cap \partial_1 \Lambda \neq \emptyset} \frac{\varphi^T(Y)}{Y!} z^{|Y|} (\lambda_X(Y) - 1) \right). \tag{4.7}$$

A straightforward computation and an application of ii), iii) in Theorem 3 together with (4.6) leads to the final estimate:

$$\left| \frac{f_{\Lambda_1 \cup \Lambda_2}(X_1 \cup X_2)}{f_{\Lambda_1}(X_1) f_{\Lambda_2}(X_2)} - 1 \right| \leqq \exp\left(A \min_{i \in 1,2} |\partial_1 \Lambda_i| e^{-\varkappa(\beta) d(\Lambda_1, \Lambda_2)} \right) - 1, \tag{4.8}$$

valid for $\Lambda_1 \cap \Lambda_2 \neq \emptyset$ $X_1 \subset \Lambda_1$ $X_2 \subset \Lambda_2$. It is obvious that (4.8) implies (2.2), (2.3).

5. Proof of (2.2), (2.3) in the Case $z = e^{-8\beta}$, $\beta \gg 1$

Note first of all that the following theorem holds: (cfr. 7)

Theorem 5. *If* $z = e^{-8\beta}$ *and* β *is large enough, there exist only two ergodic and* τ*-invariant measures associated with the process* (z, β)*. They will be denoted* μ_+ *and* μ_-.

Furthermore the measures $f_\Lambda^+(X)$ $(f_\Lambda^-(X))$ of the atoms of $\mathscr{P}_\Lambda$ (see (1.9)) are given by (see (1.6))

$$f_\Lambda^+(X) = \lim_{M\to\infty} f_\Lambda^{+(M)}(X) = \lim_{M\to\infty} \frac{\sum\limits_{\substack{T\subset M\setminus\Lambda \\ T\supset \partial_1^- M}} e^{4\beta[T]} e^{4\beta[X]} z^{|X|} z^{|T|} e^{4\beta i(X|T)}}{\sum\limits_{\substack{Y\subset M \\ Y\supset\partial_1^- M}} e^{4\beta[Y]} z^{|Y|}}, \tag{5.1}$$

where M is a square box with the same center as Λ and such that $\Lambda\subset M$; $\partial_1^- M$ is the inner layer along the boundary ∂M.

Similar relation holds for f_Λ^- which is calculated on an increasing sequence of square boxes $M_1 \subset M_2 \subset$ ecc. using a formula similar to (5.1) in which the sets T, Y are subject to the restriction that $T\cap\partial_1^- M = \emptyset$, $Y\cap\partial_1^- M=\emptyset$.

We give now an alternative way of describing the configurations $X\in K_\Theta$, $X=\{x_1 \ldots x_p\}$. For each point $x\in X$ we construct a unit square with center x. The collection of all such squares forms a region; its boundary is a set of closed connected lines i.e. contours $\gamma_1, \gamma_2 \ldots \gamma_n$ which uniquely determine X.[2] The lines $\gamma_1 \ldots \gamma_n$ are, by construction mutually non intersecting or, as we shall say, compatible. Let $\Theta(\gamma)$ be the set of lattice points inside the outer boundary of γ. Let us write, for a given configuration X, $\Gamma(X)\equiv\{\gamma_1 \ldots \gamma_n\}$ (since we identify X with the collection of the associated contours $\{\gamma_1 \ldots \gamma_n\}$). Among the contours $\{\gamma_1, \gamma_2 \ldots \gamma_n\}$ $\equiv \Gamma(X)$ associated with X we call "outer" those which can be connected to the boundary of Θ by a broken line without crossing other contours.

We can now write (5.1) in terms of contours i.e.

$$f_\Lambda^+(X) = \lim_{M\to\infty} f_\Lambda^{+(M)}(X) = \lim_{M\to\infty} \frac{\sum\limits_{\substack{T\subset M\setminus\Lambda \\ T\supset\partial_1^- M}} e^{-2\beta|\Gamma(X\cup T)|}}{\sum\limits_{\substack{Y\subset M \\ Y\supset\partial_1^- M}} e^{-2\beta|\Gamma(Y)|}}, \tag{5.2}$$

where $|\Gamma(X)| = \sum\limits_{\gamma\in\Gamma(X)} |\gamma|$, and $|\gamma|$ = length of the contour γ. In order to prove (2.2), (2.3), we first give, for this new setting, some technical results similar to the ones in Section 4.

Let $\mathscr{I}$ be the space of the real valued functions defined on the set $\mathscr{C}_0$ of finite collections of contours $\Gamma=\{n_{\gamma_1}\gamma_1, n_{\gamma_2}\gamma_2, \ldots, n_{\gamma_s}\gamma_s\}$, where now $\gamma_1 \ldots \gamma_s$ are allowed to be incompatible and n_{γ_i} is the multiplicity of the contour γ_i.

---

[2] From now on we suppose that one of the two "boundary conditions"

$T\supset\partial_1^- M$, $Y\supset\partial_1^- M$ or $T\cap\partial_1^- M=\emptyset$, $Y\cap\partial_1^- M=\emptyset$ in (5.1) is fixed.

Let $\tilde{\Gamma}$ be the set composed of all distinct elements of Γ, each taken only once. We put $|\Gamma| = \sum_{\gamma\in\tilde{\Gamma}} n_\gamma|\gamma|$, $N(\Gamma) = \sum_{\gamma\in\tilde{\Gamma}} n_\gamma$,

$$\Gamma! = \prod_{\gamma\in\tilde{\Gamma}} n_\gamma!, \quad \text{and} \quad \tilde{\mathscr{I}} = \left\{\psi|\psi\in\mathscr{I}, \sup_{\Gamma:N(\Gamma)=n} |\psi(\Gamma)| < \infty, \forall\, n\right\}.$$

Let now χ be a character function on $\mathscr{C}_0$, i.e. a function on $\mathscr{C}_0$ such that $\chi(\Gamma) = \prod_{\gamma\in\tilde{\Gamma}} \chi(\gamma)^{n_\gamma}$, and define $\varphi\in\tilde{\mathscr{I}}$ by:

$$\varphi(\Gamma) = e^{-2\beta \sum_{\gamma_i\in\Gamma} |\gamma_i|}$$

for $\Gamma! = 1$ and $\Gamma = \{\gamma_1 \dots \gamma_n\}$ = set of compatible contours i.e. $\forall\, \gamma_i, \gamma_j \in \Gamma$ $\theta(\gamma_i)\cap\theta(\gamma_j) = \emptyset$;

$$\varphi(\Gamma) = 0$$

otherwise.

Then the following theorem holds:

Theorem 6. *There exists a function $\varphi^T \in \tilde{\mathscr{I}}$ such that*

$$\sum_\Gamma \varphi(\Gamma)\,\chi(\Gamma) = \exp\sum_\Gamma \varphi^T(\Gamma)\,\chi(\Gamma), \tag{5.3}$$

provided

$$\sum_\Gamma |\varphi(\Gamma)\,\chi(\Gamma)| < \infty.$$

Furthermore the function φ^T has the following properties:

i) $\varphi^T(\Gamma) = \varphi^T(\Gamma+\xi)$, $\quad\forall\, \xi\in Z^2$, *(translation invariance)*,

ii) $\qquad \varphi^T(\Gamma)$ *is independent of* χ,

iii) $$\sum_{\substack{\Gamma\circ p\\ \theta(\Gamma)\cap\Lambda\neq\emptyset}} \frac{|\varphi^T(\Gamma)|}{\Gamma!} \leqq A(\beta)\, e^{-\alpha(\beta)\, d(p,\Lambda)} \quad \textit{for} \quad 3e^{-2\beta} < 1, \tag{5.4}$$

where $\Gamma\circ p$ means that among the contours in Γ there is at least one γ such that $\theta(\gamma)$ encloses the lattice point p.

A proof of this theorem can be found in [15]. The machinery of Appendix 1 transposed to the setting and hypotheses of this section may be taken as a short proof of this result.

We are now able to give the main steps of the proof of the inequalities (2.2), (2.3). We start by observing that the following useful relation holds:

$$f_\Lambda(X) = \sum_{\Gamma\in G_\Lambda} P_\Lambda(\Gamma)\, f_\Lambda(X|\Gamma), \tag{5.5}$$

where

$$G_\Lambda = \{\Gamma | \forall\, \gamma\in\Gamma\ \theta(\gamma)\cap\Lambda\neq\emptyset \ \text{and}\ \forall\, \gamma_i, \gamma_j \in\Gamma\ \theta(\gamma_i)\cap\theta(\gamma_j) = \emptyset\};$$

$P_\Lambda(\Gamma)$ is the probability for finding a configuration such that Γ is the set of outer contours intersecting Λ;

$f(X|\Gamma)$ is the conditional probability "for finding X in Λ given that Γ is the set of outer contours intersecting Λ".

Using the relation (5.5) the l.h.s. of (2.2) can now be written

$$\sum_{X_1\subset\Lambda_1}\sum_{X_2\subset\Lambda_2}\sum_{\Gamma\in G_{\Lambda_1\cup\Lambda_2}}\Big|P_{\Lambda_1\cup\Lambda_2}(\Gamma)\, f_{\Lambda_1\cup\Lambda_2}(X_1\cup X_2|\Gamma) - \sum_{\substack{\Gamma_1\in G_{\Lambda_1}\\ \Gamma_2\in G_{\Lambda_2}}} P_{\Lambda_1}(\Gamma_1)\,P_{\Lambda_2}(\Gamma_2)\, f_{\Lambda_1}(X_1|\Gamma_1)\, f_{\Lambda_2}(X_2|\Gamma_2)\Big|. \tag{5.6}$$

Let G^* be the collection of sets of outer contours extending not too far from $\partial\Lambda_1$ and $\partial\Lambda_2$; more precisely

$$G^*=\{\Gamma_1\cup\Gamma_2|\Gamma_i\in G_{\Lambda_i};\ i\in(1,2)\ \text{such that}\ d(\theta(\Gamma_1),\Lambda_1)\leqq\tfrac{1}{3}d(\Lambda_1,\Lambda_2)\ \text{and}\ d(\theta(\Gamma_2),\Lambda_1)\geqq\tfrac{2}{3}d(\Lambda_1,\Lambda_2)\}.$$

Then we can split (5.6) in Part I and Part II: the terms which involve $\Gamma\in G^*$ yield Part I; those with $\Gamma\notin G^*$ Part II.

Here we give an upper bound to the first one i.e. ($\Gamma_1\cup\Gamma_2\in G^*$). In this case

$$f_{\Lambda_1\cup\Lambda_2}(X_1\cup X_2|\Gamma_1\cup\Gamma_2)=f_{\Lambda_1}(X_1|\Gamma_1)\, f_{\Lambda_2}(X_2|\Gamma_2),$$

so we find

$$|I|\leqq\sum_{X_1\subset\Lambda_1}\sum_{X_2\subset\Lambda_2}\sum_{\Gamma_1\cup\Gamma_2\in G^*} f_{\Lambda_1}(X_1|\Gamma_1)\, f_{\Lambda_2}(X_2|\Gamma_2)$$

$$\cdot|P_{\Lambda_1\cup\Lambda_2}(\Gamma_1\cup\Gamma_2)-P_{\Lambda_1}(\Gamma_1)P_{\Lambda_2}(\Gamma_2)|\leqq\sup_{\Gamma_1\cup\Gamma_2\in G^*}\left|\frac{P_{\Lambda_1\cup\Lambda_2}(\Gamma_1\cup\Gamma_2)}{P_{\Lambda_1}(\Gamma_1)\,P_{\Lambda_2}(\Gamma_2)}-1\right|.$$

Notice that by the definition of $P_\Lambda(\Gamma)$ and the relation (5.3) we have

$$\left|\frac{P_{\Lambda_1\cup\Lambda_2}(\Gamma_1\cup\Gamma_2)}{P_{\Lambda_1}(\Gamma_1)\,P_{\Lambda_2}(\Gamma_2)}-1\right|=\left|\exp\sum_{\substack{\Gamma':\theta(\Gamma')\cap(\theta(\Gamma_1)\cup\Lambda_1)\neq\emptyset\\ :\theta(\Gamma')\cap(\theta(\Gamma_2)\cup\Lambda_2)\neq\emptyset}}\varphi^T(\Gamma')-1\right|. \tag{5.7}$$

Then it holds

$$|I|\leqq\exp\sum_{\substack{\Gamma':\theta(\Gamma')\cap Q_{d/3}\neq\emptyset\\ :\theta(\Gamma')\cap(Q_{2/3\,d})^c\neq\emptyset}}|\varphi^T(\Gamma')|-1,$$

where $Q_\alpha=\{p\in Z^2|d(p,\Lambda_1)\leqq\alpha\}$, and $Q_\alpha^c=$ complement of Q_α in Z^2.

Finally by (5.4) we have

$$|I|\leqq\exp B(\beta)\,|\partial\Lambda_1|\,e^{-\varkappa'(\beta)\,d(\Lambda_1,\Lambda_2)}-1. \tag{5.8}$$

Remark. The argument above can of course be used for Λ_2 instead of Λ_1.

For the Part II (to which contribute big contours i.e. small terms) we find in Appendix 2 the following upper bound:

$$|II| \leqq C(\beta)\, |\partial \Lambda_1|\, e^{-\varkappa''(\beta) d(\Lambda_1, \Lambda_2)}\,. \tag{5.9}$$

The inequalities (5.8), (5.9) together with the previous remark give the inequalities (2.2), (2.3).

6. Concluding Remarks

Notice that (2.2), (2.3) imply that if $\Theta \subset Z^\nu$ is finite and τ_g is a one-dimensional (i.e. with one generator) subgroup of τ the process, (on (K, μ)), $(\mathscr{P}_\Theta, \tau_g)$ is a weak Bernoulli scheme in the sense of [3] and, therefore, it is a Bernoulli scheme, [3].

In particular, if

$$\theta_n = \{\xi \mid |\xi_i| \leqq n \;\; i = 2, 3, \dots \nu;\, \xi_1 = 0\},$$

and $\tau_{(1)} = \{\tau_\xi\}_{\xi_i = 0;\, i = 2, 3, \dots \nu}$ is the subgroup of the translations in the direction 1, the processes $(\mathscr{P}_{\theta_n}, \tau_{(1)})$ are Bernoulli schemes. Since the algebras $\mathscr{A}_n = \bigvee_{-\infty}^{+\infty} \tau^i_{(1)} \mathscr{P}_{\theta_n}$ are $\tau_{(1)}$-invariant and increasing to the algebra $\mathscr{A}$ of all the μ measurable sets we deduce that the process $\left(\tau_g, \bigvee_0^{+\infty} \mathscr{P}_{\theta_n}\right)$ is a generalized Bernoulli shift [13]. It is easy to see that if $E(\mathscr{P}, \tau)$ is the entropy of the process $(\mathscr{P}, \tau)$, then the entropy of $(\mathscr{P}_{\theta_n}, \tau_{(1)})$ is $nE(\mathscr{P}, \tau)$: hence $\left(\tau_g, \bigvee_0^{\infty} \mathscr{P}_{\theta_n}\right)$ has infinite entropy ($E(\mathscr{P}, \tau) \neq 0$ in the cases we are considering).

A particularly interesting process is the process $(\mathscr{P}, \tau_{(1)})$: it seems very interesting to study the properties of this process when (2.2), (2.3) are not valid. It should be noticed that, on physical grounds, it seems to be possible that (2.2), (2.3) are always valid except for a certain critical process (z_c, β_c) (which if $\nu = 2$ is given by $z_c = e^{-8\beta_c}$, $sh 2\beta_c = 1$) corresponding, in the physical language, to an Ising ferromagnet at the critical point.

For the process (z_c, β_c) it is known, that the measure has very long range correlations (roughly decaying as $d^{-1/4}$) [14] and it would not be surprising if, is this case, the system were not a Bernoulli scheme: this situation might be general and the non Bernoullicity of a Gibbs' process could be associated with the critical points and could be used to give an abstract definition of them. If the non Bernoullicity of the Markov processes at the critical point were true one would also have found a number of examples of K-systems[3] which are not Bernoulli schemes but are isomorphic to their inverses.

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Appendix 1

Proof of Theorem 3:

1) $\hat{Z}_0^\nu =$ space of sequences $X = \{n_\xi\}$ $\xi \in Z^\nu$ of non negative integers such that $\sum_\xi n_\xi < \infty$, $X \in \hat{Z}_0^\nu$ is interpreted as a finite subset of $\hat{Z}^\nu$ with multiply occupied sites;

2) if $X \in \hat{Z}_0^\nu$ $\tilde{X} = \{\xi \mid \xi \in Z^\nu, n_\xi \geqq 1\} =$ set of "occupied sites";

3) $X! = \prod_{\xi \in Z^\nu} n_\xi!$;

4) $|X| = \sum_\xi n_\xi$;

5) $\mathscr{I} =$ space of the functions on $\hat{Z}_0^\nu$;

6) $\mathscr{I}_0 = \{f \mid f \in \mathscr{I}, f(\emptyset) = 0\}$;

7) $\mathscr{I}_1 = \{f \mid f \in \mathscr{I}, f(\emptyset) = 1\}$.

8) If $f, g \in \mathscr{I}$ and $\sum_{X_1 \cup X_2 = X}$ denotes the sum over the ordered pairs (X_1, X_2) such that $X_1 \cup X_2 = X$ we define

$$(f \circ g)(X) = \sum_{X_1 \cup X_2 = X} f(X_1)\, g(X_2)\,.$$

9) The function $\mathbf{1}(X) = 0$ if $|X| \geqq 1$ and $\mathbf{1}(\emptyset) = 1$ is the identity for the product in 8.

1) If $\chi \in \mathscr{I}$ is of the form $\chi(X) = \prod_{\xi \in \tilde{X}} \chi(\xi)^{n_\xi}$ (here $0^0 = 0$), so that $\chi(X_1 \cup X_2) = \chi(X_1)\, \chi(X_2)$, then

$$\langle \chi, f \circ g \rangle \equiv \sum_X \frac{\chi(X)\,(f \circ g)(X)}{X!} = \langle \chi, f \rangle \langle \chi, g \rangle\,.$$

[3] It is known that the τ-invariant Markov processes (z, β) are K-systems for $z \neq e^{-8\beta}$ or for $z = e^{-8\beta}$ $\beta \ll \beta_c$. In two dimensions it is also known that they are K-systems if they are ergodic and $\beta \gg \beta_c$ (see: R.L. Dobruschin in [5]) the results of this paper obviously imply that in any dimension the τ-invariant ergodic processes $(\beta, e^{-8\beta})$ with $\beta \gg \beta_c$ are K-systems. The situation is unclear in the remaining region ($z = e^{-8\beta}$ $\beta > \beta_c$ but not very large).

11) If $f \in \mathscr{I}_0$, we define:

$$(\mathrm{Exp} f)(X) = \sum_{n=0}^{\infty} \frac{1}{n!} f^n(X),$$

(where $f^0 = \mathbf{1}$, and f^n is understood in the sense of product 8))

$$(\mathrm{Log}(\mathbf{1} + f))(X) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} f^n(X),$$

$$(\mathbf{1} + f)^{-1}(X) = \sum_{n=0}^{\infty} (-1)^n f^n(X);$$

12) $(D_X f)(Y) = f(X \cup Y)$.

13) Define

$$\varphi(X) = \begin{cases} 0 & \text{if } X! > 1 \quad (\text{i.e. if } \tilde{X} \neq X) \\ e^{+4\beta[X]} & \text{if } X! = 1 \end{cases}$$

$$\chi_\Lambda^z(X) = \prod_{\xi \in \tilde{X}} z^{n_\xi} \chi_\Lambda(\xi)^{n_\xi},$$

where $\chi_\Lambda(\xi)$ is the characteristic function of Λ. It makes sense to consider $\mathrm{Log}\,\varphi$ and φ^{-1} because $\varphi \in \mathscr{I}_1$ (i.e. $\varphi = \mathbf{1} + \varphi'$, $\varphi' \in \mathscr{I}_0$).

14) Define

$$\varphi^T = \log \varphi \qquad (\text{hence } \mathrm{Exp}\,\varphi^T = \varphi)$$

and suppose that

$$\sum_{\tilde{X} \subset \Lambda} \frac{|\varphi^T(X)|}{X!} z^{|X|} < +\infty.$$

15) Then

$$\sum_{X \subset \Lambda} z^{|X|} \varphi(X) = \langle \chi_\Lambda^z, \varphi \rangle = \exp \langle \chi_\Lambda^z, \varphi^T \rangle$$

$$= \exp \sum_{\tilde{X} \subset \Lambda} \frac{\varphi^T(X)}{X!} z^{|X|};$$

16) To show the convergence of the series in (14) consider the functions

$$\Delta_X(Y) = (\varphi^{-1} \cdot D_X \varphi)(Y),$$

and remark that $\varphi^T(X \cup \xi) = \Delta_\xi(X)$ (this follows from the rule $D_\xi \,\mathrm{Exp}\,\varphi^T = (D_\xi \varphi^T) \cdot \mathrm{Exp}\,\varphi^T$).

The above functions verify some equations which can be derived as follows (remember that $\varphi(X) \neq 0$ only if $X! = 1$, see 13)): Put $X^{(1)} = X \setminus \{x_1\}$ and $K(x_1, T) = \prod_{\xi \in T} (e^{4\beta i(x_1|\xi)} - 1)$;

$$\begin{aligned}
\frac{\Delta_X(Y)}{Y!} &= \sum_{X_1 \cup X_2 = Y} \frac{\varphi^{-1}(X_1)}{X_1!} \frac{\varphi(X \cup X_2)}{X_2!} \\
&= \sum_{\substack{X_1 \cup X_2 = Y \\ X_2! = 1}} \frac{\varphi^{-1}(X_1)}{X_1!} z^{|X| + |X_2|} e^{4\beta [X \cup X_2]} \\
&= \sum_{\substack{X_1 \cup X_2 = Y \\ X_2! = 1}} \frac{\varphi^{-1}(X_1)}{X_1!} z z^{|X^{(1)}| + |X_2|} e^{4\beta [X^{(1)} \cup X_2]} e^{4\beta i(X_1 | X^{(1)})} e^{4\beta i(X_1 | X_2)} \\
&= z e^{4\beta i(x_1 | X^{(1)})} \sum_{\substack{X_1 \cup X_2 = Y \\ X_2 \cap X = \emptyset}} \frac{\varphi^{-1}(X_1)}{X_1!} \varphi(X^{(1)} \cup X_2) \prod_{\xi \in X_2} e^{4\beta i(X_1 | \xi)} \\
&= z e^{4\beta i(x_1 | X^{(1)})} \sum_{\substack{X_1 \cup X_2 = Y \\ X_2 \cap X = \emptyset}} \frac{\varphi^{-1}(X_1)}{X_1!} \varphi(X^{(1)} \cup X_2) \prod_{\xi \in X_2} (e^{4\beta i(X_1 | \xi)} - 1 + 1) \\
&= z e^{4\beta i(x_1 | X^{(1)})} \sum_{\substack{X_1 \cup X_2 = Y \\ X_2 \cap X = \emptyset}} \frac{\varphi^{-1}(X_1)}{X_1!} \varphi(X^{(1)} \cup X_2) \sum_{T \subset X_2} K(x_1, T) \\
&= z e^{4\beta i(x_1 | X^{(1)})} \sum_{\substack{T \subset \tilde{Y} \\ T \cap X = \emptyset}} K(x_1, T) \sum_{\substack{X_1 \cap X_2 = Y \\ X_2 \cap X = \emptyset \\ X_2 \supset T}} \frac{\varphi^{-1}(X_1)}{X_1!} \varphi(X^{(1)} \cup X_2) \\
&= z e^{4\beta i(x_1 | X^{(1)})} \sum_{\substack{T \subset \tilde{Y} \\ T \cap X = \emptyset}} K(x_1, T) \sum_{\substack{X_1 \cup s = Y \setminus T \\ s \cap (X \cup T) = \emptyset}} \frac{\varphi^{-1}(X_1)}{X_1!} \varphi(X^{(1)} \cup S \cup T) \\
&= z e^{4\beta i(x_1 | X^{(1)})} \sum_{\substack{T \subset \tilde{Y} \\ T \cap X = \emptyset}} K(x_1, T) \left(\frac{\Delta_{X^{(1)} \cup s}(Y \setminus T)}{(Y \setminus T)!} - \frac{\Delta_{X \cup s}(Y \setminus T \cup X_1)}{(Y \setminus T \cup X_1)!} \right).
\end{aligned}$$

17) Define

$$I_n = \sup_{\substack{h+k=n \\ h \geq 1 \\ k \geq 0}} \sup_{\substack{X \in \hat{Z}_0^\nu \\ |X| = h}} \sum_{\substack{Y \in Z_0^\nu \\ |Y| = k}} \frac{|\Delta_X(Y)|}{Y!}.$$

Then the Eq. (16) imply, recursively,

$$I_{n+1} \leq z c(\beta) I_n \quad \text{with} \quad c(\beta) = (\exp(e^{8\nu\beta} - 1) - 1).$$

Hence

$$I_{n+1} \leq z (z c(\beta))^n \quad \text{and} \quad z_0(\beta) = \frac{1}{c(\beta)}.$$

For more details see [6].

Appendix 2

Here we give an upper bound for

$$\sum_{X_1 \subset \Lambda_1} \sum_{X_2 \subset \Lambda_2} \Bigg| \sum_{\Gamma \in G_{\Lambda_1 \cup \Lambda_2} \setminus G^*} P_{\Lambda_1 \cup \Lambda_2}(\Gamma)\, f_{\Lambda_1 \cup \Lambda_2}(X_1 \cup X_2 | \Gamma) - \sum_{(\Gamma_1, \Gamma_2) \in G_{\Lambda_1} \times G_{\Lambda_2} \setminus G^*} P_{\Lambda_1}(\Gamma_1)\, P_{\Lambda_2}(\Gamma_2)\, f_{\Lambda_1}(X_1 | \Gamma_1)\, f_{\Lambda_2}(X_2 | \Gamma_2) \Bigg| ,$$

i.e. for the terms which involves contours which do not belong to G^* (Part II of the text of Section 5).

Let us begin with some improvement on the definition of the sets of contours which contribute to Part II.

We put

$$\bar{G}_1 = \{\Gamma \in G_{\Lambda_1} | \theta(\Gamma) \cap (Q_{d/3})^c \neq \emptyset\} ; \qquad \bar{G}_2 = \{\Gamma \in G_{\Lambda_2} | \theta(\Gamma) \cap (Q_{2d/3}) \neq \emptyset\} ;$$
$$\bar{G} = \{\Gamma_1 \cup \Gamma_2 | \exists\, i \in (1,2) : \Gamma_i \in \bar{G}_i\} ; \qquad \bar{\bar{G}} = \{(\Gamma_1, \Gamma_2) | \exists\, i \in (1,2) : \Gamma_i \in \bar{G}_i\} ;$$

It is worthwhile to observe that the first and the second element of the ordered pair in $\bar{\bar{G}}$ are mutually incompatible. Moreover 2^n elements of $\bar{\bar{G}}$ correspond to each element in $\bar{G}$ which contains n contours $\gamma_1, \gamma_2 \ldots \gamma_n$ such that $\forall_i\ \theta(\gamma_i) \cap \Lambda_1 \neq \emptyset\ \theta(\gamma_i) \cap \Lambda_2 \neq \emptyset$.

Next we note that $\forall\, \Gamma, \sum_{X \subset \Lambda} f_\Lambda(X | \Gamma) = 1$; so we can write

$$\begin{aligned} |II| &\leqq \sum_{\Gamma \in \bar{G}} P_{\Lambda_1 \cup \Lambda_2}(\Gamma) + \sum_{(\Gamma_1, \Gamma_2) \in \bar{\bar{G}}} P_{\Lambda_1}(\Gamma_1)\, P_{\Lambda_2}(\Gamma_2) \\ &\leqq 2 \sum_{\Gamma \in \bar{G}_1} P_{\Lambda_1}(\Gamma) + 2 \sum_{\Gamma \in \bar{G}_2} P_{\Lambda_2}(\Gamma_2) . \end{aligned} \tag{AII.1}$$

The first term of the r.h.s. of (AII.1) is twice the probability for finding an outer contour γ such that

$$\theta(\gamma) \cap \Lambda_1 \neq \emptyset \quad \text{and} \quad \theta(\gamma) \cap (Q_{d/3})^c \neq \emptyset .$$

The second term of r.h.s. of (AII.1) can be interpreted similarly.

We put $\Pi(\gamma)$ = probability for finding the outer contour γ.

It is known (cfr. [16]) that

$$\Pi(\gamma) \leqq e^{-2\beta|\gamma|} . \tag{AII.2}$$

In order to perform an explicit calculation of the right-hand side of (AII.1) let p be a point on $\partial \Lambda_1$ and r an outward straight line starting at p such that $\forall\, q \in r$ the following relation holds:

$$d(q, p) \leqq d(q, \partial \Lambda_1) ;$$

let also L be the segment of r enclosed by $\partial \Lambda_1$ and $\partial Q_{d/3}$.

It is now easy to verify that the following relation holds:

$$\begin{aligned}\sum_{\Gamma\in\bar{G}_1} P_{\Lambda_1}(\Gamma) \leqq \sum_{q\in\partial\Lambda_1} \sum_{\substack{\gamma\ni q\\ |\gamma|\geqq\frac{2}{3}d}} \Pi(\gamma) + \sum_{\substack{q\in L\\ d(q,\partial\Lambda_1)<\frac{d}{6}}} \sum_{\substack{\gamma\ni q\\ |\gamma|>\frac{d}{3}}} \pi(\gamma)\\ + \sum_{\substack{q\in L\\ d(q,\partial\Lambda_1)\geqq\frac{d}{6}}} \sum_{\substack{\gamma\ni q\\ |\gamma|>\frac{d}{3}}} \Pi(\gamma) + \sum_{\substack{q\in r\\ q\notin L}} \sum_{\substack{\gamma\ni q\\ |\gamma|>2d(q,\partial\Lambda_1)}} \Pi(\gamma).\end{aligned} \tag{AII.3}$$

The inequality (AII.2) together with the fact that the number of contours of length l is less then 3^l, give

$$\text{r.h.s. of (AII.3)} \leqq (2|\partial\Lambda_1| + \tfrac{8}{3})\, e^{-2\varkappa d/3} + \tfrac{2}{3} d e^{-\varkappa d/3} \tag{AII.4}$$

where $\varkappa = 2\beta - \log 3$, $d \equiv d(\Lambda_1, \Lambda_2)$ and β, for sake of semplicity, is chosen such that $3e^{-2\beta} < \frac{1}{2}$. In the same way we derive for the second term of (AII.1)

$$\sum_{\Gamma_2\in\bar{G}_2} P_{\Lambda_2}(\Gamma) \leqq (2|\partial\Lambda_1| + 16/3\, d + 4)\, e^{-2\varkappa d/3} + \tfrac{2}{3} d e^{-\varkappa d/3}\,. \tag{AII.5}$$

Collecting together (AII.4) and (AII.5) we obtain the following upper bound:

$$\text{r.h.s. of (AII.1)} \leqq c|\partial\Lambda_1|\, e^{-\varkappa d/4}\,.$$

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F. di Liberto
Istituto di Fisica Teorica dell'Università
Mostra d'Oltremare
Pad. 19
I-80125 Napoli, Italy