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Finite Dimensional Representations of the $SU(2)$-Current Lie Algebra

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Abstract. We give a complete classification of the finite dimensional solutions for the Lie functional equations of $SU(2)$.

I. Introduction

In 1966 Dashen and Gellman posed the following [1] problem which originated from their study of current algebras;

"Let L be a Lie algebra characterized by the CR

$$[J_i,J_j] = h_{ij}^k J_k \tag{1.1}$$

classify all the solutions of the functional equations

$$[J_i(x), J_j(y)] = h_{ij}^k J_k(x+y) \qquad x, y \in R^{n''} . \tag{1.2}$$

Several authors [2] attempted to solve this problem with the additional (physical) requirement of Lorentz covariance on the operators $\boldsymbol{J}(x) = (J_1(x), J_2(x), J_3(x))$. However, with this additional requirement the problem acquires a new dimension of complexity. This is due to the fact that the unitary representations of the Lorentz group are of infinite dimension and hence the nature of the underlying Hilbert space may give rise to complicated problems.

In this paper we analyze the system (1.2) for the $SU(2)$ case [i.e. when the original commutation relations (CR) (1.1) are those of $SU(2)$] and give a complete classification of their finite dimensional solutions without imposing the additional condition of Lorentz covariance.

We note that within the context of the original physical problem which gave rise to the system (1.2) this classification might be considered unsatisfactory as the Lorentz invariance of the required solution is essential. Nevertheless, we feel that our solution is a useful first step toward the solution of the more complicated problem.

We add that our work is interesting also from the mathematical point of view as the solutions to (1.2) form a natural generalization of the representation problem for Lie algebra.

Our first step toward the solution is to observe that $J_i(0)$ must satisfy the CR of the original Lie algebra and, therefore, must form a representation of it.

Having made this observation, we approach the general problem in three steps. In Section II we find all the solutions of (1.2) when $\boldsymbol{J}(0)$ form an irreducible representation of $SU(2)$ (in its standard form). We remark that the results of this section are not new [3] but we believe that the method of their derivation is new and paves the way to the treatment of the reducible case. In Section III we classify all the solutions when $\boldsymbol{J}(0)$ is a finite dimensional reducible representations of $SU(2)$ in the form

$$\boldsymbol{J}(0) = \sum_{\tau, l} \oplus D_l^{\tau}$$

where l is the highest weight of the representation and τ is a degeneracy index in case we have to deal with several representations with the same l. In relation to the material of this section, we note that Joseph [4] attempted a similar classification with the additional requirement that $J_i(x)$ be in the form

$$J_i(x) = \sum_{j=1}^{n} a_{ij}(x) H_j \qquad n < \infty$$

where H_j are matrices of constant entries. In this paper, however, no apriori assumption on the form of the solutions is made and closed exact expressions for the matrix elements of the operators $\boldsymbol{J}(x)$ are derived. Finally in Section IV we deal with the general finite dimensional case i.e. when $\boldsymbol{J}(0)$ is any finite dimensional representation of $SU(2)$ not necessarily in its standard form.

Before closing this section, we make the following two remarks:

1. Our assumption that $\boldsymbol{J}(0)$ is finite dimensional implies that l, τ have a finite and discrete range. Moreover, this assumption means that $\boldsymbol{J}(x)$ are finite dimensional matrix operators and hence all functional analysis complications that appear when the range of these indices is infinite (this happens naturally when we impose the Lorentz covariance condition on the solutions) do not arise.

2. The proof of Theorem 2 depends strongly on the results given in pp. 118–125 of Ref. [5]. It is recommended that the reader will familiarize himself with this material before the reading of Section III.

II. The Irreducible Case

The basic $SU(2)$ nonzero commutation relations (CR) in the raising lowering basis are

$$\begin{gathered}[J_+, J_3] = -J_+, [J_-, J_3] = J_- \\ [J_+, J_-] = 2J_3 \,; \end{gathered} \tag{2.1}$$

these relations lead to the functional equations

$$[J_+(x), J_3(y)] = -J_+(x+y), [J_-(x), J_3(y)] = J_-(x+y)$$
$$[J_+(x), J_-(y)] = 2J_3(x+y) \tag{2.2}$$

and

$$[J_+(x), J_+(y)] = [J_-(x), J_-(y)] = [J_3(x), J_3(y)] = 0 . \tag{2.3}$$

If we assume that $J_+(0), J_-(0), J_3(0)$ form an irreducible representation of $SU(2)$ of dimension $(2l+1)$ in its standard form [5], we then have that

$$(J_3(0))_{ij} = i\delta_{ij} \qquad (J_+(0))_{ij} = \alpha_i \delta_{i,j+1}$$
$$(J_-(0))_{ij} = \alpha_{i+1}\delta_{i,j-1} \tag{2.4}$$
$$i, j = l, \ldots, -l .$$

where

$$\alpha_i = \sqrt{(l+i)(l-i+1)}$$

(note the nonstandard indexing of the rows and columns).

We shall solve the functional equations (2) which involve $3(2l+1)^2$ functions by reducing the problem in several steps to a scalar functional equation

Lemma 1. $$(J_3(x))_{ij} = A_i(x)\,\delta_{ij} . \tag{2.5}$$

Proof. If we denote the matrix elements of $J_3(x)$ by $a_{ij}(x)$ then the relation

$$[J_3(x), J_3(0)] = 0$$

implies that

$$(j-i)\,a_{ij}(x) = 0$$

and therefore

$$a_{ij}(x) = A_i(x)\,\delta_{ij} .$$

Lemma 2. $$(J_+(x))_{ij} = B_i(x)\,\delta_{i,j+1}$$
$$(J_-(x))_{ij} = C_i(x)\,\delta_{i,j-1} . \tag{2.6}$$

Proof. Let us denote the matrix elements of $J_+(x), J_-(x)$ by $b_{ij}(x)$ and $c_{ij}(x)$ respectively, the relations

$$[J_3(0), J_+(x)] = J_+(x)$$
$$[J_-(x), J_3(0)] = J_-(x)$$

then imply that

$$(i-j-1)\,b_{ij}(x) = 0$$
$$(j-1-i)\,c_{ij}(x) = 0$$

and therefore

$$b_{ij}(x) = B_i(x)\,\delta_{i,j+1}$$
$$c_{ij}(x) = C_i(x)\,\delta_{i,j-1} .$$

Now we want to relate $B_i(x), C_i(x)$ to $A_i(x)$.

Lemma 3.
$$B_i(x) = \alpha_i(A_i(x) - A_{i-1}(x))$$
$$C_i(x) = \alpha_{i+1}(A_{i+1} - A_i(x)) . \tag{2.7}$$

Proof. We use the CR

$$[J_3(x), J_+(0)] = J_+(x)$$
$$[J_-(0), J_3(x)] = J_-(x)$$

to obtain the desired formulas.

Now we relate the A_i to each other

Lemma 4. *The $A_i(x)$ satisfy the following recursive relation*

$$\alpha_i^2 A_{i-1}(x) + \alpha_{i+1}^2 A_{i+1}(x) = (\alpha_i^2 + \alpha_{i+1}^2 - 2) A_i(x) . \tag{2.8}$$

Proof. We simply use the CR

$$[J_+(x), J_-(0)] = 2J_3(x)$$

and (2.7) to obtain this formula.

From (2.8) we see that it is enough to fix $A_l(x)$

Lemma 5.
$$A_l(x) = le^{\alpha \cdot x} \qquad \alpha \in R^n . \tag{2.9}$$

Proof. We use the CR

$$[J_+(x), J_-(y)] = 2J_3(x+y)$$

to find that

$$2A_i(x+y) = (B_i(x)\, C_{i-1}(y) - C_i(x)\, B_{i+1}(x))$$

which for $i = l$ gives that

$$2A_l(x+y) = \alpha_l^2 (A_l(x) - A_{l-1}(x))\,(A_l(y) - A_{l-1}(y))$$

but by (2.8)

$$\alpha_l^2 A_{l-1}(x) = (\alpha_l^2 - 2)\, A_l(x)$$

and therefore $A_l(x)$ must satisfy the functional equation

$$A_l(x)\, A_l(y) = l A_l(x+y)$$

whose solutions of the form

$$A_l(x) = le^{\alpha \cdot x} , \qquad \alpha \in R^n .$$

Theorem 1. *The solutions to functional equations* (2.2)–(2.3) *when* $\boldsymbol{J}(0)$ *is an irreducible finite dimensional representation of* $SU(2)$ *in its standard form are in the form*

$$\boldsymbol{J}(x) = e^{\alpha \cdot x} \boldsymbol{J}(0) .$$

Proof. We proved $A_l(x) = le^{\alpha \cdot x}$. We now use the relations (2.8) to show that $J_3(x) = e^{\alpha \cdot x} J_3(0)$ and the relations (2.7) to give the same for $J_+(x), J_-(x)$.

III. The Reducible Case

We consider now the case when $\boldsymbol{J}(0)$ is a direct sum of irrdeucible representations in their standard form i.e.

$$\boldsymbol{J}(0) = \sum_{l,\tau} \oplus \boldsymbol{D}_l^\tau$$

where the matrix elements of $\boldsymbol{D}_l^\tau$ are given by (2.4).

Theorem 2. *The general form of* $J_3(x), J_+(x), J_-(x)$ *is*

$$\begin{aligned}(J_3(x))_{l'lm'm} = {} & c_{l-1,l}^{\tau'\tau}(x) \sqrt{l^2 - m^2}\, \delta_{mm'} \delta_{l,l'+1} \\ & + c_{ll}^{\tau'\tau}(x)\, m \delta_{mm'} \delta_{ll'} + c_{l+1,l}^{\tau'\tau}(x) \sqrt{(l+1)^2 - m^2}\, \delta_{mm'} \delta_{l,l'-1}\,,\end{aligned} \tag{3.1}$$

$$\begin{aligned}(J_+(x))_{l'l,m'm} = {} & c_{l-1,l}^{\tau'\tau}(x) \sqrt{(l-m)(l-m-1)}\, \delta_{l'l-1} \delta_{m',m+1} \\ & + c_{ll}^{\tau'\tau}(x) \sqrt{(l-m)(l+m+1)}\, \delta_{ll'} \delta_{m',m+1} \\ & - c_{l+1,l}^{\tau'\tau}(x) \sqrt{(l+m+1)(l+m+2)} \cdot \delta_{l',l+1} \delta_{m,m+1}\,.\end{aligned} \tag{3.2}$$

$$\begin{aligned}(J_-(x))_{l'l,m',m} = {} & -c_{l-1,l}^{\tau'\tau}(x) \sqrt{(l+m)(l+m-1)}\, \delta_{l',l-1} \delta_{m',m-1} \\ & + c_{ll}^{\tau'\tau}(x) \sqrt{(l+m)(l-m-1)}\, \delta_{ll'} \delta_{m',m-1} \\ & + c_{l+1,l}^{\tau'\tau}(x) \sqrt{(l-m+1)(l-m+2)}\, \delta_{l',l+1} \delta_{m',m-1}\,.\end{aligned} \tag{3.3}$$

Proof. For fixed x the following CR must hold

$$\begin{gathered}[J_3(x), J_3(0)] = 0\,, \quad [J_3(x), J_+(0)] = J_+(x) \\ [J_3(x), J_-(0)] = J_-(x)\,,\end{gathered} \tag{3.4}$$

$$\begin{gathered}[J_+(x), J_3(0)] = -J_+(x)\,, \quad [J_+(x), J_+(0)] = 0 \\ [J_+(x), J_-(0)] = 2J_3(x)\,,\end{gathered} \tag{3.5}$$

$$\begin{gathered}[J_-(x), J_3(0)] = J_-(x)\,, \quad [J_-(x), J_+(0)] = -2J_3(x) \\ [J_-(x), J_-(0)] = 0\,.\end{gathered} \tag{3.6}$$

It is now important to realize that for any fixed x these CR are exactly the same CR investigated by Gelfand [3], p. 118–125. Thus the general form of the operators $J_+(x), J_-(x), J_3(x)$ must be exactly the same as those for L_3, L_+, L_- found by him. In our case, however, the constants $c_{ll}^{\tau'\tau}, c_{l,l+1}^{\tau'\tau}, c_{l-1,l}^{\tau'\tau\tau}$ will depend on x and hence the formulas (3.1)–(3.3) (we use the notations introduced by Gelfand).

Theorem 3. *The functions* $c^{\tau'\tau}_{l,l+1}(x), c^{\tau'\tau}_{l-1,l}(x), c^{\tau'\tau}_{ll}(x)$ *must satisfy the following functional equations*

$$-c^{\tau'\tau}_{l+2,l+1}(x)\,c^{\tau'\tau}_{l+1,l}(y)+c^{\tau'\tau}_{l+2,l+1}(y)\,c^{\tau'\tau}_{l+1,l}(x)=0\,, \tag{3.7}$$

$$c^{\tau'\tau}_{l-2,l-1}(y)\,c^{\tau'\tau}_{l-1,l}(x)-c^{\tau'\tau}_{l-2,l-1}(x)\,c^{\tau'\tau}_{l-1,l}(y)=0\,, \tag{3.8}$$

$$\begin{aligned}&c^{\tau'\tau}_{l+1,l}(y)\,[c^{\tau'\tau}_{l+1,l+1}(x)\,(l-m+2)-c^{\tau'\tau}_{ll}(x)\,(l-m)]\\&\quad+c^{\tau'\tau}_{l+1,l}(x)\,[c^{\tau'\tau}_{l+1,l+1}(x)\,(l+m+2)-c^{\tau'\tau}_{ll}(x)\,(l+m)]\\&\quad=2c^{\tau'\tau}_{l+1,l}(x+y)\,,\end{aligned} \tag{3.9}$$

$$\begin{aligned}&c^{\tau'\tau}_{l-1,l}(y)\,[c^{\tau'\tau}_{ll}(x)\,(l+m+1)-c^{\tau'\tau}_{l-1,l-1}(x)\,(l+m-1)]\\&\quad+c^{\tau'\tau}_{l-1,l}(x)\,[c^{\tau'\tau}_{ll}(y)\,(l-m+1)-c^{\tau'\tau}_{l-1,l-1}(y)\,(l-m-1)]\\&\quad=2c^{\tau'\tau}_{l-1,l}(x+y)\,,\end{aligned} \tag{3.10}$$

$$\begin{aligned}2m(c^{\tau'\tau}_{ll}(x+y)-c^{\tau'\tau}_{ll}(x)\,c^{\tau',\tau}_{ll}(y))&=c^{\tau'\tau}_{l-1,l}(y)\,c^{\tau',\tau}_{l,l-1}(x)\,(l+m-1)\,(l+m)\\&\quad-c^{\tau'\tau}_{l,l+1}(y)\,c^{\tau'\tau}_{l+1,l}(x)\,(l+m+1)\,(l+m+2)\\&\quad+c^{\tau'\tau}_{l,l+1}(x)\,c^{\tau'\tau}_{l+1,l}(y)\,(l-m+2)\\&\quad\cdot(l-m+1)-c^{\tau'\tau}_{l-1,l}(x)\,c^{\tau',\tau}_{l,l-1}(y)\,(l-m)\,(l-m-1)\,.\end{aligned} \tag{3.11}$$

Proof. We use the relation

$$[J_+(x), J_-(y)]=2J_3(x+y)$$

and formulas (3.1)–(3.3) to obtain these functional equations.

To analyze these relations we note that (3.8), (3.7) imply that

$$\begin{aligned}\frac{c^{\tau'\tau}_{l+2,l+1}(x)}{c^{\tau'\tau}_{l+1,l}(x)}&=\frac{c^{\tau'\tau}_{l+2,l+1}(y)}{c^{\tau'\tau}_{l+1,l}(y)}\\\frac{c^{\tau'\tau}_{l-2,l-1}(x)}{c^{\tau'\tau}_{l-1,l}(x)}&=\frac{c^{\tau'\tau}_{l-2,l-1}(y)}{c^{\tau'\tau}_{l-1,l}(y)}\end{aligned} \tag{3.12}$$

for all x, y. Moreover, since l is dummy, we can rewrite these relations in the form

$$\frac{c^{\tau'\tau}_{l+1,l}(x)}{c^{\tau'\tau}_{l,l-1}(x)}=\frac{1}{A_l}\,,\quad\frac{c^{\tau'\tau}_{l-1,l}(x)}{c^{\tau'\tau}_{l,l+1}(x)}=B_l\,. \tag{3.13}$$

To proceed, we observe that $c_{ll}(x)$, etc. are independent of m and therefore the functional equations (3.10), (3.9) must also be independent of m. This implies that either $c^{\tau'\tau}_{l+1,l}(x)=c^{\tau'\tau}_{l-1,l}(x)=0$ or

$$c^{\tau'\tau}_{ll}(x)=c^{\tau'\tau}_{l+1,l+1}(x)\,. \tag{3.14}$$

The first possibility is trivial as the general solution is then simply $\Sigma\oplus e^{\alpha_i x}\boldsymbol{D}_i$ where α_i are independent. [This can be either from Theorem I or the relation (3.11).] We proceed therefore to investigate the other possibility. In this case (3.9)–(3.11) reduce to [using (3.13)].

$$c^{\tau'\tau}_{l+1,l}(y)\,c^{\tau'\tau}_{ll}(x)+c^{\tau'\tau}_{l+1,l}(x)\,c^{\tau'\tau}_{ll}(y)=c^{\tau'\tau}_{l+1,l}(x+y)\,, \tag{3.15}$$

$$c^{\tau'\tau}_{l-1,l}(y)\,c^{\tau'\tau}_{ll}(x)+c^{\tau'\tau}_{l-1,l}(x)\,c^{\tau'\tau}_{ll}(y)=c^{\tau'\tau}_{l-1,l}(x+y)\,, \tag{3.16}$$

$$
\begin{aligned}
2m(c_{ll}^{\tau'\tau}(x+y)-c_{ll}^{\tau'\tau}(x)\,c_{ll}^{\tau'\tau}(y))
&= c_{l,l+1}^{\tau'\tau}(y)\,c_{l+1,l}^{\tau'\tau}(x)\cdot\{A_l B_l(l+m-1)(l+m)\\
&\quad -(l+m+1)(l+m+2)\} \qquad (3.17)\\
&\quad + c_{l,l+1}^{\tau'\tau}(x)\,c_{l+1,l}^{\tau'\tau}(y)\,\{(l-m+2)(l-m+1)\\
&\quad - A_l B_l(l-m)(l-m-1)\}\,.
\end{aligned}
$$

Thus we see that $c_{l-1,l}^{\tau'\tau}(x)$, $c_{l+1,l}^{\tau'\tau}(x)$ satisfy the same functional equation.

To solve (3.15)–(3.17) let us be given a chain of representations which we want to couple and let L be the maximal weight in the chain then for this L (3.17) reduce to

$$c_{LL}^{\tau'\tau}(x+y)=c_{LL}^{\tau'\tau}(x)\,c_{LL}^{\tau'\tau}(y)$$

whose solution is

$$c_{LL}^{\tau'\tau}(x)=e^{\alpha\cdot x}\delta_{\tau'\tau}$$

(since $c_{LL}^{\tau'\tau}(0)=\delta_{\tau'\tau}$). Equation (3.14) then implies that for all representations in the chain

$$c_{ll}^{\tau'\tau}(x)=e^{\alpha\cdot x}\delta_{\tau'\tau}\,. \qquad (3.18)$$

To find $c_{l,l+1}(x)$ and $c_{l+1,l}(x)$, we substitute (3.18) in (3.17) and rewrite the coefficients of $c_{l,l+1}^{\tau'\tau}(y)\,c_{l+1,l}^{\tau'\tau}(x)$, $c_{l,l+1}^{\tau'\tau}(x)\,c_{l+1,l}^{\tau'\tau}(y)$ in powers of m, then it is easy to see that either $c_{l,l+1}^{\tau'\tau}(x)$ or $c_{l+1,l}^{\tau'\tau}(x)$ is zero (since the relation must be true for all m and A_l, B_l are constants) or

$$c_{l+1,l}(x)=R_l c_{l,l+1}(x)\,. \qquad (3.19)$$

In the latter case we still have to solve (3.15), but if we substitute (3.18) then we find that

$$c_{l,l+1}^{\tau'\tau}(x)=P_l^\tau\,|x|\,e^{\alpha\cdot x}\delta_{\tau,\tau'} \qquad |x|=\Sigma x_i \qquad (3.20)$$

where P_l^τ is a constant. The relation (3.19) then implies that

$$c_{l+1,l}(x)=Q_l^\tau\,|x|\,e^{\alpha\cdot x}\delta_{\tau,\tau'}\,. \qquad (3.21)$$

A similar analysis now applied to the other two CR's shows that they finally lead to the same equations and hence impose no new restrictions on the c's.

Thus we proved the following theorem:

Theorem 4. *If* $\boldsymbol{J}(0)=\sum_{l,\tau}\oplus \boldsymbol{D}_l^\tau$ *then the general solution of the functional equation* (2.2)–(2.3) *is given by either of* 1., 2. *given below or a combination of them:*

1. $\boldsymbol{J}(x)=\sum_{l,\tau}\oplus\exp(\alpha_l^\tau\cdot x)\,\boldsymbol{D}_l^\tau$, $\alpha_l^\tau\in R^n$, *where* α_l^τ *are unrelated.*

2. *In the form* (3.1)–(3.3) *where* $c_{ll}^{\tau'\tau}(x)$, $c_{l+1,l}^{\tau'\tau}(x)$, $c_{l,l+1}^{\tau'\tau}(x)$ *are given by* (3.18), (3.20), (3.21) *where* α *is fixed for each chain of coupled representations.*

Remark. Note that we did not require hermiticity of $J_1(x)$, $J_2(x)$, $J_3(x)$ but if we make this requirement then $Q_l^\tau=P_l^\tau$.

IV. The General Case

Theorem 5. *If* $\boldsymbol{J}(0)$ *is a representation of* $SU(2)$ *such that*

$$\boldsymbol{J}(0)=U\sum_{\tau,l}\oplus D_l^\tau\ U^{-1} \tag{4.1}$$

where D_l^τ *are irreducible representation of* $SU(2)$ *in standard form, then the general solution of* (2.2)–(2.3) *with the boundary value* $\boldsymbol{J}(0)$ *is of the form*

$$J(x)=UD(x)\,U^{-1} \tag{4.2}$$

where $D(x)$ *is any solution of* (2.2)–(2.3) *with* $D(0)=\Sigma\oplus D_l^\tau$.

Proof. Let $D(x)$ be a solution of (2.2)–(2.3) with $D(0)=\Sigma\oplus D_l^\tau$ then obviously $\boldsymbol{J}(x)=UD(x)\,U^{-1}$ satisfies (4.1) and, moreover,

$$\begin{aligned}[J_i(x),J_j(y)]&=UD_i(x)\,U^{-1}UD_j(y)\,U^{-1}\\&\quad-UD_j(y)\,U^{-1}UD_i(x)\,U^{-1}=i\varepsilon_{ijk}UD_k(x+y)\,U^{-1}\\&=i\varepsilon_{ijk}J_k(x+y)\end{aligned} \tag{4.3}$$

thus $\boldsymbol{J}(x)$ is a solution of (2.2)–(2.3) with the proper boundary conditions.

On the other hand suppose $J(x)$ is any solution of (2.2)–(2.3) satisfying (4.1) then

$$D(x)=U^{-1}J(x)\,U$$

satisfies

$$D(0)=U^{-1}J(0)\,U=\Sigma\oplus D_l^\tau$$

and is a solution of (2.2)–(2.3), therefore, it must be one of given by Theorem 4 which implies

$$J(x)=UD(x)\,U^{-1}$$

which is the desired result.

References

1. Dashen, R., Gellman, M.: Phys. Rev. Lett. **17**, 340 (1967)
2. For a recent review article see Mendes, R. V., Ne'eman, Y.: Springer Tracts in Modern Phys. **60** p. 18, Berlin Heidelberg New York: Springer 1971
3. Roffman, E. H.: J. Math. Phys. **8**, 1954 (1967)
4. Joseph, A.: Commun. math. Phys. **19**, 106 (1970)
5. Gel'fand, I. M., *et al.*: Representations of the rotation and Lorentz groups and their applications New York: Pergamon Press 1963

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