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The Symmetries of Kerr Black Holes*

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Abstract. The Kerr solution describes, in Einstein's theory, the gravitational field of a rotating black hole. The axial symmetry and stationarity of the solution are shown here to arise in a simple way from properties of the curvature tensor.

I. Introduction

In the course of analyzing gravitational fields, special attention is often conferred upon the symmetries of the spacetime and upon the algebraic characterization of the curvature tensor. For example, the Kerr solution is axially symmetric and stationary, with an algebraically special vacuum curvature tensor of type {22}. On the former property, in part, is based the physical interpretation of the Kerr field; the exploitation of the latter property was tantamount to the discovery of the field [1].

In the case of the Kerr solution the symmetries can be inferred by an argument centering around properties of the curvature tensor. A more general inference of this sort holds, in fact, for the entire class of type {22} vacuum gravitational fields, as well as for those Einstein-Maxwell fields for which the electromagnetic field is of type {11} with its principal null rays aligned with those of the gravitational field[1]. It is in this more general context that the argument for the Kerr field will be delineated.

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[1] Kinnersley [2], in his careful exhaustive examination of these type {22} spacetimes, observed that all the solutions admit at least two Killing vectors.

It will prove useful during the discussion to have noted that the class of spacetimes under consideration is specified by the existence of a normalized spinor dyad[2],

$$\{o_A, \iota_A : o_A \iota^A = 1\},$$

each member of which being tangent to a shearfree congruence of null geodesics, together with a pair of complex scalars Υ and α, such that upon inspection of the curvature tensor there can be constructed the following *table of basic fields* [4–6]:

Basic fields	Field equations
$\phi_{AB} := \Upsilon^{2/3} o_{(A}\iota_{B)}$	$\nabla^A_{A'} \phi_{AB} = 0$ Maxwell equation
$X_{AB} := \Upsilon^{-1/3} o_{(A}\iota_{B)}$	$\nabla_{A'(A} X_{BC)} = 0$ Twistor equation
$\Upsilon_{ABCD} := \Upsilon\, o_{(A} o_B \iota_C \iota_{D)}$	$\nabla^A_{A'} \Upsilon_{ABCD} = 0$ Spin two zero rest mass field equation
$\Psi_{ABCD} := \alpha \Upsilon_{ABCD}$	$\nabla^A_{A'} \Psi_{ABCD} = \nabla^{B'}_{(B} \Phi_{CD)A'B'}$ Einstein-Maxwell Bianchi identity
$\Phi_{ABA'B'} := \phi_{AB} \bar{\phi}_{A'B'}$	

II. A Complex Killing Vector

Some information of significance can be extracted from the Ricci identities, which, for arbitrary symmetric Γ^{AB} here read

$$\nabla^{A'(C} \nabla^{D)}_{A'} \Gamma^{AB} = 2\gamma\, \Upsilon^{ECD(A} \Gamma^{B)}{}_E ;$$
$$\nabla^{C(A'} \nabla^{B')}_{C} \Gamma^{AB} = 2(\gamma - 1)(\alpha - 1)^{-1} \Phi^{A'B'C(A} \Gamma^{B)}{}_C ,$$

where for a vacuum $\gamma := 1$, and for an Einstein-Maxwell spacetime $\gamma := \alpha$.

With the choice $\Gamma_{AB} = X_{AB}$ there can straightforwardly be obtained the contracted identities

$$\nabla^{A'A} \nabla^B_{A'} X_{AB} = 0 ;$$
$$\nabla^{(B'}_{(B} \nabla^{A')C} X_{A)C} = 0 ,$$

where, in arriving at the latter of these, use is made of the twistor equation for X_{AB} in the alternative form

$$\nabla_{A'A} X_{BC} = \tfrac{2}{3} \nabla^D_{A'} X_{D(B} \varepsilon_{C)A} .$$

[2] The Battelle conventions [3] are used throughout for the denotation and manipulation of tensor and spinor indices.

Making the definition

$$\xi_{A'A} := \nabla^B_{A'} X_{BA}$$

the contracted identities for X_{AB} can be rewritten in the form

$$\nabla^a \xi_a = 0 ; \qquad \nabla^{(B'}_{(B} \xi^{A')}_{A)} = 0 .$$

Thus, inasmuch as

$$\nabla_{(a} \xi_{b)} = \nabla_{(A'(A} \xi_{B)B')} + \tfrac{1}{4} g_{ab} \nabla^c \xi_c ,$$

the vector ξ_a is a Killing vector:

$$\nabla_{(a} \xi_{b)} = 0 .$$

III. An Aside on Quadratic Killing Tensors

It is a direct consequence of the twistor equation for X_{AB} that the tensor

$$K_{ab} := X_{AB} \bar{X}_{A'B'} + \tfrac{1}{4} K g_{ab}$$

satisfies, for some scalar K, the quadratic Killing equation

$$\nabla_{(a} K_{bc)} = 0$$

precisely in the event that

$$\Phi_{ab} \nabla^b (\Upsilon \bar{\Upsilon})^{-1} = \tfrac{3}{4} \nabla_a K .$$

This condition can be formulated in a more concise way: First, note that the Einstein-Maxwell Bianchi identity equation can be reexpressed [6] in the form

$$\Phi_{ab} \nabla^b (\Upsilon^{-1}) = -\tfrac{1}{4} \nabla_a \alpha ;$$

and, from this relation, obtain, in turn, that

$$\Phi_{ab} \nabla^b (\Upsilon \bar{\Upsilon})^{-1} = -\tfrac{1}{4} [\bar{\Upsilon}^{-1} \nabla_a \alpha + \Upsilon^{-1} \nabla_a \bar{\alpha}] .$$

Observe, therefore, that the Killing tensor condition can be integrated, the resulting requirement of integrability being that there should exist a function $g(\bar{\Upsilon})$ and a *real* function $f(\Upsilon, \bar{\Upsilon})$ such that

$$\alpha = \int \Upsilon^{-2} f(\Upsilon, \bar{\Upsilon}) \, d\Upsilon + g(\bar{\Upsilon}) .$$

This requirement will be brought into consideration in Section IV.

IV. The Degeneracy of the Complex Killing Vector

It may be the case that the Killing vector constructed in Section II is degenerate in this sense: having renormalized X_{AB} with a suitable complex constant phase factor, it is found that

$$\xi_a = \bar{\xi}_a .$$

This degeneracy occurs, in fact, in the instance of the Kerr solution; thus, its consequences will be examined in some detail.

First it is necessary to express the Einstein-Maxwell Bianchi identity equation in still another of its manifestations,

$$-\tfrac{1}{4}\nabla_a \alpha = \bar{\phi}_{A'B'}\xi_A^{B'} .$$

Applying the degeneracy condition for ξ_a renders this statement into the form

$$\tfrac{1}{3}\nabla_a \alpha = \nabla_a \bar{\Upsilon}^{1/3}$$

which has as its solution

$$\alpha = 3\bar{\Upsilon}^{1/3} + \beta$$

where β is a constant. Thus, by virtue of the integrability condition of Section III having been manifestly satisfied, one has at his disposal a Killing tensor K_{ab}.

V. The Second Killing Vector

At this point it is possible to arrive, finally, in a relatively compact way at the second Killing vector of the Kerr solution. Significant in the construction is the fact that, on account of the twistor equation

$$\nabla_{A'A} X_{BC} = \tfrac{2}{3}\xi_{A'(B}\varepsilon_{C)A} ,$$

the spinor X_{BC} is parallel along ξ^a:

$$\xi^a \nabla_a X_{BC} = -\tfrac{2}{3}\xi_{(C}^{A'}\xi_{B)A'} = 0 .$$

By virtue of the first Ricci identity, X_{BC} is also Lie invariant along ξ^a:

$$\mathcal{L}_\xi X_{BC} = \xi^a \nabla_a X_{BC} + X_{A(B}\nabla_{C)A'}\xi^{AA'} = 0 .$$

The Killing tensor K_{ab} is constructed entirely from the metric and X_{AB}, from which it is seen that it, too, is both parallel and Lie invariant along the Killing field ξ^a.

If the vector field η_a is defined[3] by

$$\eta_a := K_{ab}\xi^b ,$$

then from the Killing equation for K_{ab},

$$\nabla_{(a}K_{b)c} = -\tfrac{1}{2}\nabla_c K_{ab} ,$$

[3] In Schwarzschild's spacetime the Killing vector ξ^a is orthogonal to K_{ab}. From this degeneracy one can infer the spherical symmetry [7].

it follows that

$$\nabla_{(a}\eta_{b)} = \tfrac{1}{2}\mathcal{L}_\xi K_{ab} - \xi^c \nabla_c K_{ab}\,.$$

Thus

$$\nabla_{(a}\eta_{b)} = 0\,.$$

It is worth remarking, incidentally, that, as an immediate consequence of the invariance of K_{ab} with respect to Lie derivation by ξ^a, the two Killing vectors commute:

$$\mathcal{L}_\xi \eta_a = 0\,.$$

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