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Existence and Uniqueness of Equilibrium States for Some Spin and Continuum Systems

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Abstract. The one to one correspondence between the existence of a unique equilibrium state and the differentiability of the free energy density with respect to the external field previously shown for Ising ferromagnets is extendend to higher valued spin systems as well as to continuum systems satisfying the Fortuin, Kasteleyn and Ginibre inequalities. In particular this is shown to hold for a mixture of $A-B$ particles in which there is no interaction between like particles and a repulsion between unlike particles. Where the derivative of the free energy is discontinuous there are at least two equilibrium states.

1. Introduction

In a previous paper [1] we considered lattice spin systems with Hamiltonians

$$H = -\sum_{i<j} J(i,j)\, S_i S_j - \sum_{i<j} \gamma(i,j)\, S_i^2 S_j^2 - \sum_i h(i)\, S_i - \sum_i \mu(i)\, S_i^2 \qquad (1.1)$$

where the summation is over all sites of the lattice Z^ν with spacing δ, contained in a region $\Lambda \subset R^\nu$, and S_i, the spin variable at the ith site can take on the integer values $p, p-2, \ldots, -p+2, -p$. For such a system it was shown that the FKG inequalities [2] hold whenever

$$J(i,j) \geqq (2p-2)^2\, |\gamma(i,j)|\,, \qquad \text{for all} \qquad i,j \in \Lambda \qquad (1.2)$$

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i.e., if $f(S_{i_1}, \ldots S_{i_n})$ and $g(S_{j_1} \ldots S_{j_m})$ are both either non-decreasing or non-increasing functions of their arguments then they are positively correlated,

$$\langle fg\rangle - \langle f\rangle \langle g\rangle \geqq 0 \tag{1.3}$$

where the brackets denote the usual thermal average taken with the measure $Z^{-1}\exp[-\beta H]$. It follows directly from (1.3) that if f is a non-decreasing function then $\frac{\partial\langle f\rangle}{\partial h(i)} = \langle fS_i\rangle - \langle f\rangle \langle S_i\rangle \geqq 0$ since S_i is an increasing function.

In the present note as in [1] we are particularly interested in the case $p=2$, $S_i = 2, 0, -2$. Such a system is isomorphic to a two component lattice gas, $A-B$ system, with $S_i = 2(-2)$ corresponding to the presence of an $A(B)$ particle at the ith site and $S_i = 0$ to the site being empty. This lattice gas system can go over to a continuum $A-B$ system when the lattice spacing $\delta \to 0$ and the interactions appearing in (1.1) are related to continuum functions,

$$\begin{aligned} J(i,j) &= J(r_i - r_j), \qquad & \gamma(i,j) &= \gamma(r_i - r_j), \\ h(i) &= h(r_i), \qquad & \exp[\beta\mu(i)] &= \delta^\nu \exp[\beta\mu(r_i)] \end{aligned} \tag{1.4}$$

where r_i and r_j are the position vectors of the ith and jth lattice sites and $J(r), \gamma(r), h(r)$, and $\mu(r)$ are defined for $r \in R^\nu$ (independent of δ). The lattice inequalities remain valid for the continuum as was shown in [1]. We shall here use these inequalities to establish some results about the existence and uniqueness of the correlation functions for the continuum systems.

The paper is arranged in the following way. In Section 2 we show (in analogy with the results of Lebowitz and Martin-Lof [3] for a ferromagnetic spin one system, $p=1$) that when (1.2) is satisfied the existence of a unique equilibrium state is determined entirely by the differentiability of the thermodynamic free energy with respect to the external magnetic field h. In Section 3 we show how to carry over the results for the lattice to the continuum.

2. Lattice Systems

Using lattice gas language we can describe our system in terms of occupation numbers $\varrho_i = \frac{1}{2}(S_i + p) = 0, 1, \ldots, p$. Since each ϱ_i is an increasing function of S_i and is non-negative any product of the ϱ_i's i.e. $\prod_{i\in\omega} (\varrho_i)^{n_i}$, n_i a positive integer, is also an increasing function. It is also easy to verify that for any set $\omega \subseteqq \Lambda$ the function

$$\sum_{i \in \omega} \varrho_i - \prod_{i \in \omega} (\varrho_i/p)^{n_i}, \tag{2.1}$$

where n_i is a positive integer is also an increasing function.

Using the above increasing functions we can transcribe all the results of [3] for ferromagnetic Ising spin systems, $p=1$, which depended only on the validity of the FKG inequalities, i.e. $J(i,j) \geqq 0$, to the more general class of systems whose interactions satisfy (1.2). In particular we consider the thermodynamic limit $\Lambda \to \infty$, for a system in which $J(i,j)$ and $\gamma(i,j)$ are translation invariant and $h(i) = h + h_b(i)$, $\mu(i) = \mu + \mu_b(i)$, with $h_b(i)$ and $\mu_b(i)$ being due to the specification of the values of the S_j's or ϱ_j's for $j \in \bar{\Lambda}$, the complement of Λ in $\mathbb{Z}^\nu$. Thus

$$h_b(i) = \sum_{j \in \bar{\Lambda}} J(i,j)\, \bar{S}_j\,, \qquad \mu_b(i) = \sum_{j \in \bar{\Lambda}} \gamma(i,j)\, \bar{S}_j^2 \tag{2.2}$$

where we have placed a bar on the spins in $\bar{\Lambda}$. We shall assume here that $\gamma(i,j)$ and $J(i,j)$ are of a finite range and designate by $\langle f/h, \mu, b, \Lambda\rangle$ the expectation value of f for a given h, μ, Λ and boundary condition b. We let $b = +(-)$ denote the boundary conditions corresponding to $\varrho_j = p(0)$ for all $j \in \bar{\Lambda}$, (the arguments h and μ will be left out when they are not necessary).

We observe that by letting $h(i) \to +\infty(-\infty)$ we can assure that $S_i = p(-p)$. Hence we clearly have as in the case $p=1$ that for any increasing function f

$$\langle f \,|\, \pm, \Lambda\rangle \gtreqless \langle f \,|\, \pm, \Lambda'\rangle \quad \text{for} \quad \Lambda \subset \Lambda' \tag{2.3}$$

i.e. f is a monotonically non-decreasing (non-increasing) function of the size of the system when we have $-(+)$ boundary conditions. This proves the existence of the thermodynamic limit of all correlation functions for $\pm$ boundaries,

$$\lim_{\Lambda \to \infty} \left\langle \prod_{i \in \omega} (\varrho_i)^{n_i} \,|\, \pm, \Lambda \right\rangle = \left\langle \prod_{i \in \omega} (\varrho_i)^{n_i} \,|\, \pm \right\rangle \tag{2.4}$$

Furthermore for any increasing bounded function f one can show (see appendix A)

$$\langle f \,|\, -\rangle \leqq \lim_{\Lambda \to \infty} \langle f \,|\, b, \Lambda\rangle \leqq \langle f \,|\, +\rangle \tag{2.5}$$

Therefore using the increasing functions $\prod_i (\varrho_i)^{n_i}$ and $\sum_{i \in \omega} \varrho_i = \prod_{i \in \omega} (\varrho_i/p)^{n_i}$ we have

$$\left\langle \prod_{i \in \omega} (\varrho_i)^{n_i} | - \right\rangle \leqq \left\langle \prod_{i \in \omega} (\varrho_i)^{n_i} | b \right\rangle \leqq \left\langle \prod_{i \in \omega} (\varrho_i)^{n_i} | + \right\rangle \tag{2.6}$$

and

$$0 \leqq \left\langle \prod_{i \in \omega} (\varrho_i)^{n_i} | + \right\rangle - \left\langle \prod_{i \in \omega} (\varrho_i)^{n_i} | - \right\rangle \leqq \left(\prod_{i \in \omega} (p)^{n_i} \right) \sum_{i \in \omega} [\langle \varrho_i | + \rangle - \langle \varrho_i | - \rangle] \tag{2.7}$$

Thus as in [3] $\langle \varrho_i | + \rangle = \langle \varrho_i | - \rangle$ implies that *all* correlation functions are the same for *all* boundary conditions, i.e. the system has a unique equilibrium state.

A transcription of the arguments developed in [3] further shows that when (1.2) holds

$$\begin{aligned} \langle f | h, \mu, + \rangle &= \lim_{h' \to h^+} \langle f | h', \mu, + \rangle \\ \langle f | h, \mu, - \rangle &= \lim_{h' \to h^-} \langle f | h', \mu, - \rangle \end{aligned} \tag{2.8}$$

for f a non-decreasing function, and that calling $\Psi(h, \mu)$ the thermodynamic limit of the free energy/site (which is independent of boundary conditions) then

$$\langle S_i | h, \mu, + \rangle = \lim_{h' \to h^+} \frac{\partial \Psi(h', \mu)}{\partial h'}, \qquad \langle S_i | h, \mu, - \rangle = \lim_{h' \to h^-} \frac{\partial \Psi(h', \mu)}{\partial h'} .$$

Hence $\langle \varrho_i | h, \mu, + \rangle = \langle \varrho_i | h, \mu, - \rangle$ and the state is unique if and only if the derivative of the free energy with respect to the external field h exists, i.e. if the magnetization is continuous.

We note here that the results of the section will also hold if we set

$$\begin{aligned} J(l, h) &= \hat{J}(l, h)/p^2 ; \qquad & \gamma(l, h) &= \hat{\gamma}(l, h)/p^h \\ \mu(i) &= \hat{\mu}(i)/p^2 ; & h &= \hat{h}/p \end{aligned}$$

and then let $p \to \infty$ obtaining a continuous Ising spin system with $S_i \in [1, -1]$. The condition (1.2) for the F.K.G. inequalities is now simply

$$\hat{J}(l, h) \geqq 4 |\hat{\gamma}(i, j)| .$$

3. Continuum Systems — Thermodynamic Limit

We now consider the special continuum analog of the lattice system when $p = 2, S = 2, 0, -2$. As stated in the introduction we shall define an A particle as being present in the site i when $S_i = 2$, and a B particle when $S_i = -2$. If the site i is occupied by an A particle we put $\varrho_A(i) = 1$ and if i is not occupied by an A particle we put $\varrho_A(i) = 0$. Similarly we define $\varrho_B(i)$ and we get:

$$\begin{aligned} \varrho_A(i) &= \frac{1}{2} \varrho_i(\varrho_i - 1) = \frac{S_i(S_i + 2)}{8} \\ \varrho_B(i) &= \frac{1}{2} (\varrho_i - 1)(\varrho_i - 2) = \frac{S_i(S_i - 2)}{8} \end{aligned} \tag{3.1}$$

In the continuum limit of [1], discussed in the introduction, the system goes over into a two component A, B system with activities

$$\begin{aligned} z_A(\boldsymbol{r}) &= \exp[4\mu(\boldsymbol{r}) + 2h(\boldsymbol{r})] \\ z_B(\boldsymbol{r}) &= \exp[4\mu(\boldsymbol{r}) - 2h(\boldsymbol{r})] \end{aligned} \tag{3.2}$$

and with pair interaction:

$$\begin{aligned} V_{AA}(\boldsymbol{r}) = V_{BB}(\boldsymbol{r}) &= -4J(\boldsymbol{r}) - 16\gamma(\boldsymbol{r}) \\ V_{AB}(\boldsymbol{r}) = V_{BA}(\boldsymbol{r}) &= 4J(\boldsymbol{r}) - 16\gamma(\boldsymbol{r}) \end{aligned} \tag{3.3}$$

The two functions $J(\boldsymbol{r})$ and $\gamma(\boldsymbol{r})$ cannot be arbitrary: they will be restricted so that (3.3) defines a stable interaction [4]; we shall require also (1.2). The stability requirement constrains us to consider only models for which $V_{AA} = V_{BB} = 0$ and $V_{AB} \geqq 0$. A simple example of such an interaction is the Widom-Rowlinson model [5] obtained by choosing $J(\boldsymbol{r}) = -4\gamma(\boldsymbol{r}) = +\infty$ if $|\boldsymbol{r}| \leqq a$ and $J(\boldsymbol{r}) = \gamma(\boldsymbol{r}) = 0$ otherwise (i.e. $V_{AA} = V_{BB} = 0$, $V_{AB} = +\infty$ if $|\boldsymbol{r}| \leqq a$, $V_{AB} = 0$ if $|\boldsymbol{r}| \geqq a$). This model is particularly interesting since it has been shown to have a phase transition [6].

The correlation functions of the lattice gas go over into these of the continuum system via the transcription

$$\begin{aligned} &\lim_{\delta\to 0}\left\langle \prod_{j=0}^{l_1}\Big(\sum_{i\in\omega_j}\varrho_A(i)\Big)\prod_{p=1}^{l_2}\Big(\sum_{i\in\omega_p'}\varrho_B(i)\Big)\Big| h,\mu,b,\Lambda\right\rangle \\ &\quad = \int_{\omega_1\times\omega_2\cdots\omega_{l_1}} d\boldsymbol{x}_1 \dots d\boldsymbol{x}_{l_1} \int_{\omega_1'\times\omega_2'\dots\times\omega_{l_2}'} d\boldsymbol{y}_1 \dots d\boldsymbol{y}_{l_2} \\ &\qquad \cdot n_{l_1,l_2}(\boldsymbol{x}_1 \dots \boldsymbol{x}_{l_1}, \boldsymbol{y}_1 \dots \boldsymbol{y}_{l_2} | h, \mu, b, \Lambda) \end{aligned} \tag{3.4}$$

where $\omega_i, \omega_j' \subset \mathbb{R}^\nu$ are disjoint open regions and

$$n_{l_1,l_2}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_l; \boldsymbol{y}_1 \dots \boldsymbol{y}_{l_2} | h, \mu, b, \Lambda)$$

is the joint distribution function for l_1 A-particles at $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{l_1}$ and l_2 B-particles at $\boldsymbol{y}_1, \dots, \boldsymbol{y}_{l_2}$. The arguments h, μ refer now to the continuum functions in (3.1) and b has to be interpreted as a boundary condition corresponding to an arbitrary specification of A and B particles in $\bar{\Lambda}$.

In the remainder of this section we shall discuss the thermodynamic limit for the functions $n_{l_1,l_2}(\boldsymbol{x}_1 \dots \boldsymbol{x}_{2_1}; \boldsymbol{y}_1 \dots \boldsymbol{y}_{l_2} | h, \mu, b, \Lambda\rangle$ using the FKG inequalities.

It is convenient to introduce the following functions defined on the configurations X of $A-B$ particles

$$I^n(X, A, \omega) = \begin{cases} 1 & \textit{if} \ \#\ (A \text{ particles in } \omega) \geqq n \\ 0 & \text{if} \ \#\ (A \text{ particles in } \omega) < n \end{cases} \tag{3.5}$$

and $I^n(X, B, \omega)$ is defined symmetrically; here ω is an arbitrary open region contained in $\mathbb{R}^\nu$ and (3.5) makes sense both in the discrete and continuum case.

We have:

$$\#\,(A\text{-particles in }\omega) = \sum_{n=1}^{\infty} I^n(X, A, \omega)\,. \tag{3.6}$$

Furthermore the function defined on the lattice $A-B$ configurations

$$\begin{aligned} F(n_1 \ldots n_{l_1}; m_1 \ldots m_{l_2}; \omega_1 \ldots \omega_{l_1}; \omega'_1 \ldots \omega'_{l_2}) \\ = \prod_{i=1}^{l_1} I^{n_i}(X, A, \omega_i) \prod_{j=1}^{l_2} [1 - I^{m_j}(X, B, \omega_j)] \end{aligned} \tag{3.7}$$

is increasing. This implies using the results of § 2

$$\langle F | h, \mu, -, \Lambda\rangle \leqq \langle F | h, \mu, b, \Lambda\rangle \leqq \langle F | h, \mu, +, \Lambda\rangle \tag{3.8}$$

and

$$\langle F | h, \mu, \pm, \Lambda\rangle \gtreqless \langle F | h, \mu, \pm, \Lambda'\rangle \quad \text{if} \quad \Lambda \subset \Lambda'\,. \tag{3.9}$$

The inequalities (3.8) and (3.9) obviously go over unchanged to the continuum limit.

Therefore, for continuum systems, the limits $\lim_{\Lambda\to\infty} \langle F | h, \mu, \pm, \Lambda\rangle$ exist, and are monotonically attained. Hence the limits:

$$\begin{aligned} \lim_{\Lambda\to\infty} \left\langle \prod_{i=1}^{l_1} I^{n_i}(X, A, \omega_i) \prod_{j=1}^{l_2} I^{m_j}(X, B, \omega'_j) | h, \mu, \pm, \Lambda \right\rangle \\ = \left\langle \prod_{i=1}^{l_1} I^{n_i}(X, A, \omega_i) \prod_{j=1}^{l_2} I^{m_j}(X, B, \omega'_j) | h, \mu, \pm \right\rangle \end{aligned} \tag{3.10}$$

exist for all integers $n_1, n_2 \ldots n_{l_1}, m_1, m_2, \ldots m_{l_2}$ (but are not monotonically reached, at least in principle, unless l_1 or l_2 are zero).

To deduce the existence of the thermodynamic limit for the correlation functions n_{l_1,l_2} we use (3.6) and the following a priori estimates:

$$\begin{aligned} n_{l_1,l_2}(\boldsymbol{x}_1 \ldots \boldsymbol{x}_{l_1}; \boldsymbol{y}_1 \ldots \boldsymbol{y}_{l_2} | h, \mu, b, \Lambda) \leqq z_A^{l_1} z_B^{l_2} \\ \left\langle \prod_{i=1}^{l_1} I^{n_i}(X, A, \omega_i) \prod_{j=1}^{l_2} I^{m_j}(X, B, \omega'_j) | h, \mu; b, \Lambda \right\rangle \leqq \prod_{i=1}^{l_1} \xi_{n_i} \prod_{j=1}^{l_2} \xi_{m_j} \end{aligned} \tag{3.11}$$

where

$$\xi_{n_i} = \left(\sum_{k=n_i}^{\infty} \frac{(|\omega_i|\, z_A)^k}{k!}\right) e^{z_A|\omega_i|}\,; \qquad \xi_{m_j} = \left(\sum_{k=m_j}^{\infty} \frac{(|\omega'_j|\, z_B)^k}{k!}\right) e^{z_B|\omega'_j|} \tag{3.12}$$

for the proof see appendix B and C.

We need as well the Mayer Montroll equations valid when the points $x_1, \ldots, x_{l_1}; y_1 \ldots \ldots y_{l_2}$ are further than the range of V_{AB} from $\partial\Lambda$ (see appendix D)

$$n_{l_1,l_2}(x_1 \ldots x_{l_1}; y_1 \ldots y_{l_2} | h, \mu, b, \Lambda)$$

$$= z_A^{l_1} z_B^{l_2} \exp\left[-\sum_{i=1}^{l_1}\sum_{j=1}^{l_2} V_{AB}(x_i - y_j)\right] \tag{3.13}$$

$$\cdot \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \int \frac{d\xi_1 \ldots d\xi_n}{n!} \frac{d\eta_1 \ldots d\eta_m}{m!} n_{n,m}(\xi_1 \ldots \xi_n, \eta_1 \ldots \eta_m | h, \mu, b, \Lambda)$$

$$\cdot K(x_1 \ldots x_{l_1} | \eta_1 \ldots \eta_m) K(y_1 \ldots y_{l_2} | \xi_1 \ldots \xi_n)$$

where

$$K(x_1 \ldots x_{l_1} | \eta_1 \ldots \eta_m) = \prod_{j=1}^{m} \left(e^{-\sum_{i=1}^{l_1} V_{AB}(x_i - \eta_j)} - 1\right)$$

and equations (3.13) make sense because of (3.11).

From (3.12) and (3.6) we deduce:

$$\lim_{\Lambda\to\infty} \int_{\omega_1 \times \cdots \times \omega_{l_1}} dx_1 \ldots dx_{l_1} dy_1 \ldots dy_{l_2} n_{l_1,l_2}(x_1 \ldots x_{l_1}; y_1 \ldots y_{l_2} | h, \mu, \pm, \Lambda)$$

$$\equiv \lim_{\Lambda\to\infty} \sum_{n_1 \ldots n_{l_1}} \sum_{m_1 \ldots m_{l_2}} \left\langle \prod_{i=1}^{l_1} I^{n_i}(X, A, \omega_i) \prod_{j=1}^{l_2} I^{m_j}(X, B, \omega'_j) | h, \mu, \pm, \Lambda \right\rangle$$

$$= \sum_{n_1 \ldots n_{l_1}} \sum_{m_1 \ldots m_{l_2}} \left\langle \prod_{i=1}^{l_1} I^{n_i}(X, A, \omega_i) \prod_{j=1}^{l_2} I^{m_j}(X, B, \omega'_j) | h, \mu, \pm \right\rangle \tag{3.14}$$

The existence of the limit (3.14) for disjoint regions $\omega_i \ldots \omega_{l_1}, \omega'_1 \ldots \ldots \omega'_{l_2}$, together with the bound (3.11), implies the existence of the limit

$$\lim_{\Lambda\to\infty} \int f(x_1 \ldots x_{l_1}; y_1 \ldots y_{l_2}) n_{l_1,l_2}(x_1 \ldots x_{l_1}, y_1 \ldots y_{l_2} | h, \mu, \pm, \Lambda\rangle$$
$$dx_1 \ldots dx_{l_1} dy_1 \ldots dy_{l_2}$$

for all $f \in L_1(dx_1 \ldots \ldots dx_{l_1}, dy_1 \ldots dy_{l_2})$; hence using (3.15) with $f = K(x_1 \ldots x_{l_1} | \eta_1 \ldots \eta_m) \cdot K(y_1 \ldots y_{l_2} | \xi_1 \ldots \xi_n)$ and Eq. (3.13) and (3.11) we deduce that:

$$\lim_{\Lambda\to\infty} \frac{n_{l_1,l_2}(x_1 \ldots x_{l_1}, y_1 \ldots y_{l_2} | h, \mu, \pm, \Lambda)}{\exp\left[-\sum_i\sum_j V_{AB}(x_i - y_j)\right]} = \frac{n_{l_1,l_2}(x_1 \ldots x_{l_1}, y_1 \ldots y_{l_2} | h, \mu, \pm)}{\exp\left[-\sum_i\sum_j V_{AB}(x_i - y_j)\right]} \tag{3.15}$$

exist and define $n_{l_1,l_2}(\boldsymbol{x}_1 \ldots \boldsymbol{x}_{l_1}, \boldsymbol{y}_1 \ldots \boldsymbol{y}_{l_2} | h, \mu, \pm\rangle$ which are continuos functions "outside" the hard cores and discontinuity points of $\sum_{i,j} V_{AB}(\boldsymbol{x}_i - \boldsymbol{y}_j)$.

4. Continous Case Differentiability in *h* and Boundary Condition Independence

In this section we extend to the continuum case the result, already remarked in the discrete case, that the infinite volume correlation functions are boundary condition independent if and only if the free energy $\Psi(h, \mu)$ is differentiable with respect to h. This, in turn, will be shown to happen if and only if $n_{1,0}(\boldsymbol{x}/h; \mu, +) = n_{1,0}(\boldsymbol{x}, h, \mu, -)$.

Notice first that the correlation functions in the right hand side of (3.16) are translation invariant as a consequence of the translation invariance of the limit $\lim_{\Lambda \to \infty} \langle F | h, \mu, \pm, \Lambda\rangle$ (see sec. 3 for the definition of F), which, in turn, follows by standard arguments [3] from (3.9). Hence $n_{1,0}(\boldsymbol{x}/h, \mu, \pm)$ is x-independent. The condition $n_{1,0}(\boldsymbol{x}/h, \mu, +) = n_{1,0}(\boldsymbol{x}/h, \mu, -)$ could also be written: $n_{0,1}(\boldsymbol{x}/h, \mu, +) = n_{0,1}(\boldsymbol{x}/h, \mu, -)$ as will appear in what follows.

The proof of the quoted results proceeds much along the same lines as in sec. 2 and [3].

First notice that

$$\langle F | h, \mu, \pm\rangle = \lim_{h' \to h^{\pm}} \langle F | h', \mu, \pm\rangle \tag{4.1}$$

which is proven as (2.7) and implies, using (3.14) and (3.13)

$$n_{l_1,l_2}(\boldsymbol{x}_1 \ldots \boldsymbol{x}_{l_1}, \boldsymbol{y}_1 \ldots \boldsymbol{y}_{l_2} | h, \mu, \pm) = \lim_{h' \to h^{\pm}} n_{l_1,l_2}(\boldsymbol{x}_1 \ldots \boldsymbol{x}_{l_1}; \boldsymbol{y}_1 \ldots \boldsymbol{y}_{l_2} | h', \mu, \pm) \tag{4.2}$$

at the continuity points of n_{l_1,l_2}.

Hence it follows as in [3] and as in sec. 2:

$$\left.\frac{\partial \Psi(h, \mu)}{\partial h}\right/_{h^{\pm}} = n_{1,0}(\boldsymbol{x}/h, \mu, \pm) - n_{0,1}(\boldsymbol{x}/h, \mu, \pm) \tag{4.3}$$

which shows that differentiability of Ψ implies $n_{1,0}(\boldsymbol{x}/h, \mu, +) - n_{0,1}(\boldsymbol{x}/h, \mu, +) = n_{1,0}(\boldsymbol{x}/h, \mu, -) - n_{0,1}(\boldsymbol{x}/h, \mu, -)$ and, together with $n_{1,0}(\boldsymbol{x}/h, \mu, +) \geqq n_{1,0}(\boldsymbol{x}/h, \mu, -)$ and $n_{0,1}(\boldsymbol{x}/h, \mu, +) \leqq n_{0,1}(\boldsymbol{x}/h, \mu, -)$, implies (and is implied since $\Psi(h, \mu)$ is convex in h):

$$n_{1,0}(\boldsymbol{x}/h, \mu, +) = n_{1,0}(\boldsymbol{x}/h, \mu, -)$$

and

$$n_{0,1}(\boldsymbol{x}/h, \mu, +) = n_{0,1}(\boldsymbol{x}/h, \mu, -)$$

Clearly if the infinite volume correlation functions are boundary condition independent we must have, in particular, $n_{1,0}(+)=n_{1,0}(-)$.

Viceversa, assuming $n_{1,0}(+)=n_{1,0}(-)$ and $n_{0,1}(+)=n_{0,1}(-)$ we shall show that in this case:

$$\langle F|h,\mu,+\rangle=\langle F|h,\mu,-\rangle \tag{4.4}$$

which, together with the results of the preceding sections (see in particular (3.8), will imply that

$$\lim_{\Lambda\to\infty} n_{l_1,l_2}(\boldsymbol{x}_1\dots\boldsymbol{x}_{l_2}\boldsymbol{y}_1\dots\boldsymbol{y}_{l_2}|h,\mu,b,\Lambda)=n_{l_1,l_2}(\boldsymbol{x}_1\dots\boldsymbol{x}_{l_1};\boldsymbol{y}_1\dots\boldsymbol{y}_{l_2}|h,\mu,+) \tag{4.5}$$

for all the continuity points $\boldsymbol{x}_1\dots\boldsymbol{x}_{l_1},\boldsymbol{y}_1\dots\boldsymbol{y}_{l_2}$.

Notice that $n_{1,0}(\boldsymbol{x}/h,\mu,+)=n_{1,0}(\boldsymbol{x}/h,\mu,-)$ implies

$$\langle I^n(X,A,\omega)|h,\mu,+\rangle=\langle I^n(X,A,\omega)|h,\mu,-\rangle$$

and a similar relation with B in place of A; consider next the function:

$$\begin{aligned}\sum_{i=1}^{n} I^{n_i}(X,A,\omega_i)+\sum_{j=1}^{m}(1-I^{m_j}(X,B,\omega_j'))&\\ -\prod_{i=1}^{n} I^{n_i}(X,A,\omega_i)\prod_{j=1}^{m}(1-I^{m_j}(X,B,\omega_j'))&\end{aligned} \tag{4.6}$$

which is increasing. Hence, proceeding as in sec. 2 (see (2.6)):

$$\begin{aligned}0\leqq&\left\langle\prod_{i=1}^{n} I^{n_i}(X,A,\omega_i)\prod_{j=1}^{m}(1-I^{m_j}(X,B,\omega_j'))|h,\mu,+\right\rangle\\ &-\left\langle\prod_{i=1}^{n} I^{n_i}(X,A,\omega_i)\prod_{j=1}^{m}(1-I^{m_j}(X,B,\omega_j')|h,\mu,-\right\rangle\\ \leqq&\sum_{i=1}^{n}(\langle I^{n_i}(X,A,\omega_i|h,\mu,+\rangle-\langle I^{n_i}(X,A,\omega_i)|h,\mu,-\rangle)\\ &-\sum_{j=1}^{m}(\langle I^{m_j}(X,B,\omega_j'|h,\mu,+\rangle-\langle I^{m_j}(X,B,\omega_j'|h,\mu,-))\end{aligned} \tag{4.7}$$

which concludes our proof.

5. Concluding Remarks

The results obtained in the previous sections both for lattice and continuum systems rely mainly on monotonicity properties of the thermal averages of suitably chosen functions of the configurations.

One may wonder about the different choice of increasing functions that we have made in the lattice and in the continuum case. It is certainly possible to get an unified treatment of binary mixtures in terms of the I^n by simply shrinking the ω's to points on the lattice for the discrete case; but in dealing with higher spin systems the procedure to construct monotonic functions becomes rather involved.

On the other hand, if we insist in using the ϱ_A's, a mere transcription from the discrete to the continuum case works only for the homogeneous distributions (i.e. for the $n_{l,0}$ and $n_{0,l}$). A possible way out would be to consider the conditional expectations of $\sum_{i\in\omega} \varrho_A(i)$ given that $\varrho_B(j)=1$ for some $j \notin \omega$; for the realization of this program it is important to note that relations of the following type hold:

$$\lim_{\delta\to 0} \left\langle \left(\sum_{i\in\omega} \varrho_A(i) \,|\, b, \Lambda, \varrho_B(j)=1 \right\rangle \equiv \lim_{\delta\to 0} \frac{\left\langle \varrho_B(j) \sum_{i\in\omega} \varrho_A(i) \,|\, b, \Lambda \right\rangle}{\langle \varrho_B(j) \,|\, b, \Lambda \rangle} = \frac{\int_\omega n_{1,1}(x, j/b, \Lambda)\, dx}{n_{0,1}(j\,|\,b, \Lambda)} \tag{5.1}$$

from which it is not hard to get the thermodynamic limit of the mixed distribution functions. In this context it is not, however, clear how to recover the analog of equation (4.8) and, therefore, of the theorem on the boundary conditions independence.

The number of signatures under the titre of this paper requires some explanations.

Initially J.M. in collaboration with J.L.L. obtained most of the results presented in this paper: a gap however remained in the proof of the relationship between the boundary condition independence and the differentiability of the free energy. They were using the method described above.

Independently M.C. obtained many of their results using the I functions technique and this enabled J.M. and J.L.L. to complete their proof.

The continuity properties of the correlation functions are due to G.G.

Appendix A — Proof of (2.5)

Consider an arbitrary boundary condition b for a system enclosed in a box Λ and consider a system enclosed in a box $\Lambda' \subset \Lambda$ whose boundary lies at a distance l from the boundary $\partial\Lambda$. We assume that l exceeds the range of the potentials $J(i,j)$ and $\gamma(i,j)$.

Let f be an arbitrary increasing function and let all the sites in $\Lambda\backslash\Lambda'$ be occupied either by a maximal spin or by a minimal spin; then we find, respectively:

$$\langle f\,|\,+, \Lambda' \rangle \geqq \langle f\,|\,b, \Lambda \rangle \quad \text{and} \quad \langle f\,|\,-, \Lambda' \rangle \leqq \langle f\,|\,b, \Lambda \rangle \tag{A.1}$$

Therefore in the limit Λ and $\Lambda' \to \infty$:

$$\langle f|-\rangle \leqq \lim_{\Lambda \to \infty} \langle f|b, \Lambda\rangle \leqq \langle f|+\rangle \tag{A.2}$$

In the special case of an $A-B$ system ($p=2$) one could get stronger results (see (A.3) below) even in the case in which the range of the potential is not finite.

In fact let us examine this case and use the $A-B$ particle language. From (1.1) we see that $-\sum_i h_i S_i - \sum_i \mu_i S_i^2$ can be written as

$$-\sum_i (4\mu_i + 2h_i)\,\varrho_A(i) - \sum_i (4\mu_i - 2h_i)\,\varrho_B(i)\,.$$

Now, given an increasing function f, we consider an A-type of boundary condition, i.e. we set $4\mu_i + 2h_i = +\infty$ and $4\mu_i - 2h_i = -\infty$ on the boundary sites. Afterwards we lower $4\mu_i + 2h_i$ on some selected boundary sites while keeping $4\mu_i - 2h_i$ constant. In this process $\langle f\rangle$ decreases because of the inequality

$$\frac{\partial\langle f\rangle}{\partial(4\mu_i + 2h_i)} = \langle f\varrho_A(i)\rangle - \langle f\rangle\,\langle\varrho_A(i)\rangle \geqq 0\,.$$

When $4\mu_i + 2h_i = -\infty$ we are in a situation in which the considered boundary sites are all empty; then we start increasing $4\mu_i - 2h_i$ keeping, now, $h\mu_i + 2h_i = -\infty$ constant. Since

$$\frac{\partial\langle f\rangle}{\partial(4\mu_i - 2h_i)} = \langle f\varrho_B(i)\rangle - \langle f\rangle\,\langle\varrho_B(i)\rangle \leqq 0$$

the value $\langle f\rangle$ decreases further and when $4\mu_i - 2h_i = +\infty$ we reach a situation in which the boundary sites involved in the transformation are occupied by B-particles.

Clearly the above arguments imply:

$$\langle f|+, \Lambda\rangle \geqq \langle f|b, \Lambda\rangle \geqq \langle f|-, \Lambda\rangle \tag{A.3}$$

Appendix B — Proof of (2.11)

By definition:

$$n_{l_1,l_2}(\boldsymbol{x}_1, \dots \boldsymbol{x}_{l_1}, \boldsymbol{y}_1 \dots \boldsymbol{y}_{l_2}) \tag{B.1}$$

$$= z_A^{l_1} z_B^{l_2} \frac{\sum\limits_{m,n}^{0,\infty} z_A^m z_B^n \int \frac{d\boldsymbol{\eta}_1 \dots d\boldsymbol{\eta}_n}{n!} \frac{d\boldsymbol{\xi}_1 \dots d\boldsymbol{\xi}_m}{m!} \exp[-U(\boldsymbol{x}_1 \dots \boldsymbol{x}_{l_1}, \boldsymbol{\xi}_1 \dots \boldsymbol{\xi}_m, \boldsymbol{y}_1 \dots \boldsymbol{y}_{l_2}, \boldsymbol{\eta}_1 \dots \boldsymbol{\eta}_n)]}{\sum\limits_{m,n}^{0,\infty} z_A^m z_B^n \int \frac{d\boldsymbol{\eta}_1 \dots d\boldsymbol{\eta}_n}{n!} \frac{d\boldsymbol{\xi}_1 \dots d\boldsymbol{\xi}_m}{m!} \exp[-U(\boldsymbol{\xi}_1 \dots \boldsymbol{\xi}_m, \boldsymbol{\eta}_1 \dots \boldsymbol{\eta}_n)]}$$

where

$$U(\xi_1 \dots \xi_m, \eta_1 \dots \eta_n) = \sum_{i=1}^{m} \sum_{j=1}^{n} V_{A,B}(\xi_i - \eta_j)$$

and therefore:

$$U(x_1 \dots x_{l_1}, \xi_1 \dots \xi_m, y_1 \dots y_{l_2}, \eta_1 \dots \eta_n)$$
$$\geqq U(x_1 \dots x_{l_1}, y_1 \dots y_{l_2}) + U(\xi_1 \dots \xi_m, \eta_1 \dots \eta_n)$$

Hence:

$$n_{l_1,l_2}(x_1 \dots x_{l_1}, y_1 \dots y_{l_2}) \leqq z_A^{l_1} z_B^{l_2} \exp\left[-\sum_{i=1}^{l_1} \sum_{j=1}^{l_2} V_{AB}(x_i - y_i)\right] \leqq z_A^{l_1} z_B^{l_2} \quad \text{(B.2)}$$

Appendix C — Proof of (3.12)

We treat only the simple case $\langle I^n(X, A, \omega)\rangle$. The general case can be treated along the same lines.

The probability for finding inside ω exactly p A-particles is given, in terms of the n_{l_1,l_2}, by

$$P(p, \omega) = \sum_{m=0}^{\infty} \int_{\omega} n_{p+m_1 0}(x_1 \dots x_p, x'_1 \dots x'_m)(-1)^m \frac{dx_1 \dots dx_p}{p!} \frac{dx'_1 \dots dx'_m}{m!}$$
$$\leqq \sum_{m=0}^{\infty} \frac{|\omega|^{m+p}}{m!\,p!} z_A^{m+p} = \frac{|\omega|^p z_A^p}{p!} e^{|\omega| z_A} \quad \text{(C.1)}$$

the first equality in (C.1) has an obvious probabilistic meaning [7].

Hence it follows that:

$$\langle I^n(X, A, \omega)\rangle = \sum_{p \geqq n} P(p, \omega) \leqq e^{|\omega| z_A} \left(\sum_{p \geqq n} \frac{|\omega|^p z_A^p}{p!}\right) \quad \text{(C.2)}$$

Appendix D — Proof of (3.13)

This proof follows the usual steps (see for instance [8]). Using the definitions (B.1) and (B.2) we find:

$$n_{l_1,l_2}(x_1 \dots x_{l_1}; y_1 \dots y_{l_2}) = z_A^{l_1} z_B^{l_2} \left(\exp\left[-\sum_{i=1}^{l_1} \sum_{j=1}^{l_2} V_{AB}(x_i \dots y_j)\right]\right.$$
$$\cdot \sum_{m,n} z_A^m z_B^n \int \frac{d\xi_1 \dots d\xi_m}{m!} \frac{d\eta_1 \dots d\eta_n}{n!} \quad \text{(D.1)}$$
$$\cdot \exp\left[-\sum_{i=1}^{l_1} \sum_{j=1}^{n} V_{AB}(x_i - \eta_j) - \sum_{i=1}^{l_2} \sum_{j=1}^{m} V_{AB}(y_i - \xi_j)\right]$$

Define next

$$K(\boldsymbol{x}_1 \ldots \boldsymbol{x}_{l_1} | \boldsymbol{\eta}_1 \ldots \boldsymbol{\eta}_p) = \prod_{j=1}^{p} \left(\exp\left[-\sum_{i=1}^{l_1} V_{AB}(\boldsymbol{x}_i - \boldsymbol{\eta}_j) \right] - 1 \right) \tag{D.2}$$

and notice

$$\begin{aligned} &\exp\left[-\sum_{i=1}^{l_1} \sum_{j=1}^{n} V_{AB}(\boldsymbol{x}_i - \boldsymbol{\eta}_j) - \sum_{i=1}^{l_2} \sum_{j=1}^{m} V_{AB}(\boldsymbol{y}_i - \boldsymbol{\xi}_j) \right] \\ &= \sum_{s} \sum_{(i_1 \ldots i_s)} K(\boldsymbol{x}_1 \ldots \boldsymbol{x}_{l_1} | \boldsymbol{\eta}_{i_1} \ldots \boldsymbol{\eta}_{i_s}) \cdot \sum_{t} \sum_{(f_1 \ldots f_t)} K(\boldsymbol{y}_1 \ldots \boldsymbol{y}_{l_2} | \boldsymbol{\xi}_{j_1} \ldots \boldsymbol{\xi}_{j_t}) \end{aligned} \tag{D.3}$$

Combining this identity with (D.1) and properly interchanging sums and integrations one finds: (3.13).

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