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A Continuity Property of the Entropy Density for Spin Lattice Systems

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Abstract. The entropy density of spin lattice systems is known to be a weak* upper semi-continuous functional on the set of the lattice invariant states. (It is even weak* discontinuous.) However we prove here that it is continuous with respect to the norm topology on those states.

I. Preliminaries

We consider a lattice $\mathbb{Z}^d$ of N spin states per lattice site. By $\Lambda \subset \mathbb{Z}^d$ we will always mean a non-empty finite volume and by $V(\Lambda)$ the number of points in it.

To $\Lambda \subset \mathbb{Z}^d$ we associate the local algebra $\mathscr{A}_\Lambda$ of observables:

$$\mathscr{A}_\Lambda = \mathscr{B}(\mathscr{H}_\Lambda) \quad \text{where} \quad \mathscr{H}_\Lambda = \bigotimes_{i \in \Lambda} \mathscr{H}_i \quad \text{and each} \quad \mathscr{H}_i$$

is an isomorphic copy of the N-dimensional Hilbert space $\mathbb{C}^N$. For $\Lambda_1 \subset \Lambda_2$ we trivially get an isometric embedding of $\mathscr{A}_{\Lambda_1}$ in $\mathscr{A}_{\Lambda_2}$ which maps A into $A \otimes 1_{\Lambda_2 \setminus \Lambda_1}$. This allows us to construct the C^*-algebra $\mathscr{A}$ of quasilocal observables:

$$\mathscr{A} = \overline{\bigcup_{\Lambda \subset \mathbb{Z}^d} \mathscr{A}_\Lambda}^{\,n} .$$

The natural translation mappings $\tau_x : \mathscr{A}_\Lambda \to \mathscr{A}_{\Lambda + x}$, $x \in \mathbb{Z}^d$ extend to a group $\{\tau_x \mid x \in \mathbb{Z}^d\}$ of automorphisms of $\mathscr{A}$. A state ω on $\mathscr{A}$ is called lattice invariant if $\omega \circ \tau_x = \omega$ for all $x \in \mathbb{Z}^d$. Let $\mathscr{E}$ denote the set of all lattice invariant states on $\mathscr{A}$.

For each state ω on $\mathscr{A}$ and for each $\Lambda \subset \mathbb{Z}^d$ there exists a unique density matrix $\varrho_\Lambda \in \mathscr{B}(\mathscr{H}_\Lambda)$ such that $\forall\, A \in \mathscr{A}_\Lambda \;\; \omega(A) = \operatorname{Tr} \varrho_\Lambda A$.

The local entropy density of the state ω is given by

$$s_\Lambda(\omega) = -\frac{1}{V(\Lambda)} \operatorname{Tr} \varrho_\Lambda \log \varrho_\Lambda .$$

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The existence of $s(\omega)=\lim_{\Lambda\to\infty} s_\Lambda(\omega)$ has been proved for $\omega\in\mathscr{E}$ in the sense of growing cubes [1].

The continuity properties of the map $\omega\to s(\omega)$ are important with respect to the variational formulation of statistical mechanics [1] and explicit calculations of the entropy density [2]. It has been proved that the map is upper semi-continuous for the weak* topology on $\mathscr{E}$ [1]; however it can easily be proved that it is not weak* continuous [2]. Here we prove the norm continuity.

II. A Continuity Property of the Entropy Density

Lemma 1 (Lidskii [3]). *Let A, B be selfadjoint operators on the n-dimensional Hilbert space H. Denote by α_k, β_k and γ_k, $k=1,\ldots,n$ the repeated eigenvalues of A, B and $A-B$ in ascending order. Then there exist numbers σ_{kj}, $k,j=1,\ldots,n$ such that:*

1) $\sigma_{kj}\geqq 0$.
2) $\sum_k \sigma_{kj}=\sum_j \sigma_{kj}=1$.
3) $\alpha_k-\beta_k=\sum_j \sigma_{kj}\gamma_j$.

Lemma 2. *The function $f:[0,1]\to[0,1/e]: x\to f(x)=-x\log x$ satisfies the inequality:*

$$|f(x)-f(y)|\leqq 2|y-x|+f(|y-x|)\,.$$

Proof. For $0\leqq x\leqq y\leqq 1$ we have:

$$\begin{aligned}
&|f(x)-f(y)|\\
&=|-x(\log x-1)+y(\log y-1)+y-x|\\
&\leqq|-x(\log x-1)+y(\log y-1)|+y-x\\
&=\left|\int_x^y dt\,\log t\right|+y-x\\
&=-\int_x^{x+(y-x)} dt\,\log t+y-x\\
&\leqq-\int_0^{y-x} dt\,\log t+y-x=2(y-x)+f(y-x)\,. \quad \square
\end{aligned}$$

Theorem. *The function $s:\mathscr{E}\to[0,\log N]:\omega\to s(\omega)$ is continuous with respect to the norm topology on $\mathscr{E}$.*

Proof. Let $\omega_1,\omega_2\in\mathscr{E}$ and $\varrho_\Lambda^1,\varrho_\Lambda^2$, $\Lambda\subset\mathbb{Z}^d$ the corresponding density matrices.

Denote by $\lambda_k^1, \lambda_k^2, \mu_k$, $k=1, \ldots, N^{V(\Lambda)}$ the repeated eigenvalues of $\varrho_\Lambda^1, \varrho_\Lambda^2$ and $\varrho_\Lambda^1 - \varrho_\Lambda^2$ in ascending order. By Lemma 1

$$\begin{aligned}\sum_k |\lambda_k^1 - \lambda_k^2| &= \sum_k |\sum_j \sigma_{kj}\mu_j| \leqq \sum_k \sum_j \sigma_{kj}|\mu_j| \\ &= \sum_j |\mu_j| = \operatorname{Tr}|\varrho_\Lambda^1 - \varrho_\Lambda^2| = \sup_{\substack{\|A\| \leqq 1 \\ A \in \mathscr{A}_\Lambda}} |\operatorname{Tr}(\varrho_\Lambda^1 - \varrho_\Lambda^2) A| \\ &= \sup_{\substack{\|A\| \leqq 1 \\ A \in \mathscr{A}_\Lambda}} |(\omega_1 - \omega_2)(A)| \leqq \sup_{\substack{\|A\| \leqq 1 \\ A \in \mathscr{A}}} |(\omega_1 - \omega_2)(A)| \\ &= \|\omega_1 - \omega_2\| .\end{aligned} \tag{1}$$

Using Lemma 2

$$\begin{aligned}|s_\Lambda(\omega_1) - s_\Lambda(\omega_2)| &= \frac{1}{V(\Lambda)} \left|\sum_k f(\lambda_k^1) - f(\lambda_k^2)\right| \\ &\leqq \frac{1}{V(\Lambda)} \sum_k |f(\lambda_k^1) - f(\lambda_k^2)| \\ &\leqq \frac{1}{V(\Lambda)} \sum_k 2|\lambda_k^1 - \lambda_k^2| + f(|\lambda_k^1 - \lambda_k^2|) .\end{aligned}$$

Let $\varepsilon_k = |\lambda_k^1 - \lambda_k^2|$, then

$$|s_\Lambda(\omega_1) - s_\Lambda(\omega_2)| \leqq \frac{1}{V(\Lambda)} \sum_k 2\,\varepsilon_k + f(\varepsilon_k)$$

and by (1)

$$\sum_k \varepsilon_k = a \leqq \|\omega_1 - \omega_2\| .$$

The case $a=0$ implies $\varepsilon_k = 0$ for all k and hence

$$|s_\Lambda(\omega_1) - s_\Lambda(\omega_2)| = 0 .$$

If $a > 0$ then by the convexity of the logarithm

$$\begin{aligned}\frac{1}{V(\Lambda)} \sum_k 2\,\varepsilon_k + f(\varepsilon_k) &= \frac{2a}{V(\Lambda)} + \frac{a}{V(\Lambda)} \sum_{\varepsilon_k \neq 0} \frac{\varepsilon_k}{a} \log \frac{1}{\varepsilon_k} \\ &\leqq \frac{2a}{V(\Lambda)} + \frac{a}{V(\Lambda)} \log \sum_{\varepsilon_k \neq 0} \frac{1}{a} \leqq \frac{2a}{V(\Lambda)} + \frac{a}{V(\Lambda)} \log\left(\frac{N^{V(\Lambda)}}{a}\right) \\ &= \frac{2a - a\log a}{V(\Lambda)} + a \log N .\end{aligned}$$

Hence

$$\begin{aligned}|s(\omega_1) - s(\omega_2)| &= \lim_{\Lambda \to \infty} |s_\Lambda(\omega_1) - s_\Lambda(\omega_2)| \\ &\leqq \|\omega_1 - \omega_2\| \log N . \quad \square\end{aligned}$$

References

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