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Return to Equilibrium

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Abstract. The problem of return to equilibrium is phrased in terms of a C^*-algebra $\mathfrak{A}$, and two one-parameter groups of automorphisms τ, τ^P corresponding to the unperturbed and locally perturbed evolutions. The asymptotic evolution, under τ, of τ^P-invariant, and τ^P-K.M.S., states is considered. This study is a generalization of scattering theory and results concerning the existence of limit states are obtained by techniques similar to those used to prove the existence, and intertwining properties, of wave-operators. Conditions of asymptotic abelianness provide the necessary dispersive properties for the return to equilibrium. It is demonstrated that the τ^P-equilibrium states and their limit states are coupled by automorphisms with a quasi-local property; they are not necessarily normal with respect to one another. An application to the $X-Y$ model is given which extends previously known results and other applications, and examples, are given for the Fermi gas.

I. Introduction

We examine general properties of systems whose dynamics have been locally perturbed and illustrate these properties with examples. Our specific interest is whether systems, that have been perturbed in this manner, return to equilibrium under the unperturbed evolution. In this context we consider the behaviour of states which are invariant, or satisfy the K.M.S. condition, for the perturbed dynamics. We demonstrate that this type of problem is tractable with methods which are a natural generalization of scattering theory.

For simplicity of formulation we work in an algebraic setting and assume that the kinematic observables of our system form a C^*-algebra $\mathfrak{A}$. The dynamics is specified by a one-parameter group of automorphisms τ of $\mathfrak{A}$ which we take to be strongly continuous, i.e.

$$\|\tau_t(A) - A\| \xrightarrow[|t|=0]{} 0 .$$

These assumptions could be weakened at the cost of introducing more detailed structure which is not directly relevant to the problem under discussion.

In Section 2 we define a second group of automorphisms τ^P of $\mathfrak{A}$ which is to be considered as a group arising from a local perturbation of the Hamiltonian. We point out some properties which are stable under this perturbation and some that are not. In Section 3 we consider the evolution under τ of states which are τ^P-invariant, or satisfy the K.M.S. condition with respect to τ^P. In Section 4 we discuss the notion of quasilocal automorphisms of $\mathfrak{A}$ and their relevance to the problems of return to equilibrium. We give applications of our results to the $X-Y$ model and the Fermi gas in Section 5.

II. Local Dynamical Perturbations

In the following $\mathfrak{A}$ is always a C^*-algebra with identity and τ a strongly continuous one-parameter group of automorphisms of $\mathfrak{A}$.

Definition 1. *Given the pair $(\mathfrak{A}, \tau)$ and $P = P^* \in \mathfrak{A}$ the perturbed evolution τ^P is defined, as a strongly continuous one-parameter group of automorphisms of $\mathfrak{A}$, by the uniformly convergent series expansions*

$$\tau_t^P(A) = \tau_t(A) + \sum_{n \geqq 1} i^n \int_{0 \leqq s_1 \leqq \cdots \leqq s_n \leqq t} ds_1 \dots ds_n \left[\tau_{s_1}(P), \left[\dots \left[\tau_{s_n}(P), \tau_t(A)\right]\right]\right]$$

and

$$\tau_t^P(A) = \tau_t(A) + \sum_{n \geqq 1} i^n \int_{t \leqq s_n \leqq \cdots \leqq s_1 \leqq 0} ds_1 \dots ds_n \left[\tau_{s_1}(P), \left[\dots \left[\tau_{s_n}(P), \tau_t(A)\right]\right]\right]$$

for all $A \in \mathfrak{A}$ and for $t \geqq 0$, and $t \leqq 0$, respectively.

Note that the n-th term of these series is bounded by $(2|t|\,\|P\|)^n \|A\|/n!$ which ensures the convergence and allows one to check that τ^P is a group of automorphisms etc. Alternatively one can calculate from these series that

$$\frac{d}{dt} \tau_t^P \tau_{-t}(A) = i \tau_t^P \tau_{-t}([\tau_t(P), A])$$

and hence deduce the integral relations

$$\tau_{t_2}^P \tau_{-t_2}(A) - \tau_{t_1}^P \tau_{-t_1}(A) = i \int_{t_1}^{t_2} ds\, \tau_s^P \tau_{-s}([\tau_s(P), A]) \tag{$*$}$$

and

$$\tau_{t_1}^P \tau_{t_2}(A) - \tau_{t_1+t_2}(A) = i \int_0^{t_1} ds\, \tau_s^P \tau_{-s}([\tau_s(P), \tau_{t_1+t_2}(A)])\,. \tag{$**$}$$

We next show in what sense τ^P is a local perturbation of τ and simultaneously deduce that covariance of representations of $\mathfrak{A}$ is stable under such perturbations. Recall that a representation π of $\mathfrak{A}$, on a Hilbert space $\mathscr{H}_\pi$, is said to be τ-covariant if there exists a strongly continuous group of unitary operators U on $\mathscr{H}_\pi$ which implement the group of automorphisms τ in the following way

$$\pi(\tau_t(A)) = U(t)\, \pi(A)\, U(-t)\,, \qquad t \in \mathbb{R},\ A \in \mathfrak{A}\,.$$

Theorem 1. *Let* $(\mathfrak{A}, \tau, \tau^P)$ *be defined as above and let* π *be a representation of* $\mathfrak{A}$. *The following are equivalent;*

1. π *is* τ*-covariant.*
2. π *is* τ^P*-covariant.*

If these latter conditions are fulfilled and U, U_P, *denote the groups of unitaries implementing the* τ-, *and* τ^P*-automorphisms, respectively, then the infinitesimal generators* H, *and* H_P, *of these groups satisfy the relation*

$$H_P = H + \pi(P).$$

Proof. To establish the equivalence it suffices to prove that condition 1 implies condition 2 because τ can be expressed as a perturbation of the evolution τ^P. But to show that 1 implies 2 it is sufficient to construct U_P in terms of U and π. We sketch this construction for $t \geqq 0$, the case $t \leqq 0$ is similar.

Consider the integral equation

$$U_P(t) = U(t) + i \int_0^t ds\, U_P(s)\, \pi(P)\, U(-s)\, U(t).$$

This can be solved by iteration and one finds

$$U_P(t) = W(t)\, U(t)$$

where

$$W(t) = 1 + \sum_{n \geqq 1} i^n \int_{0 \leqq s_1 \leqq \cdots \leqq s_n \leqq t} ds_1 \dots ds_n \pi(\tau_{s_1}(P)) \dots \pi(\tau_{s_n}(P)).$$

It is then easy to check that

$$U_P(t)\, \pi(A)\, U_P(-t) = \pi(\tau_t^P(A)), \qquad A \in \mathfrak{A},\ t \in \mathbb{R}_+ .$$

The proof of the second statement of the theorem can be extracted from [1], pages 496—497.

In the foregoing there is a symmetry between τ and τ^P, each group is a local perturbation of the other with perturbations P, and $-P$, respectively. There are, however, many properties of the pair $(\mathfrak{A}, \tau)$ which are not stable under perturbations of the above kind and hence introduce an asymmetry between τ and τ^P. We mention two such properties which are of use in the analysis of the return to equilibrium of perturbed states.

There exist triplets $(\mathfrak{A}, \tau, \tau^P)$ *such that* $(\mathfrak{A}, \tau)$ *is asymptotically abelian, i.e.*

$$\|[A, \tau_t(B)]\| \xrightarrow[|t| = \infty]{} 0, \qquad A, B \in \mathfrak{A}$$

but $(\mathfrak{A}, \tau^P)$ *is not asymptotically abelian.*

It is easy to construct examples of this nature, several are given in Section 5. We show in the following two sections that this asymmetry has a physical origin and arises whenever the perturbation isolates a finite subsystem, either by strict isolation in space or by the formation of bound states.

Next let ω be a state over $\mathfrak{A}$ and denote by N_ω the set of states which are normal with respect to ω, i.e. the uniform closure of the convex hull of the set of states $\{\omega_A; A \in \mathfrak{A}\}$ where

$$\omega_A(B) = \omega(A^* B A)/\omega(A^* A), \qquad B \in \mathfrak{A}, \ \omega(A^* A) > 0 .$$

The set of states N_ω is often viewed as the set of local perturbations of ω. It is known that these states have properties of return to equilibrium under quite general circumstances (see for example [2], Chapter VI). We cite, as an illustration, the result that if ω is a τ-invariant factor state, $(\mathfrak{A}, \tau)$ is asymptotically abelian, and $\omega' \in N_\omega$ then it follows that

$$\lim_{|t| \to \infty} \omega'(\tau_t(A)) = \omega(A) .$$

It might appear that this property would be of use in studying the evolution, under τ, of τ^P-invariant states. In general the situation is however more complicated.

There exist triplets $(\mathfrak{A}, \tau, \tau^P)$, with $(\mathfrak{A}, \tau)$ asymptotically abelian, and a τ^P-invariant state ω' such that ω' is not contained in any set N_ω associated with a τ-invariant state ω in the above manner.

The set of states one is led to consider in the framework of dynamical perturbations is not directly connected to the set of normal states but is related to sets obtained by the action of certain automorphisms which we discuss in Sections 3 and 4.

They are two physical reasons why the perturbed and unperturbed equilibrium states are not normal with respect to each other and these can be roughly explained as follows. As the perturbation P is bounded the states differ by a finite amount of energy. This energy can be transferred either by a finite number of particles or an infinite number of particles with infinitesimally small energy. In the first case normality is usually respected but non-normality can arise if there is a superselection rule in the theory. In the second case the infra-red phenomena is expected to destroy normality. Examples of both these phenomena are given in Section 5.

III. Return to Equilibrium

In this section we examine the evolution under τ of states which are τ^P-invariant. Firstly we consider a special subclass of such states, the states that satisfy the Kubo-Martin-Schwinger (K.M.S.) condition with respect to τ^P. To introduce this condition it is necessary to extend the automorphisms τ, and τ^P, to imaginary times. We denote by $\tilde{\mathfrak{A}}$, and $\tilde{\mathfrak{A}}^P$ the dense τ- and τ^P-invariant, sub $*$-algebras of $\mathfrak{A}$ formed by the τ-, and τ^P, analytic elements (cf. [3] and [4]; we follow Definition 1 of [4]). For simplicity we take $P \in \tilde{\mathfrak{A}}$, then the definition of τ^P implies that $\hat{\mathfrak{A}} = \hat{\mathfrak{A}}^P$ and the extension of τ^P to positive imaginary times is given by

$$\tau^P_{i\beta}(A) = \tau_{i\beta}(A) + \sum_{n \geqq 1} \int_{0 \leqq s_1 \leqq \cdots \leqq s_n \leqq t} ds_1 \ldots ds_n [\tau_{is_1}(P), [\ldots [\tau_{is_n}(P), \tau_{i\beta}(A)]]]$$

for

$$A \in \hat{\mathfrak{A}} \quad \text{and} \quad \beta \geqq 0 .$$

The state ω is defined to satisfy the τ^P-K.M.S. condition, at inverse temperature $\beta \geqq 0$, if

$$\omega(\tau^P_{i\beta}(A)\, B) = \omega(BA), \qquad A, B \in \tilde{\mathfrak{A}}^P .$$

If this condition is satisfied then ω is automatically τ^P-invariant [5]. A similar definition and conclusion holds for τ.

Theorem 2. *Let ω be a state over $\mathfrak{A}$ satisfying the τ^P-K.M.S. condition at inverse temperature β, where $P \in \tilde{\mathfrak{A}}$. Further let $\omega_\pm$ denote weak $*$ limit points of $\tau_t\omega$ as t tends to $\pm\infty$.*

If $(\mathfrak{A}, \tau)$ is asymptotically abelian it follows that $\omega_\pm$ satisfy the τ-K.M.S. condition, at inverse temperature β. In particular, if there is a unique state ω_β with this latter property then it follows that

$$\omega_\beta(A) = \lim_{|t| \to \infty} \omega(\tau_t(A)), \qquad A \in \mathfrak{A} .$$

Proof. Take $A, B \in \hat{\mathfrak{A}} \subset \tilde{\mathfrak{A}}^P$. From $(**)$ and the τ^P-invariance of ω we have

$$\begin{aligned} &|\omega(\tau^P_{i\beta}(\tau_t(A))\, \tau_t(B)) - \omega(\tau_{t+i\beta}(A)\, \tau_t(B))| \\ &= \left| \int_0^\beta ds\, \omega(\tau^P_{is}([P, \tau_{t+i(\beta-s)}(A)])\, \tau_t(B)) \right| \\ &= \left| \int_0^\beta ds\, \omega([P, \tau_{t+i(\beta-s)}(A)]\, \tau^P_{-is}\tau_t(B)) \right| . \end{aligned}$$

But using the τ^P-K.M.S. condition for ω we then find

$$\begin{aligned} &|\omega(\tau_t(\tau_{i\beta}(A)\, B)) - \omega(\tau_t(BA))| \\ &\leqq \int_0^\beta ds\, \|[P, \tau_{t+i(\beta-s)}(A)]\|\, \|\tau^P_{-is}\tau_t(B)\| . \end{aligned}$$

The first factor of the integrand converges to zero as $|t| \to \infty$, because $(\mathfrak{A}, \tau)$ is asymptotically abelian, and the second factor has a bound, uniform in t, which can be calculated from the definition of τ^P using

$B \in \tilde{\mathfrak{A}}$. Thus the first conclusion of the theorem is established. The second conclusion follows because the assumed uniqueness ensures that all limit points $\omega_{\pm}$ are identical, i.e. the limit exists.

Remark. A slight elaboration of this proof establishes the more general result that if ω' is a state normal with respect to a τ^P-K.M.S. state, i.e. $\omega' \in N_{\omega}$, then all weak * limit points of $\tau_t \omega'$ as $|t| \to \infty$ are τ-K.M.S.

The existence of the weak * limit of $\tau_t \omega$ is not established in the above theorem, except by the implicit assumption of a unique τ-K.M.S. state. It is unclear whether asymptotic abelianness of $(\mathfrak{A}, \tau)$ is sufficient to establish that these limits exist but conditions of this type can be given which are both necessary and sufficient.

Theorem 3. *Take $P = P^* \in \mathfrak{A}$ and let ω be a τ^P-invariant state. The following pairs of conditions are equivalent.*

$1_{\pm}$) *The limit*

$$\omega_{\pm}(A) = \lim_{t \to \pm\infty} \omega(\tau_t(A))$$

exists for all $A \in \mathfrak{A}$.

$2_{\pm}$) *The function*

$$t \in \mathbb{R} \to \omega([P, \tau_t(A)]) \in \mathbb{C}$$

is integrable at $\pm\infty$ for all A in some uniformly dense subset $\hat{\mathfrak{A}}$ of $\mathfrak{A}$.

If these conditions are satisfied then

$$\omega_+(A) - \omega_-(A) = i \int_{-\infty}^{\infty} dt\, \omega([P, \tau_t(A)]), \qquad A \in \hat{\mathfrak{A}}$$

and $\omega_+ = \omega_-$ if, and only if, the integral on the right hand side is zero for all $A \in \hat{\mathfrak{A}}$.

Proof. The cases $t > 0$ and $t < 0$ are similar; consider the former. The τ^P-invariance of ω and $(*)$ imply that

$$\begin{aligned} \omega(\tau_{t_2}(A)) - \omega(\tau_{t_1}(A)) &= \omega(\tau^P_{-t_2} \tau_{t_2}(A)) - \omega(\tau^P_{-t_1} \tau_{t_1}(A)) \\ &= i \int_{t_1}^{t_2} dt\, \omega([P, \tau_t(A)]) \end{aligned}$$

for all $A \in \mathfrak{A}$. The equivalence of 1_+) and 2_+) follows immediately. If 2_+) and 2_-) are valid the relation between ω_+ and ω_- is established by the same method.

The conditions of integrability introduced in the above theorem are not usually of great practical use because they involve the state ω. In certain examples (see Section 5) stronger conditions of integrability are valid. We introduce these conditions into our general framework by the following definition.

Definition 2. *The pair* $(\mathfrak{A}, \tau)$ *is defined to be asymptotically integrable with respect to* $P \in \mathfrak{A}$ *if the function*

$$t \in \mathbb{R} \mapsto \|[P, \tau_t(A)]\| \in \mathbb{R}$$

is integrable at infinity for all A *in some uniformly dense subset* $\hat{\mathfrak{A}}$ *of* $\mathfrak{A}$.

This condition is of interest for a number of reasons. If it is satisfied then the problems under discussion are simplified by the following purely algebraic result.

Theorem 4. *Take* $P = P^* \in \mathfrak{A}$ *and assume that* $(\mathfrak{A}, \tau)$ *is asymptotically integrable with respect to* P. *It follows that the limits*

$$\gamma_\pm(A) = \lim_{t \to \pm\infty} \tau^P_{-t}\tau_t(A), \qquad A \in \mathfrak{A}$$

exist in the uniform topology on $\mathfrak{A}$ *and define epi-morphisms of* $\mathfrak{A}$. *These epi-morphisms* $\gamma_\pm$ *satisfy the intertwining relations*

$$\gamma_\pm \tau_t = \tau^P_t \gamma_\pm .$$

Proof. If $A \in \hat{\mathfrak{A}}$ we deduce directly from (*) that

$$\begin{aligned} &\|\tau^P_{-t_2}\tau_{t_2}(A) - \tau^P_{-t_1}\tau_{t_1}(A)\| \\ &\leqq \int_{t_1}^{t_2} ds \, \|[P, \tau_s(A)]\| \end{aligned}$$

and the existence of $\gamma_\pm$ follows immediately. It is straightforward to check that $\gamma_\pm$ preserve algebraic relations, e.g.

$$\begin{aligned} \gamma_\pm(A+B) &= \gamma_\pm(A) + \gamma_\pm(B) \\ \gamma_\pm(AB) &= \gamma_\pm(A)\,\gamma_\pm(B) \end{aligned}$$

etc. but these mappings fail, in general, to be automorphisms of $\mathfrak{A}$ because their range is not necessarily dense in $\mathfrak{A}$, i.e. the mappings

$$A \in \mathfrak{A} \mapsto \gamma_\pm(A) \in \mathfrak{A}$$

are not necessarily invertible.

The intertwining property follows from the definition of $\gamma_\pm$ because

$$\begin{aligned} \gamma_\pm(\tau_t(A)) &= \lim_{s \to \pm\infty} \tau^P_{-s}\tau_{t+s}(A) \\ &= \lim_{s \to \pm\infty} \tau^P_t \tau^P_{-s}\tau_s(A) \\ &= \tau^P_t \gamma_\pm(A) . \end{aligned}$$

The epi-morphisms introduced by the above theorem are the algebraic analogues of the wave operators of scattering theory. These latter operators are in general isometric and are only invertible, i.e. unitary, in the absence of bound states. The lack of invertibility of the $\gamma_\pm$ can be understood by analogy. We will return to this point in the sequel.

The epi-morphisms $\gamma_\pm$ lead by transposition to mappings $\omega \mapsto \omega_{\gamma_\pm}$ of the states of $\mathfrak{A}$, i.e.

$$\omega_{\gamma_\pm}(A) = \omega(\gamma_\pm(A)).$$

If ω is a τ^P-invariant state then

$$\omega_{\gamma_\pm}(A) = \lim_{t \to \pm\infty} \omega(\tau_t(A)), \qquad A \in \mathfrak{A}.$$

Thus $\omega_{\gamma_\pm}$ are τ-invariant. This mapping conserves ergodicity properties.

Theorem 5. *Consider the same assumptions as Theorem 4 and let ω be a τ^P-ergodic (τ^P-weakly mixing), (τ^P-strongly clustering) state. The limit states*

$$\omega_{\gamma_\pm}(A) = \lim_{t \to \pm\infty} \omega(\tau_t(A)), \qquad A \in \mathfrak{A}$$

exist and are τ-ergodic, (τ-weakly mixing), (τ-strongly clustering).

Proof. Recall that ω is τ^P-ergodic, (τ^P-weakly mixing), (τ^P-strongly clustering) if the function

$$t \in \mathbb{R} \mapsto \omega(A\tau_t^P(B)) - \omega(A)\,\omega(B)$$

has mean value zero, (its modulus has mean value zero), (it tends to zero as $|t| \to \infty$) for all $A, B \in \hat{\mathfrak{A}}$. The properties of τ-invariant states are similarly characterized.

From $(*)$ and the definition of ω_{γ_+} one finds

$$|\omega_{\gamma_+}(A\tau_t(B)) - \omega(\tau_s(A)\,\tau_{t+s}(B))|$$
$$\leqq \|A\| \int_{s+t}^{\infty} dr\,\|[P, \tau_r(B)]\| + \|B\| \int_{s}^{\infty} dr\,\|[P, \tau_r(A)]\|.$$

But from $(**)$ one concludes that

$$|\omega(\tau_s(A)\,\tau_{t+s}(B)) - \omega(\tau_s(A)\,\tau_t^P\,\tau_s(B))|$$
$$\leqq \|A\| \int_{s}^{s+t} dr\,\|[P, \tau_r(B)]\|.$$

Thus

$$|\omega_{\gamma_+}(A\tau_t(B)) - \omega(\tau_s(A)\,\tau_t^P\,\tau_s(B))|$$
$$\leqq \|A\| \int_{s}^{\infty} dr\,\|[P, \tau_r(B)]\| + \|B\| \int_{s}^{\infty} dr\,\|[P, \tau_r(A)]\|.$$

Hence if ω is τ^P-ergodic

$$\left|\lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} dt\,\{\omega_{\gamma_+}(A\tau_t(B)) - \omega(\tau_s(A))\,\omega(\tau_s(B))\}\right|$$
$$\leqq \|A\| \int_{s}^{\infty} dr\,\|[P, \tau_r(B)]\| + \|B\| \int_{s}^{\infty} dr\,\|[P, \tau_r(A)]\|.$$

Taking the limit $s \to \infty$ one deduces that ω_{γ_+} is τ-ergodic. The other statements of the theorem follow in a similar manner.

Although the condition that $(\mathfrak{A}, \tau)$ is asymptotically integrable with respect to P simplifies the discussion of the evolution of τ^P-invariant states it does not shed any light on the question whether

$$\omega_{\gamma_+} = \omega_{\gamma_-} .$$

The only criterion that we have developed for this equality is that of Theorem 2, i.e. the uniqueness of the τ-K.M.S. state.

Finally we give a criterion for the stability of asymptotic abelianness. It is clear that $(\mathfrak{A}, \tau^P)$ cannot be asymptotically abelian (except if $\mathfrak{A}$ is abelian) when the automorphism group τ^P has fixed points, i.e. there are $A \in \mathfrak{A}$ such that $\tau_t^P(A) = A$, $t \in \mathbb{R}$. In this case it is also evident that the epi-morphisms $\gamma_\pm$ are not invertible; the fixed points of τ^P are not in the range of $\gamma_\pm$. This presence of fixed points is the analogue for an infinite system of the existence of bound states in scattering theory. It corresponds to an isolation of a subsystem, the subsystem described by the algebra of observables constituting the fixed points of τ^P.

Theorem 6. *Let $(\mathfrak{A}, \tau, \tau^P)$ be such that the following limits exist in the uniform topology of $\mathfrak{A}$*

$$\gamma_\pm(A) = \lim_{t \to \pm\infty} \tau_{-t}^P \tau_t(A), \qquad A \in \mathfrak{A} .$$

If the epi-morphisms $\gamma_\pm$ defined in this manner are invertible, i.e. if $\gamma_\pm$ are automorphisms, then the following conditions are equivalent

1. $(\mathfrak{A}, \tau)$ *is asymptotically abelian;*
2. $(\mathfrak{A}, \tau^P)$ *is asymptotically abelian.*

Proof. The proof is a consequence of the intertwining property of the $\gamma_\pm$. One has

$$[A, \tau_t^P(B)] = [A, \gamma_\pm \tau_t \gamma_\pm^{-1}(B)] .$$

Thus

$$\|[A, \tau_t^P(B)]\| = \|[\gamma_\pm^{-1}(A), \tau_t(\gamma_\pm^{-1}(B))]\| .$$

Remark. We can rephrase this in a slightly more general form. The ranges of $\gamma_\pm$ in $\mathfrak{A}$ form C^*-subalgebras $\mathfrak{A}_\pm$ of $\mathfrak{A}$. Asymptotic abelianness of $(\mathfrak{A}, \tau)$ implies asymptotic abelianness of $(\mathfrak{A}_\pm, \tau^P)$. The $\mathfrak{A}_\pm$ correspond to the dispersive observables of the perturbed system.

IV. Quasi-local Automorphisms

In this section we discuss a different question to that of the previous section. We have seen that under a variety of circumstances τ^P-invariant states ω have limits ω_+ under the evolution τ. Essentially this arises because the ω are, in some sense, "local perturbations" of the limit states $\omega_\pm$. One sees from explicit examples that the ω are not necessarily vector states of the $\omega_\pm$ or, more generally, normal states of the $\omega_\pm$. It appears, however, that these states are often interrelated by quasi-local automorphisms defined as follows.

Definition 3. *Let γ be an automorphism or, more generally, an epimorphism of $\mathfrak{A}$. γ is defined to be τ-quasi-local if*

$$\lim_{t\to\infty} \|\gamma(\tau_t(A)) - \tau_t(A)\| = 0\,, \qquad A \in \mathfrak{A}\,.$$

The physical motivation of this definition is as follows. Under a reasonable time-evolution τ one would believe that the local observable $\tau_t(A)$ gradually delocalizes spreading throughout space. If γ is an automorphism which only changes the structure in some local region then asymptotically one would expect the difference of $\tau_t(A)$ and $\gamma(\tau_t(A))$ to be small.

There are many examples of τ-quasi-local automorphisms.

1. Let γ be an inner automorphism of $\mathfrak{A}$ and let $(\mathfrak{A}, \tau)$ be asymptotically abelian then γ is τ-quasi-local.

This follows because $\gamma(A) = UAU^{-1}$ with $U \in \mathfrak{A}$ and so

$$\|\gamma(\tau_t(A)) - \tau_t(A)\| = \|[U, \tau_t(A)]\|\,.$$

2. If for $s \in \mathbb{R}$ we define γ_s by

$$\gamma_s(A) = \tau^P_{-s}\,\tau_s(A)\,, \qquad A \in \mathfrak{A}$$

and $(\mathfrak{A}, \tau)$ is asymptotically abelian then γ_s is τ-quasi-local.

This follows from $(*)$. One has

$$\|\gamma_s(\tau_t(A)) - \tau_t(A)\| \leqq \int_0^s dr\, \|[P, \tau_{t+r}(A)]\|\,.$$

3. Let γ_+ be the epi-morphism defined in Theorem 4 by

$$\gamma_+(A) = \lim_{t\to\infty} \tau^P_{-t}\,\tau_t(A)\,, \qquad A \in \mathfrak{A}\,.$$

It follows that γ_+ is τ-quasi-local.

This follows from the intertwining property of γ_+

$$\begin{aligned}\|\gamma_+(\tau_t(A)) - \tau_t(A)\| &= \|\tau^P_t\,\gamma_+(A) - \tau_t(A)\| \\ &= \|\gamma_+(A) - \tau^P_{-t}\,\tau_t(A)\|\,.\end{aligned}$$

The useful characteristic of this kind of automorphism is that the states obtained by its action automatically enjoy a property of return to equilibrium.

Theorem 7. *Let ω be a τ-invariant state and ω_γ its image under transposition of the τ-quasi-local automorphism γ, i.e. $\omega_\gamma(A)=\omega(\gamma(A))$, $A \in \mathfrak{A}$. It follows that*

$$\lim_{t\to\infty} \omega_\gamma(\tau_t(A)) = \omega(A), \qquad A \in \mathfrak{A}.$$

Proof. The proof is a direct consequence of definition 3 and the invariance of ω. One has

$$\begin{aligned}|\omega_\gamma(\tau_t(A)) - \omega(A)| &= |\omega(\gamma(\tau_t(A))) - \omega(\tau_t(A))| \\ &\leqq \|\gamma(\tau_t(A)) - \tau_t(A)\|\end{aligned}$$

and hence the conclusion.

This is essentially the mechanism by which the results of the previous section were derived and it motivates the notion that the ω_γ are local perturbations of ω.

There are one or two properties of the set of τ-quasi-local automorphisms which are easily derived. They form a group Γ. For example one has

$$\|\gamma^{-1}(\tau_t(A)) - \tau_t(A)\| = \|\gamma(\tau_t(A)) - \tau_t(A)\|$$

and

$$\|\gamma_1\gamma_2(\tau_t(A)) - \tau\,(A)\| \leqq \|\gamma_1(\tau_t(A)) - \tau_t(A)\| + \|\gamma_2(\tau_t(A)) - \tau_t(A)\|.$$

If ω is a given state of $\mathfrak{A}$ then the set of states $\{\omega_\gamma; \gamma \in \Gamma\}$ is a set of states each of which returns to ω under the evolution τ. These states are not necessarily normal with respect to ω (Section 5) and the structural properties and interrelationships of these sets remain an open problem.

V. Examples and Applications

A. The $X-Y$ Model

This model has been studied in the context of return to equilibrium in [6] and [7, 12]. It is a model of a one-dimensional spin system for which the vector space associated with each point in $\mathbb{Z}$ is two-dimensional. The kinematics of the model is describable in terms of a C^*-algebra $\mathfrak{A}$ which is the uniform closure of a set of local algebras $\mathfrak{A}_n$ corresponding to observables in the interval $[-n, n] \subset \mathbb{Z}$. The $\mathfrak{A}_n$ are generated by spin creation and annihilation operators $a_p, a_q^*, p, q \in [-n, n]$ satisfying the commutation relations

$$[a_p, a_q^*] = 0 = [a_p, a_q], \qquad p \neq q$$

$$\{a_p, a_p^*\} = 1, \qquad \{a_p, a_p\} = 0.$$

For convenience we take $\mathfrak{A}_n$ to be the algebra of even polynomials in the a_p, a_q^*, i.e. the polynomials which are even under interchanges $a_p \to -a_p$ $a_q^* \to -a_q^*$. To emphasize this particular choice we replace the notation $\mathfrak{A}_n, \mathfrak{A}$ by $\mathfrak{A}_n^e, \mathfrak{A}^e$.

The dynamics of the $X-Y$ chain are specified by the local Hamiltonians

$$H_n = \frac{J}{2} \sum_{p=-n}^{n-1} (a_p^* a_{p+1} + a_{p+1}^* a_p) + h \sum_{p=-n}^{n} a_p^* a_p$$

where $J, h \in \mathbb{R}$. One has from [8] that the following limits exist

$$\tau_t(A) = \lim_{n\to\infty} e^{iH_n t} A e^{-iH_n t}, \qquad A \in \mathfrak{A}^e, \ t \in \mathbb{R}$$

in the uniform topology on $\mathfrak{A}^e$ and define a strongly continuous one-parameter group of automorphisms of $\mathfrak{A}^e$. The following result is an application of the results of Section 3.

Application 1

Consider the $X-Y$ model and let $P = P^*$ be strictly local, i.e. $P \in \mathfrak{A}_n^e$ for some $n \in \mathbb{Z}$. Let ω be any τ^P-K.M.S. state at inverse temperature β.

It follows that the limit

$$\omega_\beta(A) = \lim_{|t|\to\infty} \omega(\tau_t(A)), \qquad A \in \mathfrak{A}^e$$

exists and ω_β is the Gibbs equilibrium state of the unperturbed model, at inverse temperature β.

It further follows that $(\mathfrak{A}^e, \tau)$ is asymptotically integrable with respect to P and the conclusions of Theorems 4 and 5 are valid.

The first conclusion is valid because of the following three properties of the unperturbed model.

1. If P is strictly local then it is τ-analytic.
2. $(\mathfrak{A}^e, \tau)$ is asymptotically abelian.
3. The Gibbs state is the unique K.M.S. state for $(\mathfrak{A}^e, \tau)$.

These properties of the model are well-known but not all of then appear explicitly in the literature. We have indicated proofs in the Appendix together with an indication of the proof of asymptotic integrability.

The result is valid for a wider class of perturbations of a quasilocal nature but for simplicity we have avoided giving the complete characterization of this class.

Note that all these properties, and consequently the proof of the result, involve calculations with the unperturbed system only; the fact that it is obtained by methods of scattering theory demonstrates why the relaxation times obtained in [6] are so long. The relaxation is coupled to the phenomena of wave dispersion.

The $X-Y$ model also gives explicit examples of the phenomena cited in the previous sections. For example if one takes P to be defined as

$$P = -\frac{J}{2} \sum_{q=-p}^{p-1} (a_q^* a_{q+1} + a_{q+1}^* a_q) - h \sum_{q=-p}^{p} a_q^* a_q, \qquad p \geqq 1$$

then the local algebra $\mathfrak{A}^e_{p-1}$ is left pointwise invariant by the perturbed automorphism group τ^P. Hence $(\mathfrak{A}^e, \tau^P)$ is not asymptotically abelian and the $\gamma_\pm$ are not invertible.

This example also illustrates that the above result describes the return to equilibrium of an isolated sub-chain brought into contact with the outer $X-Y$ chain.

Finally note that for multi-dimensional spin systems one knows, for small β and a large class of interactions Φ, that strictly local P are τ^Φ-analytic and the Gibbs state is the unique K.M.S. state for $(\mathfrak{A}, \tau^\Phi)$ [9]. Thus if $(\mathfrak{A}, \tau^\Phi)$ is asymptotically abelian statements similar to the above can be made.

B. The Free Fermi Gas

This example is described by the quasi-local algebra associated with the anti-commutation relations (see, for Example [2], Chapter VIII). We take as underlying configuration space $\mathbb{R}^\nu$ and the free evolution τ is given by its action on the generating elements of the algebra

$$\tau_t(a(f)) = a(V_t f)$$

where

$$(V_t f)(x) = \int dp\, \tilde{f}(p)\, e^{ipx - ip^2 t}, \qquad f \in L^2(\mathbb{R}^\nu)$$

($\tilde{f}$ is the Fourier transform of f). If we choose as basic algebra the C^*-algebra $\mathfrak{A}^e$ of elements which are even in $a(f)$, $a^*(g)$ then it is easy to check that

$$\|[A, \tau_t(B)]\| \xrightarrow[|t| = \infty]{} 0, \qquad A, B \in \mathfrak{A}^e.$$

It is also known [10] that the Gibbs state is the unique τ-K.M.S, state for $\beta > 0$.

Application 2

Let $P = P^* \in \mathfrak{A}^e$ be τ-analytic and ω a τ^P-K.M.S. state at inverse temperature β. The following limit exists

$$\omega_\beta(A) = \lim_{|t| \to \infty} \omega(\tau_t(A)), \qquad A \in \mathfrak{A}^e$$

and ω_β is the Gibbs equilibrium state at inverse temperature β.

If $P = P^* \in \mathfrak{A}^e$ is strictly local then $(\mathfrak{A}^e, \tau)$ is asymptotically integrable with respect to P and the conclusions of Theorems 4 and 5 are valid.

The proof of the asymptotic integrability is given by noting that if A is a monomial in the $a(f), a^*(g)$ then for $P \in \mathfrak{A}_\Lambda$ the commutator $[P, \tau_t(A)]$ is bounded by a sum of terms of the form

$$\text{const}\left[\int_\Lambda dx\, |(V_t f)(x)|^2\right]^{\frac{1}{2}}.$$

It is a well-known property of wave-packets that if $f \in L^2(\mathbb{R}^\nu) \cap L^1(\mathbb{R}^\nu)$ then this last expression is bounded by

$$\mathrm{const}\,|t|^{-\nu/2}$$

and hence is integrable at infinity for $\nu \geqq 3$. If $\nu \leqq 3$ it is necessary to choose a restricted class of f. Namely f whose Fourier transforms vanish sufficiently smoothly at the origin. One again obtains an integrable bound.

The above application can be improved. It is not necessarily that P is τ-analytic; one only needs an analytic continuation of the functions $t \mapsto \tau_t(A)$ for $0 \leqq \operatorname{Im} t \leqq \beta$. In the second part it is not necessary that P is strictly local but can be in a much larger class of quasi-local elements.

In examples such as the Fermi gas one can also exploit the group of space translations to prove that the limit states $\omega_\pm$ as $t \to \pm\infty$ are identical. If $\omega_\pm$ are invariant under space-translations it readily follows that they are equal.

C. Quasi-Local Automorphisms

One can use the Fermi algebra to give a number of examples of phenomena cited in the foregoing. If $f \in L^2(\mathbb{R}^\nu)$ is normalized and twice differentiable one can choose a local perturbation P of the form

$$P = -\lambda[a^*(\nabla^2 f)\,a(f) + a^*(f)\,a(\nabla^2 f)]\,.$$

One easily checks that

$$\tau_t^P(a(f)) = \exp\{-it(f, \nabla^2 f)\}\,a(f)\,.$$

Hence $(\mathfrak{A}, \tau^P)$ is not asymptotically abelian because $a(f)$ is a "fixed point"; the degree of freedom f is isolated by the perturbation. Now if ω_0 is the Fock state, i.e. the unique state on $\mathfrak{A}^e$ with the property

$$\omega_0(a^*(g)\,a(g)) = 0\,, \qquad g \in L^2(\mathbb{R}^\nu)$$

and ω_f is the one-particle state determined by the condition

$$\omega_f(a^*(g)\,a(g)) = |(f, g)|^2\,, \qquad g \in L^2(\mathbb{R}^\nu)$$

it follows that ω_f is τ^P-invariant. Considered as a state on the even algebra $\mathfrak{A}^e$ it is, however, not a vector state of ω_0 nor of any other τ-invariant state. Nevertheless it is the image of ω_0 under the τ-quasi-local automorphism γ_f defined by

$$\gamma_f(A) = (a^*(f) + a(f))\,A(a^*(f) + a(f))\,, \qquad A \in \mathfrak{A}^e\,.$$

This automorphism is not an inner automorphism because of our choice of the even algebra. This form of super-selection rule is not the only way in which the quasi-local automorphisms lead away from normal states. The following example, extracted from [11], demonstrates the infra-red phenomena.

Take $g \in \mathscr{S}(\mathbb{R}^\nu)$ and define P by

$$P = a^*(g)\, a(g)\,.$$

The action of τ^P is of the form

$$\tau_t^P(a(f)) = a(V_t^P f)$$

where V^P is a group of unitaries on $L^2(\mathbb{R}^\nu)$. One can calculate that $\gamma_\pm$ exist, we take the results from [11], and

$$\gamma_\pm(a(f)) = a(\Omega_\pm f)$$

where $\Omega_\pm$ is the unitary wave operator in $L^2(\mathbb{R}^\nu)$ with kernel

$$\Omega_+(p, q) = \delta(p - q) - \frac{g^*(p)\, g(q)}{p^2 - q^2 + i\varepsilon}\, h(q^2)$$

$$h(q^2) = \left(1 - \int d^\nu p \frac{|g(p)|^2}{q^2 - p^2 - i\varepsilon}\right)^{-1}.$$

It is explicitly argued in [11] that the states ω and ω_γ defined by

$$\omega_\gamma(A) = \omega(\gamma_+^{-1}(A))$$

and

$$\omega(a^*(f_n) \ldots a^*(f_1)\, a(g_1) \ldots a(g_n) = \delta_{mn} \det[(f_i, A g_i)]$$

$$A(p, q) = \theta(\mu - p^2)\, \delta(p - q)$$

generate quasi-inequivalent representations, i.e. ω_γ is not normal with respect to ω.

VI. Concluding Remarks

We have tried to demonstrate in this paper that the essential features of the return to equilibrium problem can be described by a triplet $(\mathfrak{A}, \tau, \tau^P)$. The properties of this triplet which are necessary to the discussion are given by natural generalizations of scattering theory. The comparable features are listed as follows.

The asymptotic evolution of τ^P-invariant states under the evolution τ corresponds to the existence of the wave-operator. The asymptotic behaviour is controlled by conditions of asymptotic abelianness of $(\mathfrak{A}, \tau)$ which correspond to the dispersive nature of wave-packets. The isolation by perturbation of finite subsystems is reflected by the lack of asymptotic abelianness of $(\mathfrak{A}, \tau^P)$ and this corresponds to the formation of bound states or the lack of invertibility of the wave operator. The conservation of temperature, as expressed by the K.M.S. condition, derives from the intertwining property of the wave operator. The reversibility of the system, i.e. the same asymptotic behaviour for large positive and negative times, corresponds to no scattering.

The lack of scattering appears to be a general feature arising from the equilibrium conditions, i.e. boundary conditions on the scatterer. The heuristic picture conjured up by this analogy is that of the flow of an infinite media past a local scattering centre. The absence of pervasive scattering is therefore not surprising. This analogy gives some insight into the irreversibility phenomena present in the general problem of approach to equilibrium where the interaction is altered throughout the system corresponding therefore to the flow past a scattering centre which is extended throughout space.

Appendix: The X—Y Model

Let $\mathfrak{A}_n^e$ denote the even algebra of local observables for the interval $[-n, n] \in \mathbb{Z}$. This algebra is the algebra of even polynomials in a_p, a_q^*, $p, q \in [-n, n]$ and can also be considered as the algebra generated by even polynomials in the transformed elements

$$b_p = a_p \prod_{r=-n}^{p-1} (1 - 2a_r^* a_r)$$

$$b_p^* = a_p^* \prod_{r=-n}^{p-1} (1 - 2a_r^* a_r), \qquad p \in [-n, n].$$

These latter operators satisfy the anti-commutation relations

$$\{b_p, b_q^*\} = \delta_{p,q}, \qquad \{b_p, b_q\} = 0, \qquad p, q \in [-n, n].$$

The transformation $a_p \to b_p$ complicates the inherent local structure in general, e.g. if $p < n$ then $a_p \in \mathfrak{A}_p$ but its image $b_p \notin \mathfrak{A}_p$. Out choice of the even algebras eliminates that difficulty, e.g. with $p < q$

$$b_p^* b_q = a_p^* \left(\prod_{r=p+1}^{q-1} (1 - 2a_r^* a_r) \right) a_q$$

and

$$a_p^* a_q = b_p^* \left(\prod_{r=p+1}^{q-1} (1 - 2b_r^* b_r) \right) b_q.$$

Each $\mathfrak{A}_n^e$ is generated by polynomials in elements $b_p^* b_q, b_p b_q^*, b_p^* b_q^*, b_p b_q$.

In terms of the transformed elements the $X - Y$ Hamiltonian becomes

$$H_n = \frac{J}{2} \sum_{p=-n}^{n-1} (b_p^* b_{p+1} + b_{p+1}^* b_p) + h \sum_{p=-n}^{n} b_p^* b_p.$$

This form allows one to calculate the action of τ on b_p, b_q^* and hence on a set of generating elements of $\mathfrak{A}^e$. One finds

$$\tau_t(b_p) = \sum_{q\in\mathbb{Z}} C_t(p-q)\, b_q$$

where

$$C_t(r) = \frac{1}{2\pi} \int_0^{2\pi} d\theta\, e^{ir\theta}\, e^{it(J\operatorname{Cos}\theta + h)}\,.$$

This explicit form allows the analytic extension of τ on b_p, b_q^* etc., and hence demonstrates that all strictly local elements are τ-analytic.

Asymptotic abelianness follows by using the anti-commutation relations to bound $\|[A, \tau_t(B)]\|$ with $A, B \in \mathfrak{A}_n^e$ by a finite number of terms of the form $\|\{b_p, \tau_t(b_q^*)\}\| = |C_t(p-q)|$. But this last expression tends to zero like $|t|^{-\frac{1}{2}}$ as $|t| \to \infty$ because

$$C_t(r) = e^{iht} J_r(t)$$

where $J_r(t)$ is a Bessel function. The result extends to general $A, B \in \mathfrak{A}^e$ by continuity.

This latter form of estimation is not sufficient to prove that $(\mathfrak{A}, \tau)$ is asymptotically integrable with respect to a strictly local P. If however we define for each $f = \{f_p\}_{p\in\mathbb{Z}} \in \ell^2$ elements $b(f) \in \mathfrak{A}$ by

$$b(f) = \sum_p b_p f_p$$

then with suitable chosen f we are able to conclude the result. If $P \in \mathfrak{A}_n$ and A is an even monomial in $b(f), b^*(g)$ then $\|[P, \tau_t(A)]\|$ is bounded by a finite sum of terms of the form

$$\text{constant}\left[\sum_{|p|\leqq n} |C_t * f_p|^2\right]^{\frac{1}{2}}$$

where

$$C_t * f_p = \sum_{q\in\mathbb{Z}} C_t(p-q)\, f_q$$
$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \hat{f}(\theta)\, e^{it(J\operatorname{Cos}\theta + h)}\, e^{ip\theta}\,.$$

If f is chosen such that $\tilde{f}$ vanishes smoothly at $\theta = n\pi$ then these upper bounds are integrable. This form of restriction on f still allows one to construct a dense subset of $\mathfrak{A}^*$ for which the norm of the relevant commutator is integrable.

To demonstrate that there is a unique τ-K.M.S. state over $\mathfrak{A}^e$ at any β. One proceeds as follows. First from the τ-analyticity of strictly local elements it is sufficient to consider

$$\omega(\tau_{i\beta}(A)\, B) = \omega(BA)$$

for A, B strictly local. But then it suffices to establish this relation for A, B even monomials in the b_p and b_q^*. But assuming the relation is valid one can, by a tedious calculation involving the anti-commutation relations and some distribution theory show that ω is uniquely determined. The lines of this calculation are as follows (cf. [10]).

By Fourier transformation of

$$\omega(\tau_{i\beta}(b_p^* b_q)\, b_v^* b_s) = \omega(b_v^* b_s b_p^* b_q)$$

one obtains

$$\omega(\tilde{b}_{\theta_1}^* \tilde{b}_{\theta_2} \tilde{b}_{\theta_3}^* \tilde{b}_{\theta_4})\, e^{-\beta J(\mathrm{Cos}\,\theta_1 - \mathrm{Cos}\,\theta_2)} = \omega(\tilde{b}_{\theta_3}^* \tilde{b}_{\theta_4} \tilde{b}_{\theta_1}^* \tilde{b}_{\theta_2})\,.$$

But the anti-commutation relations give

$$\omega(\tilde{b}_{\theta_3}^* \tilde{b}_{\theta_4} \tilde{b}_{\theta_1}^* \tilde{b}_{\theta_2}) = \delta(\theta_1 - \theta_4)\, \omega(\tilde{b}_{\theta_2}^* \tilde{b}_{\theta_3}) - \delta(\theta_2 - \theta_3)\, \omega(\tilde{b}_{\theta_1}^* \tilde{b}_{\theta_4}) + \omega(\tilde{b}_{\theta_1}^* \tilde{b}_{\theta_2} \tilde{b}_{\theta_3}^* \tilde{b}_{\theta_4})\,.$$

Combining the equations one has

$$\omega(\tilde{b}_{\theta_1}^* \tilde{b}_{\theta_2} \tilde{b}_{\theta_3}^* \tilde{b}_{\theta_4})\,(e^{-\beta J(\mathrm{Cos}\,\theta_1 - \mathrm{Cos}\,\theta_2)} - 1) = \delta(\theta_1 - \theta_4)\, \omega(\tilde{b}_{\theta_3}^* \tilde{b}_{\theta_2}) - \delta(\theta_2 - \theta_3)\, \omega(\tilde{b}_{\theta_1}^* \tilde{b}_{\theta_4})\,.$$

Similar relations follow from

$$\omega(\tau_{i\beta}(b_p^* b_s)\, b_r^* b_q) = \omega(b_r^* b_q b_p^* b_s)$$

and

$$\omega(\tau_{i\beta}(b_p^* b_r^*)\, b_q b_s) = \omega(b_q b_s b_p^* b_r^*)$$

and after some calculations one finds that these relations are consistent if, and only if,

$$\omega(\tilde{b}_{\theta_1}^* \tilde{b}_{\theta_2}) = \delta(\theta_1 - \theta_2)\,(1 + e^{\beta(J\,\mathrm{Cos}\,\theta_1 + h)})^{-1}$$

and

$$\omega(b_p^* b_q b_r^* b_s) = \omega(b_p^* b_q)\, \omega(b_r^* b_s) + \omega(b_p^* b_s)\, \omega(b_q b_r^*)\,.$$

Similar calculations with higher order monomials then show that ω is determined as a quasi-free state with the above two-point function.

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