

# The Use of Unitarity Bounds for a Stable Extrapolation of Low Energy Data

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**Abstract.** The properties of the scattering amplitude allow to define a function  $f(z)$  satisfying the following conditions:

- 1)  $f(z)$  is holomorphic in a simply connected domain  $\mathcal{D}$ , which can be mapped conformally onto the unit disk;
- 2)  $|\operatorname{Im} f(z)|$  is bounded by some constant  $M$  in  $\mathcal{D}$ ;
- 3)  $|\operatorname{Re} f(z)|$  is known not to exceed some constant  $m$  on a certain part  $\Gamma_1$  of the boundary  $\Gamma$  of  $\mathcal{D}$ ;  $f(z)$  is continuously extensible onto  $\Gamma$ .

Using these properties, constraints are derived on the real part of  $f(z)$  valid at any point  $z \in \mathcal{D} \cup \Gamma$ .

The result is used for performing a stable extrapolation of low energy pion-pion scattering data to any finite energy. We derive a bound on energy averaged values of the real part of the scattering amplitude. The bound depends on  $m, M$ , on the energy variable  $s$  and on the energy average interval  $s_2 - s_1$ .

Generalizations of the method are discussed.

## 1. Introduction

Upper and lower bounds on total and elastic cross-sections, scattering amplitudes, phases, etc., represent an important part of our exact knowledge in strong interaction physics. Since the pioneering work of Froisart [1], great progress has been made during the past decade, mainly due to the work of Martin and his collaborators. Important results have been obtained, especially in the following respects:

- 1) assumptions required for obtaining some bounds have been weakened;
- 2) some of the more recent bounds approach experimental curves rather closely;
- 3) a number of upper bounds on the *averages* of total cross-sections over finite energy intervals have been derived.

Details can be found in reviews [2–9] containing further references.

The average bounds are of particular interest because arbitrary unknown constants usually occurring in the bounds can be eliminated

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[10–12, 4–7]. Moments of the total cross-section  $\sigma_T(s)$  of the form

$$\bar{\sigma}_T(s_1, s_2) = \int_{s_1}^{s_2} w(s') \sigma_T(s') ds' \quad (1.1)$$

are defined and shown to be smaller than a certain known function of  $s_1$  and  $s_2$ ,  $w(s')$  being a rather arbitrary (positive) weight function, normalized over the energy range  $(s_1, s_2)$ . Bounds on averages over the imaginary part, the absolute value, etc., of the scattering amplitude are obtained analogously.

Our aim is to use these bounds for performing a stable extrapolation of experimental data from, say, the low energy region to higher energies. As an example, consider the pion-pion elastic scattering amplitude  $F(s, t)$ , in the forward direction.  $F(s, 0)$  is analytic in the complex  $s$  plane with the cuts

$$\langle -\infty, 0 \rangle \quad \text{and} \quad \langle 4\mu^2, \infty \rangle, \quad (1.2)$$

$\mu$  being the mass of the pion. We define moments of the scattering amplitude by

$$\bar{\Phi}(s) = \frac{1}{w_2(s)} \int_{s_1(s)}^{s_2(s)} w_1(s') \Phi(s') ds' \quad (1.3)$$

where  $s_1(s)$ ,  $s_2(s)$ ,  $w_1(s')$  and  $w_2(s)$  are given functions and  $\Phi(s')$  is obtained from  $F(s', 0)$  by a double subtraction:

$$\Phi(s) = \frac{1}{(s-b)^2} \left[ F(s, 0) - F(b, 0) - (s-b) \frac{\partial F}{\partial s}(b, 0) \right], \quad (1.4)$$

$b$  being assumed to be real. If the functions  $s_1$ ,  $s_2$ ,  $w_1$  and  $w_2$  are suitably chosen  $\bar{\Phi}(s)$  defined by (1.3) becomes the boundary value of a function analytic in the cut energy plane and, due to optical theorem and to boundedness of (1.1),  $\text{Im} \bar{\Phi}(s)$  becomes bounded by a known constant  $M$

$$|\text{Im} \bar{\Phi}(s)| \leq M \quad (1.5)$$

on the cuts.

Furthermore, we use the well-known consequence of the first principles that the pion-pion forward scattering amplitude satisfies dispersion relations in the cut  $s$  plane, i.e., it is an analytic function in the cut plane, polynomially bounded for  $|s| \rightarrow \infty$  and a distribution on the cuts (the order of the polynomial being, thanks to the Froissart bound, equal to 2). Then  $\text{Im} \bar{\Phi}(s)$  is bounded by  $M$  in the whole cut plane.

This turns out to be a rather powerful stabilizing condition for extrapolation of scattering data. We shall show that rough (“ex-

perimental") information on  $\text{Re } \Phi(s)$  in the form of the inequality

$$|\text{Re } \bar{\Phi}(s)| \leq m \quad (1.6)$$

in some energy interval (i.e., on some part  $\Gamma_1$  of the cut) implies boundedness of averages of the type

$$\frac{1}{s_2(s) - s_1(s)} \int_{s_1(s)}^{s_2(s)} \text{Re } \Phi(s') w_3(s', s) ds' \quad (1.7)$$

at all energies [the weight function  $w_3(s', s)$  being determined by the form of  $w_1(s')$  and  $w_2(s)$ ]. We shall find a one-parameter set  $\{L_\varrho, 0 < \varrho < 1\}$  of upper bounds on (1.7) and determine the minimal element  $L_{\varrho_0}$  of this set.

The result and the method proposed are suitable even if the experimental information is not very exact, because inequality (1.6) is all we need from experiment. Elastic scattering of pions on pions is a good example of such a situation.

In the following section, we prove a theorem specifying conditions under which inequalities (1.5) and (1.6) imply boundedness of (1.7), and give the explicit form of the set  $\{L_\varrho\}$  of bounds. The mathematical approach is based on a recent paper of Vrkoč [13] containing theorems on continuous dependence of holomorphic functions on partly given boundary values. Our result is closely related to his Theorem 6 and its proof, the main difference being that Vrkoč studies continuity conditions for  $m \rightarrow 0$ , whereas we are interested in some physical (i.e., non-vanishing) value of  $m$  and in the minimalization of  $L_\varrho$  for  $m, M$  and  $s_2 - s_1$  fixed.

Section 3 contains an application of the theorem to forward pion-pion scattering. Identifying the  $\Gamma_1$  interval with the low energy scattering region up to 1.1 GeV, say, we show that the averaged real part is bounded at any energy [see Eq. (3.13)]. Then, we find the minimal value of the bound (3.20). In Section 4, relation of the result to various extrapolation approaches is discussed.

## 2. Derivation of the Bound

We present the following

**Theorem.** *Let  $K$  be the unit disk in the complex  $z$  plane,  $|z| < 1$ ;  $\Gamma$  the boundary of  $K$ ,  $|z| = 1$ ;  $\Gamma_1 \equiv \Gamma_1(\varphi_0, \delta)$  an interval on  $\Gamma$ ,  $\Gamma_1(\varphi_0, \delta) = \{z : z = e^{i\varphi}, |\varphi - \varphi_0| \leq \delta\}$ ,  $\varphi_0 \in (-\pi, \pi)$ ;  $M, m$  be two positive numbers and  $\mathcal{F}(\Gamma_1, M, m)$  the class of functions  $f(z)$  satisfying the following conditions:*

- (i)  $f(z)$  is holomorphic in  $K$ ,

(ii)  $f(z)$  is continuously extensible onto  $\Gamma_1$  and

$$|\operatorname{Re} f(e^{i\varphi})| \leq m \quad \text{on } \Gamma_1, \tag{2.1}$$

(iii)  $0 \leq \operatorname{Im} f(re^{i\varphi}) \leq M$  in  $K$ . (2.2)

Denote

$$f_\alpha(re^{i\varphi}) = \frac{1}{2\alpha} \int_{\varphi-\alpha}^{\varphi+\alpha} f(re^{i\psi}) d\psi. \tag{2.3}$$

Then

$$|\operatorname{Re} f_\alpha(re^{i\varphi})| \leq L_\varrho(\varphi) \tag{2.4}$$

for every  $r \in \langle 0, 1 \rangle$ ,  $\varphi \in (-\pi, \pi)$ ,  $\alpha \in (0, \delta)^1$ ,  $f \in \mathcal{F}(\Gamma_1, M, m)$ , where

$$L_\varrho(\varphi) = m + M \left( \frac{1}{\alpha} \frac{1-\varrho}{\varrho} + \lambda(\varphi) \frac{4}{\pi} \frac{\varrho}{(1+\varrho)(1-\varrho)} \right) \tag{2.5}$$

$$\lambda(\varphi) = \min(|\varphi - \varphi_0|, 2\pi - |\varphi - \varphi_0|) - (\delta - \alpha), \tag{2.6}$$

$$0 < \varrho < 1.$$

*Proof.* The average function  $f_\alpha(z)$ ,  $z = re^{i\varphi}$ , given by (2.3) is holomorphic in  $K$  due to condition (i). We immediately see from condition (iii) that

$$0 \leq \operatorname{Im} f_\alpha(z) \leq M \quad \text{in } K, \tag{2.7}$$

$$\left| \frac{\partial \operatorname{Im} f_\alpha(z)}{\partial \varphi} \right| \leq \frac{M}{2\alpha} \quad \text{in } K. \tag{2.8}$$

The Cauchy-Riemann conditions imply that

$$\left| \frac{\partial \operatorname{Re} f_\alpha}{\partial r} \right| \leq \frac{M}{2\alpha r} \quad \text{in } K. \tag{2.9}$$

Condition (ii) allows to estimate the real part of  $f_\alpha(z)$  for  $z = re^{i\varphi}$ ,  $r \in (0, 1)$ ,  $|\varphi - \varphi_0| \leq \delta - \alpha$ :

$$\begin{aligned} |\operatorname{Re} f_\alpha(re^{i\varphi})| &\leq |\operatorname{Re} f_\alpha(e^{i\varphi})| + \left| \frac{\partial \operatorname{Re} f_\alpha(re^{i\varphi})}{\partial r} \right|_{r=\tilde{r}} \cdot (1-r) \\ &\leq m + \frac{M}{2\alpha} \frac{1-r}{r}, \end{aligned} \tag{2.10}$$

where  $r < \tilde{r} < 1$ . To extend this bound from  $|\varphi - \varphi_0| \leq \delta - \alpha$  to all values of  $\varphi - \varphi_0 \in (-\pi, \pi)$ , we must estimate  $(\partial \operatorname{Re} f_\alpha)/(\partial \varphi)$ . We proceed as follows.

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<sup>1</sup> For simplicity, we formulate the theorem for  $0 < \alpha \leq \delta$ . There is no difficulty in discussing the case  $\alpha > \delta$  (see the general discussion in Ref. [13]) but this case is not interesting from the physical point of view.

$\text{Im} f_x(z)$  is a harmonic function in  $K$  and can be represented in terms of the Poisson integral:

$$\text{Im} f_x(re^{i\varphi}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Im} f_x(Re^{i\theta}) \frac{R^2 - r^2}{R^2 - 2Rr \cos(\varphi - \theta) + r^2} d\theta$$

where  $r < R < 1$ . This yields a bound on the derivative:

$$\begin{aligned} \left| \frac{\partial \text{Im} f_x(re^{i\varphi})}{\partial r} \right| &\leq \frac{M}{2\pi} \int_{-\pi}^{\pi} \frac{2R|-2rR + (r^2 + R^2) \cos(\varphi - \theta)|}{(R^2 - 2rR \cos(\varphi - \theta) + r^2)^2} d\theta \\ &= \frac{4M}{\pi} \frac{R}{(R+r)(R-r)}. \end{aligned} \tag{2.11}$$

This inequality is valid for  $0 \leq r < R < 1$ ,  $\varphi \in (-\pi, \pi)$ . Since, however, the right-hand side is a continuous function of  $R$  in  $R = 1$ , the inequality is valid also for  $R = 1$ .

The Cauchy-Riemann relations imply, putting  $R = 1$  in (2.11)

$$\left| \frac{\partial \text{Re} f_x(re^{i\varphi})}{\partial \varphi} \right| \leq \frac{4M}{\pi} \frac{r}{(1+r)(1-r)}. \tag{2.12}$$

The bound on  $\text{Re} f_x(re^{i\varphi})$  for a general value of  $\varphi$  follows now immediately. Writing

$$\text{Re} f_x(re^{i\varphi}) = \text{Re} f_x(re^{i\chi}) + \int_{\chi}^{\varphi} \frac{\partial \text{Re} f_x(re^{i\psi})}{\partial \psi} d\psi \tag{2.13}$$

with  $|\chi - \varphi_0| \leq \delta - \alpha$ ,  $\delta - \alpha \leq |\varphi - \varphi_0| \leq \pi$ , we obtain from (2.10), (2.12) and (2.13)

$$|\text{Re} f_x(re^{i\varphi})| \leq m + \frac{M}{2\alpha} \frac{1-r}{r} + |\varphi - \chi| \frac{4M}{\pi} \frac{r}{(1+r)(1-r)}.$$

To make the right-hand side as small as possible we choose  $\chi$  equal to  $\varphi_0 + (\delta - \alpha)$  or  $\varphi_0 - (\delta - \alpha)$ , according to which of these values is closer to  $\varphi$ . Thus,  $|\varphi - \chi|$  is replaced by

$$\lambda(\varphi) = \min(|\varphi - \varphi_0|, 2\pi - |\varphi - \varphi_0|) - (\delta - \alpha)$$

so that

$$|\text{Re} f_x(re^{i\varphi})| \leq m + \frac{M}{2\alpha} \frac{1-r}{r} + \lambda(\varphi) \frac{4M}{\pi} \frac{r}{(1+r)(1-r)}. \tag{2.14}$$

This estimate, however, cannot be used at  $r = 1$  (where we actually need it) because it becomes infinite. We find a finite bound in the following way. We use (2.14) for  $r = \varrho$ ,  $\varrho$  being kept fixed less than 1, and express

$\operatorname{Re} f_\alpha(r e^{i\varphi})$  for  $r > \varrho$  as follows:

$$\operatorname{Re} f_\alpha(r e^{i\varphi}) = \operatorname{Re} f_\alpha(\varrho e^{i\varphi}) + (r - \varrho) \left( \frac{\partial \operatorname{Re} f_\alpha(r e^{i\varphi})}{\partial r} \right)_{r=\tilde{r}}$$

with  $\varrho < \tilde{r} < r < 1$ . The right-hand side can be estimated by using (2.14) and (2.9):

$$|\operatorname{Re} f_\alpha(r e^{i\varphi})| \leq m + \frac{M}{2\alpha} \frac{1-\varrho}{\varrho} + \lambda(\varphi) \frac{4M}{\pi} \frac{\varrho}{(1+\varrho)(1-\varrho)} + (r-\varrho) \frac{M}{2\alpha\varrho}.$$

Now the function  $f_\alpha(r e^{i\varphi})$  is continuously extensible onto  $\Gamma$  (see Ref. [13], Theorem 6). We can therefore put  $r = 1$  and obtain (2.4) with  $L_\varrho(\varphi)$  defined by (2.5), (2.6). The Theorem is proved.

### 3. Application: Forward Pion-Pion Scattering

#### A. Kinematics and Conformal Mapping

The pion-pion forward scattering amplitude is analytic in the complex  $s$  plane cut along the intervals

$$\langle -\infty, 0 \rangle, \langle 4\mu^2, \infty \rangle. \quad (3.1)$$

We consider, for simplicity, reactions which are symmetric under the interchange of  $s$  and  $u$ , for instance  $\pi^+ \pi^0 \rightarrow \pi^+ \pi^0$ . Then we can introduce, instead of  $s$ , the variable  $v$ ,

$$v = \left( \frac{s}{2\mu^2} - 1 \right)^2 \quad (3.2)$$

transforming the cuts (3.1) into a single cut  $\langle 1, \infty \rangle$ . Instead of  $v$ , we can introduce the centre-of-mass and the laboratory momentum by using the formulae

$$v = (1 + 2\mathbf{q}_{\text{c.m.}}^2/\mu^2)^2 = 1 + p^2 \quad (3.3)$$

where

$$p = |\mathbf{q}_{\text{lab}}|/\mu. \quad (3.4)$$

The complex  $v$  plane cut along  $\langle 1, \infty \rangle$  is a simply connected domain, which can be mapped conformally onto the unit disk  $K$ , the cut being transformed onto its circumference  $\Gamma$ . Denoting the transformed variable by  $z$ , we have

$$z \equiv z(v) = (1 + i\sqrt{v-1})/(1 - i\sqrt{v-1})$$

and

$$v \equiv v(z) = 4z/(1+z)^2. \quad (3.5)$$

On the cut, we have

$$v(e^{i\varphi}) = 1/\cos^2 \frac{\varphi}{2}$$

and

$$p(\varphi) = \operatorname{tg} \frac{\varphi}{2}.$$

As already mentioned in the Introduction, we apply the theorem not to the amplitude itself but to an energy average of the amplitude. Setting

$$g(z) = \Phi(s) \tag{3.7}$$

with  $\Phi(s)$  given by (1.4), we define averages  $g_\alpha(z)$  of  $g(z)$  as in (2.3)

$$g_\alpha(re^{i\varphi}) = \frac{1}{2\alpha} \int_{\varphi-\alpha}^{\varphi+\alpha} g(re^{i\psi}) d\psi. \tag{3.8}$$

For physical energies ( $s$  on the cut, i.e.,  $r = 1$ ),  $g_\alpha(re^{i\varphi})$  is an average of the form (1.3). For simplicity, we choose the length  $\alpha$  of the integration interval equal to that in (2.3) so that the resulting inequality (2.4) holds for  $|\operatorname{Re} g_{\alpha\alpha}(re^{i\varphi})|$ . The “double” average  $g_{\alpha\alpha}$  can easily be expressed in the form of a weighted simple average.

We shall check now whether conditions of the theorem presented in Section 2 are satisfied by  $g_\alpha(z)$ . Certainly,  $g_\alpha(z)$  is holomorphic in  $z$  in  $K$ . Further, the pion-pion scattering amplitude is known (from indirect measurements) in the low energy region, up to a certain laboratory momentum  $p_1$ , say. Thus, condition (2.1) is satisfied too. We can take  $\sqrt{s_1}$  slightly above 1 GeV, which corresponds to  $p_1 = 25$  approximately.

Since the low energy region  $\langle 0, p_1 \rangle$  is transformed, by (3.5), into the arc  $\Gamma_1$  which is symmetrical around the threshold momentum  $p = 0$ , we obtain

$$\begin{aligned} \varphi_0 &= 0 \\ \delta &= 2 \operatorname{arctg} p_1 \\ \lambda(\varphi) &= \varphi - \delta + \alpha \end{aligned} \tag{3.9}$$

where the angle  $\varphi \equiv \varphi(p)$  corresponds, by (3.6), to the laboratory momentum  $p$  at which we wish to know the value of the bound (2.4). The integration in (2.3) from  $\varphi - \alpha$  to  $\varphi + \alpha$  represents by itself an integration in  $p$  from  $p(\varphi - \alpha)$  to  $p(\varphi + \alpha)$ , where  $p(\varphi \pm \alpha) = \operatorname{tg}((\varphi \pm \alpha)/2)$ . We can introduce the momentum integration interval length  $2\Delta p$  given by

$$2\Delta p = p(\varphi + \alpha) - p(\varphi - \alpha) = \frac{2 \sin \alpha}{\cos \varphi + \cos \alpha}, \tag{3.10a}$$

$$\alpha = \operatorname{arctg} \left[ \frac{\Delta}{1 - \Delta^2 x^2} (1 + x \sqrt{1 + \Delta^2(1 - x^2)}) \right] \tag{3.10b}$$

with the abbreviated notation  $\Delta = \Delta p$ ,  $x = (1 - p^2)/(1 + p^2)$ . For  $(\Delta p)^2 \ll 1$ , we get

$$\alpha = \operatorname{arctg} \frac{2\Delta p}{1 + p^2}. \tag{3.10c}$$

Let us check the validity of condition (iii) of the Theorem, as given by formula (2.2). Whereas (2.2) requires the boundedness of  $\operatorname{Im} f(z)$  in the interior  $K$  of the circle, the unitarity bounds imply the boundedness of  $\operatorname{Im} g_x(z)$  on  $\Gamma$ . This is by no means equivalent: example of a function satisfying the latter but not the former requirement is given by the function  $v(z)$ : as is seen from (3.5), (3.6),  $\operatorname{Im} v(z)$  vanishes identically on  $\Gamma$  but is unbounded in  $K$ , tending to infinity at  $z \rightarrow -1$ , i.e.,  $s \rightarrow \infty$ . However, the scattering amplitude is bounded by a polynomial at  $|s| \rightarrow \infty$  [14, 5] the degree of which, due to the Froissart bound, does not exceed 2. Thus,  $|g_x(z)|$  is bounded by a constant for  $|s| \rightarrow \infty$ , i.e., for  $z$  approaching  $-1$  along any curve lying in  $K \cup \Gamma$ . In all other points of the  $s$  cut,  $g_x(e^{i\varphi})$  is the boundary value of a function which is analytic in the cut plane. As  $g_x(z)$  is continuously extensible almost everywhere onto the cut, the boundedness of  $|g_x(z)|$  on  $\Gamma$  implies boundedness of  $|\operatorname{Im} g_x(z)|$  in the whole circle. Conditions (ii) and (iii) are satisfied.

Since all assumptions of the theorem are satisfied, it follows that relation (2.4) with (2.5), (2.6) is valid for  $g_x(z)$  defined by (3.8), (3.7) and (1.4). We find now the explicit form of the bound  $L_\varrho(\varphi)$  and calculate its minimal value.

### B. Explicit Form of the Bound and its Minimal Value

The double average  $g_{xx}(z)$  of the function  $g(z)$ ,

$$g_{xx}(re^{i\varphi}) = \frac{1}{2\alpha} \int_{\varphi-\alpha}^{\varphi+\alpha} \left( \frac{1}{2\alpha} \int_{\psi-\alpha}^{\psi+\alpha} g(re^{i\chi}) d\chi \right) d\psi \tag{3.11}$$

can be written in the form

$$g_{xx}(re^{i\varphi}) = \frac{1}{2\alpha} \int_{\varphi-2\alpha}^{\varphi+2\alpha} \left( 1 - \frac{|\varphi - \psi|}{2\alpha} \right) g(re^{i\psi}) d\psi. \tag{3.12}$$

Thus, rewriting (2.4) in terms of the laboratory momentum  $p$  and the difference  $\Delta p$ , we obtain

$$\left| \frac{1}{2\alpha(p)} \int_{p(\varphi-2x)}^{p(\varphi+2x)} \left( 1 - \frac{|\operatorname{arctg} p - \operatorname{arctg} p'|}{\alpha(p)} \right) \operatorname{Re} g(re^{2i \operatorname{arctg} p'}) \frac{2dp'}{1+p'^2} \right| \leq L_\varrho(\varphi) \tag{3.13}$$



where  $p$  and  $\alpha(p)$  are given by (3.10 b, c) and (3.6). Relation (2.5) determines  $L_\varrho(\varphi)$ , with  $\lambda(\varphi)$  obtained by using (2.6), (3.8), (3.6) and (3.10). We get, writing  $L_\varrho(p)$ ,  $\lambda(p)$  instead of  $L_\varrho(\varphi)$ ,  $\lambda(\varphi)$ ,

$$\begin{aligned} \lambda(p) &= \varphi - \delta + \alpha = 2 \operatorname{arctg} p - 2 \operatorname{arctg} p_1 + \alpha(p) \\ &= 2 \operatorname{arctg} \frac{p - p_1}{1 + pp_1} + \alpha(p). \end{aligned} \tag{3.14}$$

Thus,

$$L_\varrho(p) = m + M \left( \frac{1}{\alpha} \frac{1 - \varrho}{\varrho} + \lambda(p) \frac{4}{\pi} \frac{\varrho}{(1 + \varrho)(1 - \varrho)} \right). \tag{3.15}$$

Equation

$$\frac{\partial L_\varrho(p)}{\partial \varrho} = 0 \tag{3.16}$$

giving the minimal value of  $L_\varrho(p)$ , has for  $0 < \varrho < 1$  one solution,  $\varrho = \varrho_0$ , where

$$\varrho_0^2 = (1 + 2\beta - 2\sqrt{2\beta}\sqrt{1 + \beta/2})/(1 - 4\beta) \tag{3.17}$$

with

$$\beta = \alpha\lambda(p)/\pi.$$

We conclude: the exact bound on the averaged real part of  $g(z) \equiv \Phi(s)$  has the form (3.13) with  $L_{\varrho_0}(p)$  given by (3.15),  $\lambda$ ,  $\alpha$  and  $\varrho_0$  being given by (3.14), (3.10 b) and (3.17) respectively.

To make the result more legible for physical applications, we assume  $\Delta p$ ,  $(\Delta p)^2$  to be small compared to  $p$  and 1 respectively. This allows us to neglect  $\beta$  against 1 and approximate (3.17) by

$$\varrho_0^2 \doteq 1 - 2\sqrt{2\beta}.$$

This gives for  $L_{\varrho_0}(p)$  in (3.13)

$$L_{\varrho_0}(p) \doteq m + M \sqrt{2\lambda(p)}/\sqrt{\pi\alpha}. \tag{3.18}$$

Using (3.10 c), (3.14) we obtain

$$L_{\varrho_0}(p) \doteq m + M \cdot \left( \frac{2}{\pi} \frac{1 + p^2}{\Delta p} \operatorname{arctg} \frac{p - p_1}{1 + pp_1} \right)^{1/2}. \tag{3.19}$$

The approximation  $\Delta p \ll p$  simplifies the left-hand side of (3.13) too. We obtain

$$\begin{aligned} & \left| \frac{1 + p^2}{2\Delta p} \int_{p-2\Delta p}^{p+2\Delta p} \left( 1 - \frac{1 + p^2}{2\Delta p} \left| \operatorname{arctg} \frac{p - p'}{1 + pp'} \right| \right) \operatorname{Re} g(e^{2i \operatorname{arctg} p'}) \frac{dp'}{1 + p'^2} \right| \\ & \doteq |g_{\alpha\alpha}(r e^{i\varphi})| \leq L_{\varrho_0}(p) \doteq m + M \cdot \left( \frac{2}{\pi} \frac{1 + p^2}{\Delta p} \operatorname{arctg} \frac{p - p_1}{1 + pp_1} \right)^{1/2}. \end{aligned} \tag{3.20}$$

For  $p$  close to  $p_1$  (but  $p - p_1 \gg \Delta p$ ) the right-hand side becomes

$$m + \sqrt{\frac{2}{\pi}} M \frac{p}{\sqrt{\Delta p}} \sqrt{\frac{p - p_1}{1 + pp_1}}.$$

On the other hand, for  $p$  sufficiently large ( $p \gg p_1 \gg 1$ ) we get

$$m + \sqrt{\frac{2}{\pi}} M \sqrt{\frac{p}{\Delta p}} \sqrt{\frac{p}{p_1}}.$$

So, we have obtained the bound on the averaged real part of the amplitude in a rather transparent form. The bound becomes worse with decreasing  $\Delta p$  (because the average approaches the value of the amplitude at the point  $p$ )<sup>2</sup> or with increasing  $p$  (because the extrapolation is performed to a more distant point). It is reasonable to keep  $\Delta p/p$  constant so that the “energy smearing” interval be proportional to the energy value.

#### 4. Concluding Remarks

We have shown in the case of forward pion-pion scattering how analyticity and unitarity of the scattering amplitude can be used for a stable extrapolation of low energy experimental information to any finite energy. The resulting inequality (3.20) states that the averaged real part of the scattering amplitude is bounded by an expression which depends on the low energy (“experimental”) bound  $m$  on the real part and on an over-all bound  $M$  on the averaged imaginary part.

Another modification of the problem of extrapolation “in the average” was already considered in different context (see Ref. [15]). The main difference of the present approach consists in weaker input conditions ( $M$  is the bound on the averaged imaginary part only), which are immediate consequences of first principles. We show in this way that unitarity bounds can be used as “stabilizers” of extrapolation of experimental data.

A stable extrapolation is relatively easily performed in the frame of a physical model which provides explicit energy or angle dependences containing some parameters to be determined. A much more ambitious and difficult task is, however, to proceed from the fundamental principles of  $S$  matrix theory only. As was emphasized several times (see, e.g., [16, 19–21]), analyticity alone is not sufficient to stabilize the extrapolation: indeed, assuming only that the scattering amplitude [or, more generally, a function  $f(z)$  defined by way of the amplitude and possessing the required analyticity properties] does not differ, in the measured kine-

<sup>2</sup> Note that  $M$  also may depend on  $\Delta p$ , because  $M$  is the bound on average of the imaginary part. The dependence of  $M$  on  $\Delta p$  is given by the input data of our problem, i.e., by the unitarity bound.

mathematical interval  $\Gamma_1$ , from the experimentally measured histogram  $h(z)$  by more than the “error” function  $m(z)$ ,

$$|f(z) - h(z)| < m(z), \quad z \in \Gamma_1 \quad (4.1)$$

one has to expect *any* value of  $f(z)$  outside  $\Gamma_1$ .

As an additional stabilizing condition, boundedness of the modulus of  $f(z)$ ,

$$|f(z)| < M(z) \quad z \in \Gamma_2 \quad (4.2)$$

on the remaining part  $\Gamma_2$  of the cuts is usually postulated. This condition has proved to be a very powerful stabilizing element and many interesting results have been obtained thanks to it. This is because condition (4.2), making statements about *values* of  $f(z)$ , is an excellent complement to the analyticity condition, which amounts to the existence of all derivatives (“smoothness”) of  $f(z)$  but says nothing about values.

Being a direct consequence of axiomatic field theory, the analyticity of the scattering amplitude in the energy plane provides a well-founded basis for a stable extrapolation. Contrary to this, condition (4.2) has played the rôle of a very useful but rather artificial supplement to analyticity, necessary for obtaining acceptable results.

Due to the remarkable progress made recently both in the field of the rigorous bounds and in developing stable extrapolation methods<sup>3</sup> the time is ripe for replacing conditions of the type (4.2) by some immediate consequences of first principles.

Of course, one has to be prepared to deal with conditions which are weaker than (4.2) and to refine appropriately the necessary mathematical tools. The introduction of the “averaged” quantities [4–7, 10–12, 22] satisfying rigorous bounds at all energies is a good example of this situation. The present paper shows one way how unitarity can be used for making the extrapolation stable.

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<sup>3</sup> For details, the interested reader is referred to review articles [16–20] containing further references. A recent clear exposition of fundamental problems in this field is contained in [20] and [21].

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