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Infinite Volume Limits of the Canonical Free Bose Gas States on the Weyl Algebra

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Abstract. It was shown by Araki and Woods that the infinite free Bose gas can be described by states on the Weyl algebra; they conjectured a certain family of states parameterized by temperature and density to be the infinite volume limit of the Gibbs canonical states. We show here that this conjecture is correct. We show that the volume dependent canonical states are equicontinuous in the density by a detailed calculation and a combinatorial result that gives cancellations. This allows us to develop a method of Kac that connects the canonical states explicitly with the grand canonical states which are more easily controlled in the infinite volume limit.

In 1963 Araki and Woods [1] showed that the theory of states on the Weyl algebra is a natural "quantum mechanics of infinitely many degrees of freedom" appropriate for the infinitely extended Bose gas. In this context, they conjectured a simple expression for the equilibrium state of the infinite free Bose gas at arbitrary temperature and density. In particular, their expression shows clearly the presence of the Einstein condensate above critical density. But the reasons for their conjecture were still based on the usual pre-quantum mechanical arguments of Einstein, which should be extraneous because the Gibbs canonical and grand canonical states of the finite Bose gas should simply converge in the infinite volume limit. So in the present note we give a direct proof that the Gibbs states do converge in the infinite volume limit. The canonical states converge to the state conjectured by Araki and Woods.

The main idea for the proof of convergence of the canonical states is due to Kac [2]. The infinite volume limit of the grand canonical states is easily calculated and the grand canonical state is a linear combination of canonical states at different densities. Kac showed that the coefficients of this linear combination converge to a simple distribution (in the infinite volume limit) which can be used to calculate canonical expectations from grand canonical expectations. Kac's work leaves open the technical problem of showing convergence of the canonical states themselves, but proves convergence of a wide class of linear combinations. In the present note, a slightly wider class of linear combinations is proved to converge and an equicontinuity estimate is obtained to prove convergence of the canonical states themselves.

Lewis and Pulé [3] have given a definitive treatment of Kac's work within the description of Araki and Woods. Following Kac, they began with a finite gas in a container of arbitrary shape and boundary condition, and they proved convergence of the grand canonical states. (They did not identify the canonical state as such but they noted that this state is related to the grand canonical state by the Kac density.) For the convenience of the reader, we shall give this part of the analysis also, but only for the case of periodic boundary conditions where it is quite simple (because the one particle eigenvalues are exactly known).

§ 1. Introduction and Notation

As usual, let Fock space be the completed direct sum

$$\mathscr{F} = \bigoplus_{n=0}^{\infty} \mathscr{F}^{(n)} \tag{1.1}$$

where $\mathscr{F}^{(0)} = \boldsymbol{C}$ and $\mathscr{F}^{(n)} = \mathscr{L}^2_{\mathrm{sym}}((\boldsymbol{R}^3)^n)$. Let P_n be the orthogonal projection onto $\mathscr{F}^{(n)}$ in $\mathscr{F}$. Let $\mathscr{D}$ be the incomplete direct sum

$$\mathscr{D} = \bigcup_{l=1}^{\infty} \bigoplus_{n=0}^{l} (\mathscr{F}^{(n)} \cap \mathscr{C}^{\infty}_{\mathrm{comp}}) \subset \mathscr{F}\,. \tag{1.2}$$

Let the annihilation and creation operators be defined on $\mathscr{D}$ for $f \in \mathscr{F}^{(1)}$ by

$$(a(\bar{f})\psi)^{(n)}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \sqrt{n+1} \int \bar{f}(\boldsymbol{x})\, \psi^{(n+1)}(\boldsymbol{x}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)\, d\boldsymbol{x}$$

$$(a^*(f)\psi)^{(n)}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(\boldsymbol{x}_j)\, \psi^{(n-1)}(\boldsymbol{x}_1, \ldots, \hat{\boldsymbol{x}}_j, \ldots, \boldsymbol{x}_n) \text{ (omit } \boldsymbol{x}_j),$$

where $\psi^{(n)} = P_n \psi$, for $n \geqq 0$, and zero for $n < 0$. Then $[a(\bar{f}), a^*(g)] = (f, g) I$ on $\mathscr{D}$ and

$$\chi(f) = \frac{1}{\sqrt{2}} \{a^*(f) + a(\bar{f})\} \tag{1.3}$$

is essentially self-adjoint on $\mathscr{D}$. Let

$$W(f) = e^{i\chi(f)}, \quad f \in \mathscr{F}^{(1)}. \tag{1.4}$$

The $W(f)$ are unitary operators in $\mathscr{F}$ satisfying the Weyl relations

$$W(f_1)\, W(f_2) = W(f_1 + f_2) \exp\left\{-\frac{i}{2} \operatorname{Im}(f_1, f_2)\right\}. \tag{1.5}$$

For computations the formula

$$W(f)=\exp\left\{-\frac{1}{4}\|f\|^2\right\}\exp\left\{a^*\left(\frac{i}{\sqrt{2}}f\right)\right\}\exp\left\{a\left(\frac{i}{\sqrt{2}}\bar{f}\right)\right\}, \quad (1.6)$$

valid on $\mathscr{D}$ in the sense of power expansions, is useful.

The concrete Weyl algebra $\mathfrak{A}$ will be the operator norm closure of the linear span of all $W(f)$ under the important restriction that $\operatorname{supp} f$ be compact so that $W(f)$ is localized:

$$\mathfrak{A}=\left\{\sum_j c_j W(f_j) : c_j \in \boldsymbol{C}, f_j \in \mathscr{C}^\infty_{\text{comp}}(\boldsymbol{R}^3)\right\}^{\text{op}}. \quad (1.7)$$

Thus, $\mathfrak{A}$ contains the so-called quasi-localized observables. (We could consider $\mathfrak{A}$ to be an abstract C^* algebra; but this would complicate notation. Thus we consider $\mathfrak{A}$ as an algebra of operators on $\mathscr{F}$; but the notion of state will be general.)

A *state* on $\mathfrak{A}$ is any linear function $\mathfrak{A}\to\boldsymbol{C}$, denoted $A\to\langle A\rangle$ with the properties

$$\langle A^*A\rangle \geqq 0,\ \langle I\rangle=1, \quad \text{and} \quad s\to\langle W(sf)\rangle \quad \text{continuous}, \quad (1.8)$$

for all $A\in\mathfrak{A}$, $s\in\boldsymbol{R}$, and $f\in\mathscr{C}^\infty_{\text{comp}}(\boldsymbol{R}^3)$. It follows that

$$|\langle A\rangle| \leqq \|A\| \quad (1.9)$$

and $\langle A^*\rangle=\overline{\langle A\rangle}$. A state is determined by its *generating functional*

$$f\to\langle W(f)\rangle,\quad f\in\mathscr{C}^\infty_{\text{comp}}(\boldsymbol{R}^3). \quad (1.10)$$

Furthermore, a functional $E(f)$, $f\in\mathscr{C}^\infty_{\text{comp}}(\boldsymbol{R}^3)$ corresponds to a unique state $\langle\cdot\rangle$ so that $E(f)=\langle W(f)\rangle$ if and only if

$$E(f)=\overline{E(-f)}$$

$$\sum_{ij} c_i\bar{c}_j E(f_i-f_j)\exp\left\{-\frac{i}{2}\operatorname{Im}(f_i,f_j)\right\}\geqq 0$$

$$s\to E(sf) \quad \text{continuous}$$

for all $f_j\in\mathscr{C}^\infty_{\text{comp}}(\boldsymbol{R}^3)$, $c_j\in\boldsymbol{C}$, $j=1,2,\dots,n$, and $s\in\boldsymbol{R}$. So it suffices to consider only the generating functional of a state.

Let $\boldsymbol{V}$ be the cube

$$V=\left\{\boldsymbol{x}\in\boldsymbol{R}^3 : -\frac{L}{2}\leqq \boldsymbol{x}^i\leqq\frac{L}{2},\ i=1,2,3\right\}. \quad (1.11)$$

We also let V denote the volume

$$V=L^3. \quad (1.12)$$

Let $h_V(\boldsymbol{x})$ be the characteristic function of $\boldsymbol{V} \subset \boldsymbol{R}^3$. Let

$$\begin{aligned} a_V(\boldsymbol{k}) &= a(V^{-\frac{1}{2}} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} h_V) \\ a_V^*(\boldsymbol{k}) &= a^*(V^{-\frac{1}{2}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} h_V)\,. \end{aligned} \tag{1.13}$$

If $f \in \mathscr{F}^{(1)}(\boldsymbol{V}) \equiv \mathscr{L}^2(\boldsymbol{V})$,

$$\begin{aligned} a(\bar{f}) &= \sum_{\boldsymbol{k}\in\frac{2\pi}{L}\boldsymbol{Z}^3} V^{-\frac{1}{2}} \overline{\hat{f}(\boldsymbol{k})}\, a_V(\boldsymbol{k}) \\ a^*(f) &= \sum_{\boldsymbol{k}\in\frac{2\pi}{L}\boldsymbol{Z}^3} V^{-\frac{1}{2}} \hat{f}(\boldsymbol{k})\, a_V^*(\boldsymbol{k})\,, \end{aligned} \tag{1.14}$$

in the sense of strong convergence on $\mathscr{D}$, where

$$\hat{f}(\boldsymbol{k}) = \int e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x})\, d\boldsymbol{x}\,.$$

Let $\mathscr{F}(\boldsymbol{V}) \subset \mathscr{F}$ be the Fock space over $\mathscr{F}^{(1)}(\boldsymbol{V})$, obtained by replacing $\boldsymbol{R}^3$ by $\boldsymbol{V}$ in (1.1). Write $\omega(\boldsymbol{k})$ for the one particle energy at momentum $\boldsymbol{k}$:

$$\omega(\boldsymbol{k}) = \frac{\boldsymbol{k}^2}{2m}, \qquad m = 1\,. \tag{1.15}$$

The Hamiltonian and number operator for the free Bosons in a periodic box $\boldsymbol{V}$ are now defined by

$$\begin{aligned} H_{0V} &= \sum_{\boldsymbol{k}\in\frac{2\pi}{L}\boldsymbol{Z}^3} \omega(\boldsymbol{k})\, a_V^*(\boldsymbol{k})\, a_V(\boldsymbol{k}) \\ N_V &= \sum_{\boldsymbol{k}\in\frac{2\pi}{L}\boldsymbol{Z}^3} a_V^*(\boldsymbol{k})\, a_V(\boldsymbol{k}) \end{aligned} \tag{1.16}$$

on the domain $\mathscr{D}$. They are essentially self-adjoint in $\mathscr{F}(\boldsymbol{V})$ and in $\mathscr{F}$.

We now define the *canonical state* at arbitrary temperature β^{-1} and density ϱ by the formula

$$\langle A\rangle^c_{\beta,\varrho,V} \equiv \frac{\operatorname{tr}(A P_{\varrho V} e^{-\beta H_{0V}})}{\operatorname{tr}(P_{\varrho V} e^{-\beta H_{0V}})} \tag{1.17}$$

if ϱV is an integer and in general by linear extrapolation:

$$\langle A\rangle^c_{\beta,\varrho,V} \equiv (1-\lambda)\langle A\rangle^c_{\beta,\varrho_1,V} + \lambda\langle A\rangle^c_{\beta,\varrho_2,V} \tag{1.18}$$

if $\varrho_1 V < \varrho_2 V$ are consecutive integers and

$$\varrho = (1-\lambda)\varrho_1 + \lambda\varrho_2, \qquad 0 \leqq \lambda \leqq 1\,.$$

We define the *grand canonical state* at temperature β^{-1} and mean density $\bar{\varrho}$ by

$$\langle A\rangle^{\text{g.c.}}_{\beta,\bar{\varrho},V} = \langle A\rangle^{\text{g.c.}}_{\beta,z,V} \equiv \frac{\operatorname{tr}(A z^{N_V} e^{-\beta H_{0V}})}{\operatorname{tr}(z^{N_V} e^{-\beta H_{0V}})} \tag{1.19}$$

where $z = z(V, \beta, \bar{\varrho}) \in [0, 1]$ is uniquely determined by the condition

$$\bar{\varrho} = \left\langle \frac{N_V}{V} \right\rangle^{\text{g.c.}}_{\beta, z, V} . \tag{1.20}$$

The expectations (1.17)–(1.19) define states on $\mathscr{B}(\mathscr{F}(V))$, the algebra of all bounded operators on $\mathscr{F}(V)$, which can be extended to states on $\mathscr{B}(\mathscr{F}) \supset \mathfrak{A}$. The restriction to $\mathfrak{A}$ is needed only in the limit $V \to \infty$.

If the system is in the state $\langle \cdot \rangle^{\text{g.c.}}_{\beta, \bar{\varrho}, V}$, then the probability of finding it in the state $\langle \cdot \rangle^{\text{c}}_{\beta, \varrho, V}$ is just $\langle P_{\varrho V} \rangle^{\text{g.c.}}_{\beta, \bar{\varrho}, V}$, i.e. the grand and the canonical states are connected by the formula

$$\langle A \rangle^{\text{g.c.}}_{\beta, \bar{\varrho}, V} = \sum_{\varrho = 0, \frac{1}{V}, \frac{2}{V}, \dots} \langle P_{\varrho V} \rangle^{\text{g.c.}}_{\beta, \bar{\varrho}, V} \langle A \rangle^{\text{c.}}_{\beta, \varrho, V} . \tag{1.21}$$

Let

$$K_V(\varrho, \bar{\varrho}) = \sum_{\varrho' = 0, \frac{1}{V}, \dots} \langle P_{\varrho' V} \rangle^{\text{g.c.}}_{\beta, \bar{\varrho}, V} \, \delta(\varrho - \varrho') , \tag{1.22}$$

so that (1.21) is written

$$\langle A \rangle^{\text{g.c.}}_{\beta, \bar{\varrho}, V} = \int K_V(\varrho, \bar{\varrho}) \langle A \rangle^{\text{c}}_{\beta, \varrho, V} \, d\varrho . \tag{1.23}$$

We shall call $K_V(\cdot, \bar{\varrho})$ the *Kac density* at volume V. Kac [2] found that the $V \to \infty$ limit of K_V is simply

$$K(\varrho, \bar{\varrho}) = \begin{cases} \delta(\varrho - \bar{\varrho}) \text{ if } \bar{\varrho} \leq \varrho_c(\beta) \\ \left. \begin{cases} 0 \text{ if } \varrho < \varrho_c(\beta) \\ \dfrac{1}{\bar{\varrho} - \varrho_c(\beta)} \exp\left\{ -\dfrac{\varrho - \varrho_c(\beta)}{\bar{\varrho} - \varrho_c(\beta)} \right\}, \text{ if } \varrho \geq \varrho_c(\beta) \end{cases} \right\} \text{ if } \bar{\varrho} > \varrho_c(\beta) \end{cases} \tag{1.24}$$

where $\varrho_c(\beta)$ is the *critical density*:

$$\varrho_c(\beta) \equiv G(1, \beta) \tag{1.25}$$

and

$$G(z, \beta) \equiv \int \frac{z}{e^{\beta \omega(\boldsymbol{k})} - z} \frac{d\boldsymbol{k}}{(2\pi)^3} , \tag{1.26}$$

for $z \in [0, 1]$. Therefore, if the limits of the states exist, the canonical and grand canonical states are the same below critical density, while above critical density the grand canonical state is the Laplace transform of the canonical states. Because we are considering only periodic boundary conditions, we will be able to give an especially simple proof of this result of Kac. This result is the governing idea in the proof of convergence of the canonical states.

The infinite volume limit of the canonical state as conjectured by Araki and Woods, in terms of its generating functional, is

$$\langle W(f)\rangle^{c}_{\beta,\varrho} \equiv \exp\left\{-\frac{1}{4}\|f\|^2\right\} \cdot \begin{cases} \exp\left\{-\frac{1}{2}\int \frac{z_\infty}{e^{\beta\omega}-z_\infty}|\hat{f}|^2 \frac{dk}{(2\pi)^3}\right\}, & \varrho \leqq \varrho_c(\beta) \\ \exp\left\{-\frac{1}{2}\int \frac{1}{e^{\beta\omega}-1}|\hat{f}|^2 \frac{dk}{(2\pi)^3}\right\} J_0(\sqrt{2(\varrho-\varrho_c(\beta))}\,|\hat{f}(\mathbf{0})|), & \varrho \geqq \varrho_c(\beta) \end{cases} \tag{1.27}$$

where $z_\infty \in [0,1]$ is the unique solution to

$$\varrho = G(z,\beta)\,. \tag{1.28}$$

The Bessel function

$$J_0(2\sqrt{z}) = \sum_{l=0}^{\infty} \frac{1}{l!^2}(-z)^l \tag{1.29}$$

corresponds to macroscopic occupation of the ground state [1].

Finally, to complete the notation, we note that the infinite volume grand canonical state will be given by

$$\langle W(f)\rangle^{\mathrm{g.c.}}_{\beta,\bar{\varrho}} = \begin{cases} \langle W(f)\rangle^{c}_{\beta,\bar{\varrho}}, & \text{if } \bar{\varrho} \leqq \varrho_c(\beta) \\ \exp\left\{-\frac{1}{4}\|f\|^2\right\} \exp\left\{-\frac{1}{2}\int \frac{1}{e^{\beta\omega}-1}|\hat{f}|^2 \frac{dk}{(2\pi)^3}\right\} \cdot \exp\left\{-\frac{1}{2}(\bar{\varrho}-\varrho_c(\beta))\,|\hat{f}(\mathbf{0})|^2\right\}, & \bar{\varrho} \geqq \varrho_c(\beta)\,. \end{cases} \tag{1.30}$$

§ 2. The Thermodynamic Limit

The main result of this note is the convergence of the canonical state as the volume becomes infinite:

Theorem 1. *For any* $\beta, \varrho > 0$ *and* $f \in \mathscr{C}^\infty_{\mathrm{comp}}(\boldsymbol{R}^3)$,

$$\langle W(f)\rangle^{c}_{\beta,\varrho,V} \to \langle W(f)\rangle^{c}_{\beta,\varrho} \tag{2.1}$$

for any sequence $V \to \infty$. *The convergence is uniform in* ϱ *in any compact subset of* $(0,\infty)$. *(See* (1.18) *and* (1.27).*)*

For the proof we will need a slight generalization of the corresponding result for the grand canonical states:

Theorem 2 (Kac, Lewis-Pulé). *For any* $\beta, \bar{\varrho} > 0$ *and* $f \in \mathscr{C}^\infty_{\mathrm{comp}}(\boldsymbol{R}^3)$

$$\langle W(f)\rangle^{\mathrm{g.c.}}_{\beta,\bar{\varrho},V} \to \langle W(f)\rangle^{\mathrm{g.c.}}_{\beta,\bar{\varrho}} \tag{2.2}$$

for any sequence $V \to \infty$. *(See* (1.19) *and* (1.30).*)*

The relation between the canonical and grand canonical states, given by the limit of (1.23), is

$$\langle W(f)\rangle^{\mathrm{g.c.}}_{\beta,\bar{\varrho}} = \int K(\varrho,\bar{\varrho})\,\langle W(f)\rangle^{\mathrm{c}}_{\beta,\varrho}\,d\varrho\,. \tag{2.3}$$

With the Kac density K given by (1.24), (2.3) is a simple Laplace transform above critical density and it holds by definition below critical density. This relation is useful for calculations. For example, it is trivial to calculate the reduced density matrices in the grand canonical state (see e.g. [3]); whence they are obtained in the canonical state via (2.3). This was the explicit context of Kac's work [2]. (Kac's version of Theorem 2 asserts convergence of the grand canonical reduced density matrices.)

3. The Grand Canonical State and the Kac Density

We need to calculate $\langle W(f)\rangle^{\mathrm{g.c.}}_{\beta,\bar{\varrho},V}$ and $\langle P_{\varrho V}\rangle^{\mathrm{g.c.}}_{\beta,\bar{\varrho},V}$ which appeared in (1.21). We denoted the latter as a distribution of δ functions by $K_V(\varrho,\bar{\varrho})$ in (1.22). It has the Fourier transform

$$\hat{K}_V(\xi,\bar{\varrho}) = \sum_{\varrho = 0, \frac{1}{V}, \ldots} e^{-i\varrho\xi}\langle P_{\varrho V}\rangle^{\mathrm{g.c.}}_{\beta,\bar{\varrho},V} = \left\langle e^{-i\xi\frac{N}{V}}\right\rangle^{\mathrm{g.c.}}_{\beta,\bar{\varrho},V}. \tag{3.1}$$

Using (1.6), (1.14) and the commutation relation between $a(\bar{f})$ and $a^*(f)$, one obtains

$$\begin{aligned}&\left\langle e^{-i\xi\frac{N}{V}}W(f)\right\rangle^{\mathrm{g.c.}}_{\beta,\bar{\varrho},V}\\ &\quad= \hat{K}_V(\xi,\bar{\varrho})\exp\left\{-\frac{1}{4}\|f\|^2\right\}\exp\left\{-\frac{1}{2V}\sum_{k\in\frac{2\pi}{L}Z^3}\frac{z}{e^{\beta\omega}e^{i\frac{\xi}{V}}-z}|\hat{f}(k)|^2\right\}\end{aligned} \tag{3.2}$$

with

$$\hat{K}_V(\xi,\bar{\varrho}) = \prod_{k\in\frac{2\pi}{L}Z^3}\frac{1-ze^{-\beta\omega(k)}}{1-ze^{-\beta\omega(k)}e^{i\frac{\xi}{V}}} \tag{3.3}$$

where z is determined by the condition (1.20). (This calculation can be carried out as a trivial generalization of the calculation given in the appendix of [1].) The object of this section is to determine the $V\to\infty$ limit of (3.3) and (3.2).

From (3.1) and (3.3) we obtain

$$\left\langle \frac{N}{V} \right\rangle_{\beta,z,V}^{\text{g.c.}} = \frac{1}{V} \sum_{k \in \frac{2\pi}{L} \mathbf{Z}^3} \frac{z}{e^{\beta\omega(k)} - z}, \tag{3.4}$$

which makes the condition (1.20) explicit.

Lemma 1. *Uniformly in* $z \in [0, 1]$

$$\frac{1}{V} \sum_{\substack{k \in \frac{2\pi}{L} \mathbf{Z}^3 \\ k \neq 0}} \frac{z}{e^{\beta\omega(k)} - z} \to G(z, \beta) \equiv \int \frac{z}{e^{\beta\omega} - z} \frac{dk}{(2\pi)^3}. \tag{3.5}$$

The proof is simple. For example, if we omit from the summations all terms indexed by $k : k^i = 0$ for some $i = 1, 2, 3$, then the sums are just the integrals of positive step functions that converge to the integrand of the right side monotonically from below. The z uniformity is clear since the step functions approximate the integrand uniformly in z and k outside an arbitrary neighborhood of the origin. Similarly, the omitted terms approximate two- and one-dimensional integrals with L^{-1} and L^{-2} factors; so their contribution converges uniformly to zero. This would not be valid if the ground state $k = 0$ contribution were present at $z = 1$. This lemma is more complicated if general volumes V with general boundary conditions are considered [2, 3].

Now, by (3.4) and (1.20), z is determined by the condition

$$\bar{\varrho} = \frac{1}{V} \frac{z}{1 - z} + \frac{1}{V} \sum_{\substack{k \in \frac{2\pi}{L} \mathbf{Z}^3 \\ k \neq 0}} \frac{z}{e^{\beta\omega} - z}. \tag{3.6}$$

Since $G(z, \beta)$ is strictly increasing in $z \in [0, 1]$, we have by the lemma the possibility of only two cases: For $V \to \infty$,

$$\left.\begin{array}{llll} \text{(I)} & z \to z_\infty < 1, \ \dfrac{1}{V} \dfrac{z}{1 - z} \to 0, & \text{if} & \bar{\varrho} < \varrho_c(\beta) \\[2ex] \text{(II)} & z \to z_\infty = 1, \ \dfrac{1}{V} \dfrac{z}{1 - z} \to \bar{\varrho} - \varrho_c, & \text{if} & \bar{\varrho} \geqq \varrho_c(\beta) \end{array}\right\} \tag{3.7}$$

where z_∞ is determined by (1.28): $\bar{\varrho} = G(z_\infty, \beta)$ in case (I).

Lemma 2 (Kac). *For each* $\bar{\varrho}$ *and uniformly in* ξ *in bounded sets*

$$\hat{K}_V(\xi, \bar{\varrho}) \to \hat{K}(\xi, \bar{\varrho}) \tag{3.8}$$

where

$$\hat{K}(\xi, \bar{\varrho}) = \begin{cases} \exp\{i\xi G(z_\infty, \beta)\}, & \text{(I)} \quad \bar{\varrho} \leqq \varrho_c(\beta) \\ \{1 - (\bar{\varrho} - \varrho_c(\beta)) i\xi\}^{-1} \exp\{i\xi \varrho_c(\beta)\}, & \text{(II)} \quad \bar{\varrho} \geqq \varrho_c(\beta) \end{cases} \tag{3.9}$$

which is the Fourier transform of $K(\varrho, \bar{\varrho})$ *given in* (1.24).

Proof (Kac). The proof uses only (3.7). Write

$$\hat{K}_V(\xi,\bar{\varrho}) = \frac{1-z}{1-ze^{i\frac{\xi}{V}}} \exp\left\{ -\sum_{\substack{k\in\frac{2\pi}{L}\boldsymbol{Z}^3\\ k\neq 0}} \ln \frac{1-ze^{-\beta\omega}e^{i\frac{\xi}{V}}}{1-ze^{-\beta\omega}} \right\}. \quad (3.10)$$

The first factor converges uniformly in bounded ξ:

$$\left\{1-\left(\frac{1}{V}\frac{z}{1-z}\right)\left(i\xi-\frac{1}{2}\frac{\xi^2}{V}+\dots\right)\right\}^{-1} \to \begin{cases} 1 & \text{(I)} \\ \{1-(\bar{\varrho}-\varrho_c(\beta))i\xi\}^{-1}, & \text{(II)}. \end{cases}$$

Similarly, the second factor converges to

$$\exp\{i\xi G(z_\infty,\beta)\},$$

which can be seen by expanding $e^{i\frac{\xi}{V}}$ about zero and the log about 1.

Now we come to the last factor in (3.2), which is the exponential of

$$-\frac{1}{2V}\frac{ze^{-i\frac{\xi}{V}}}{1-ze^{-i\frac{\xi}{V}}}|\hat{f}(\boldsymbol{0})|^2 - \frac{1}{2V}\sum_{\substack{k\in\frac{2\pi}{L}\boldsymbol{Z}^3\\ k\neq 0}}\sum_{n=1}^{\infty} z^n e^{-n\beta\omega(\boldsymbol{k})}e^{-in\frac{\xi}{V}}|\hat{f}(\boldsymbol{k})|^2.$$

By (3.7), the first term converges to

$$\begin{cases} 0, & \text{(I)} \\ -\frac{1}{2}\frac{\bar{\varrho}-\varrho_c}{1-(\bar{\varrho}-\varrho_c)i\xi}, & \text{(II)} \end{cases}$$

and, as in (3.5), the remaining terms converge to

$$-\frac{1}{2}\int\frac{z_\infty}{e^{\beta\omega}-z_\infty}|\hat{f}(\boldsymbol{k})|^2\frac{d\boldsymbol{k}}{(2\pi)^3}$$

uniformly in bounded ξ. If we combine this with Lemma 2, we get the following generalization of Theorem 2:

Theorem 2′. *For any* $\beta,\bar{\varrho}>0$, $f\in\mathscr{C}^\infty_{\text{comp}}(\boldsymbol{R})^3$, *and any sequence* $V\to\infty$

$$\left\langle e^{i\xi\frac{N}{V}}W(f)\right\rangle^{\text{g.c.}}_{\beta,\bar{\varrho},V}$$

$$\to\hat{K}(\xi,\bar{\varrho})\exp\left\{-\frac{1}{4}\|f\|^2\right\}\exp\left\{-\frac{1}{2}\int\frac{z_\infty}{e^{\beta\omega}-z_\infty}|\hat{f}|^2\frac{d\boldsymbol{k}}{(2\pi)^3}\right\} \quad (3.11)$$

$$\begin{cases} 1, & \text{(I)} \\ \exp\left\{-\frac{1}{2}\frac{\bar{\varrho}-\varrho_c}{1-(\bar{\varrho}-\varrho_c)i\xi}|\hat{f}(\boldsymbol{0})|^2\right\}, & \text{(II)}, \end{cases}$$

uniformly in bounded ξ. *(See* (3.7).*)*

4. Convergence of the Canonical State

Theorem 2′ is actually a statement about the Fourier transform of $K_V(\varrho,\bar{\varrho})\langle W(f)\rangle^c_{\beta,\varrho,V}$. We obtain by a trivial calculation that

$$\int e^{-i\varrho\xi} K_V(\varrho,\bar{\varrho})\langle W(f)\rangle^c_{\beta,\varrho,V}\,d\varrho = \left\langle e^{-i\xi\frac{N}{V}} W(f)\right\rangle^{\mathrm{g.c.}}_{\beta,\bar{\varrho},V}, \qquad (4.1)$$

and from (1.24) and (1.27) that

$$\int e^{-i\varrho\xi} K(\varrho,\bar{\varrho})\langle W(f)\rangle^c_{\beta,\varrho}\,d\varrho = [\text{right side of (3.11)}]. \qquad (4.2)$$

So (3.11) asserts convergence of (4.1) to (4.2) uniformly in bounded ξ:

$$(K_V\langle W(f)\rangle^c_{\beta,V})^\frown(\xi)\to(K\langle W(f)\rangle^c_\beta)^\frown(\xi) \qquad (4.3)$$

(with $\bar{\varrho}$ fixed). Since, by (1.9), $|\langle W(f)\rangle^c_{\beta,\varrho,V}|\leqq 1$ and $\int K_V(\cdot,\bar{\varrho})=1$, $|(K_V\langle W(f)\rangle^c_{\beta,V})^\frown(\xi)|\leqq 1$. Therefore if $\hat{h}\in\mathscr{L}^1(\boldsymbol{R}^3)$, we have from (4.3) that

$$\int \hat{h}(K_V\langle W(f)\rangle^c_{\beta,V})^\frown \to \int \hat{h}(K\langle W(f)\rangle^c_\beta)^\frown. \qquad (4.4)$$

From this we obtain the following:

Lemma 3. *For all $\beta,\bar{\varrho}>0$ and any sequence $V\to\infty$,*

$$\int h(\varrho)K_V(\varrho,\bar{\varrho})\langle W(f)\rangle^c_{\beta,\varrho,V}\,d\varrho\to\int h(\varrho)K(\varrho,\bar{\varrho})\langle W(f)\rangle^c_{\beta,\varrho}\,d\varrho \qquad (4.5)$$

if h has Fourier transform $\hat{h}\in\mathscr{L}^1$ (or if $h\equiv 1$).

The proof that the canonical states converge will be based on Lemma 3, which we have obtained by the method of Kac, together with the next lemma which we will prove in § 5.

Lemma 4. *Given $\beta>0$, $f\in\mathscr{C}^\infty_{\mathrm{comp}}(\boldsymbol{R}^3)$, and $[a,b]\subset(0,\infty)$, there is a constant C such that*

$$\left|\frac{d}{d\varrho}\langle W(f)\rangle^c_{\beta,\varrho,V}\right|\leqq C \qquad (4.6)$$

for a.e. $\varrho\in[a,b]$ and all V.

Now we can replace K_V by K in Lemma 3 if $h\in\mathscr{C}^\infty_0([a,b])$:

$$\int h(\varrho)K(\varrho,\bar{\varrho})\langle W(f)\rangle^c_{\beta,\varrho,V}\,d\varrho\to\int h(\varrho)K(\varrho,\bar{\varrho})\langle W(f)\rangle^c_{\beta,\varrho}\,d\varrho. \qquad (4.7)$$

In fact, with $F_V(\varrho)=\langle W(f)\rangle^c_{\beta,\varrho,V}$, since $|\hat{K}_V|\leqq 1$,

$$\left|\int\{K_V(\varrho,\bar{\varrho})-K(\varrho,\bar{\varrho})\}h(\varrho)F_V(\varrho)d\varrho\right| = \left|\int\{\hat{K}_V(-\xi,\bar{\varrho})-\hat{K}(-\xi,\bar{\varrho})\}(hF_V)^\frown(\xi)\frac{d\xi}{2\pi}\right|$$

$$\leqq\int_{-R}^{R}|\hat{K}_V(-\xi,\bar{\varrho})-\hat{K}(-\xi,\bar{\varrho})|\cdot|(hF_V)^\frown(\xi)|\frac{d\xi}{2\pi}+2\int_{|\xi|>R}|(hF_V)^\frown(\xi)|\frac{d\xi}{2\pi}.$$

The second term is dominated by

$$\frac{1}{\pi}\left(\int_{|\xi|>R}|1+\xi|^{-2}\,d\xi\right)^{\frac{1}{2}}\left(\int|(1+\xi)(hF_V)^{\widehat{}}(\xi)|^2\,d\xi\right)^{\frac{1}{2}}$$

where the second factor is V-uniformly bounded by Lemma 4 and the first factor is arbitrarily small for R large enough. The first term is arbitrarily small for V large enough by Lemma 2 and the fact that $\|hF_V\|_1 \leqq \|h\|_1$.

In case (I): $\bar{\varrho} \leqq \varrho_c(\beta)$, $K(\varrho,\bar{\varrho}) = \delta(\varrho-\bar{\varrho})$, we have immediately from (4.7) that

$$\langle W(f)\rangle^c_{\beta,\bar{\varrho},V} \to \langle W(f)\rangle^c_{\beta,\bar{\varrho}}\,. \tag{4.8}$$

In case (II): $\bar{\varrho} > \varrho_c(\beta)$, we can choose h so that

$$g(\varrho) = K(\varrho,\bar{\varrho})\,h(\varrho) \tag{4.9}$$

for any given $g \in \mathscr{C}^\infty_{\mathrm{comp}}((\varrho_c,\infty))$. So from (4.7) we have the weak convergence

$$\int g(\varrho)\,\langle W(f)\rangle^c_{\beta,\varrho,V}\,d\varrho \to \int g(\varrho)\,\langle W(f)\rangle^c_{\beta,\varrho}\,d\varrho\,. \tag{4.10}$$

Finally, for $\beta>0$, $f\in\mathscr{C}^\infty_{\mathrm{comp}}(\boldsymbol{R}^3)$, and $[a,b]\subset(0,\infty)$ fixed, we have by the inequality $|\langle W(f)\rangle^c_{\beta,\varrho,V}| \leqq 1$ and Lemma 4 that the family of functions $\langle W(f)\rangle^c_{\beta,\varrho,V}$ is precompact in the uniform topology of $\mathscr{C}([a,b])$. By (4.8) and (4.10) the only possible limit point is $\langle W(f)\rangle^c_{\beta,\varrho}$ and this proves Theorem 1.

§ 5. The Equicontinuity Estimate

In this section we will prove Lemma 4.

Using (1.14) in a straightforward calculation (see e.g. [1] for the main part of it), we obtain

$$\begin{aligned}
&\operatorname{tr}\left(P_N e^{a^*\left(\frac{i}{\sqrt{2}}f\right)} e^{a\left(\frac{i}{\sqrt{2}}\bar{f}\right)} e^{-\beta H_{0V}}\right) \qquad (5.1)\\
&= \sum_{\{\Sigma n(\boldsymbol{k})=N\}}\ \sum_{\{j(\boldsymbol{k})\leqq n(\boldsymbol{k})\}}\ \prod_{\boldsymbol{k}} e^{-\beta\omega(\boldsymbol{k})n(\boldsymbol{k})}\left(\frac{n(\boldsymbol{k})\ldots(n(\boldsymbol{k})-j(\boldsymbol{k})+1)}{j(\boldsymbol{k})!^2}\right)\\
&\quad\cdot\left(-\frac{\varrho}{2N}|\hat{f}(\boldsymbol{k})|^2\right)^{j(\boldsymbol{k})}\\
&= \sum_{\{\Sigma n(\boldsymbol{k})=N\}}\ \prod_{\boldsymbol{k}} e^{-\beta\omega(\boldsymbol{k})n(\boldsymbol{k})} L_{n(\boldsymbol{k})}\left(\frac{\varrho}{2N}|\hat{f}(\boldsymbol{k})|^2\right)
\end{aligned}$$

where $\boldsymbol{k}$ ranges over $\dfrac{2\pi}{L}\boldsymbol{Z}^3$, $N=\varrho V$, the first sum is over all functions of k that are non-negative integral valued with sum N, the second sum is over all non-negative integral valued functions $j(\boldsymbol{k}) \leqq n(\boldsymbol{k})$, and L_l is the Laguerre polynomial of order l. By (1.6), the definition of canonical state in (1.17) and (1.18), and (5.1), the inequality (4.6) of Lemma 4 becomes the inequality

$$\left| \sum_{\left\{\substack{\Sigma n(\boldsymbol{k})=N \\ \Sigma m(\boldsymbol{k})=N+1}\right\}} \left[\left\{\prod_{\boldsymbol{k}} e^{-\beta\omega(\boldsymbol{k})(n(\boldsymbol{k})+m(\boldsymbol{k}))} L_{n(\boldsymbol{k})}\left(\frac{\varrho}{2N}|\hat{f}(\boldsymbol{k})|^2\right)\right\} - \left\{\prod_{\boldsymbol{k}} e^{-\beta\omega(\boldsymbol{k})(n(\boldsymbol{k})+m(\boldsymbol{k}))} L_{m(\boldsymbol{k})}\left(\frac{\varrho}{2N}|\hat{f}(\boldsymbol{k})|^2\right)\right\}\right]\right| \leqq \frac{C\varrho}{N} \sum_{\left\{\substack{\Sigma n(\boldsymbol{k})=N \\ \Sigma m(\boldsymbol{k})=N+1}\right\}} \left\{\prod_{\boldsymbol{k}} e^{-\beta\omega(\boldsymbol{k})(n(\boldsymbol{k})+m(\boldsymbol{k}))}\right\}, \tag{5.2}$$

or

$$\left| \sum_{\{\Sigma g(\boldsymbol{k})=2N+1\}} \prod_{\boldsymbol{k}} e^{-\beta\omega(\boldsymbol{k})g(\boldsymbol{k})} \sum_{\left\{\substack{\Sigma n(\boldsymbol{k})=N \\ n(\boldsymbol{k})\leqq g(\boldsymbol{k})}\right\}} \left[\prod_{\boldsymbol{k}} L_{n(\boldsymbol{k})}\left(\frac{\varrho}{2N}|\hat{f}(\boldsymbol{k})|^2\right) - \prod_{\boldsymbol{k}} L_{g(\boldsymbol{k})-n(\boldsymbol{k})}\left(\frac{\varrho}{2N}|\hat{f}(\boldsymbol{k})|^2\right)\right]\right| \leqq \frac{C\varrho}{N} \sum_{\{\Sigma g(\boldsymbol{k})=2N+1\}} \prod_{\boldsymbol{k}} e^{-\beta\omega(\boldsymbol{k})g(\boldsymbol{k})} \sum_{\left\{\substack{\Sigma n(\boldsymbol{k})=N \\ n(\boldsymbol{k})\leqq g(\boldsymbol{k})}\right\}} 1 \tag{5.3}$$

where the first sum is over all functions g of $\boldsymbol{k}$ that are non-negative integer valued with sum $2N+1$. So it suffices to prove, for given g, that

$$\left| \sum_{\left\{\substack{\Sigma n(\boldsymbol{k})=N \\ n(\boldsymbol{k})\leqq g(\boldsymbol{k})}\right\}} \left[\prod_{\boldsymbol{k}} L_{n(\boldsymbol{k})}\left(\frac{\varrho}{2N}|\hat{f}(\boldsymbol{k})|^2\right) - \prod_{\boldsymbol{k}} L_{\Phi n(\boldsymbol{k})}\left(\frac{\varrho}{2N}|\hat{f}(\boldsymbol{k})|^2\right)\right]\right| \leqq \frac{C\varrho}{N} \sum_{\left\{\substack{\Sigma n(\boldsymbol{k})=N \\ n(\boldsymbol{k})\leqq g(\boldsymbol{k})}\right\}} 1, \tag{5.4}$$

where Φ is an arbitrary bijection

$$\Phi: \left\{\begin{matrix}\Sigma n(\boldsymbol{k})=N \\ n(\boldsymbol{k})\leqq g(\boldsymbol{k})\end{matrix}\right\} \to \left\{\begin{matrix}\Sigma m(\boldsymbol{k})=N+1 \\ m(\boldsymbol{k})\leqq g(\boldsymbol{k})\end{matrix}\right\}. \tag{5.5}$$

(Note that $n \to g-n$ was such a bijection.)

We shall prove in the Combinatorial Proposition below that there exists a bijection (5.5) with the property that for every function n there is a $\boldsymbol{k}_0$ such that

$$\begin{cases} n(\boldsymbol{k}) = \Phi n(\boldsymbol{k}) & \boldsymbol{k} \neq \boldsymbol{k}_0, \\ n(\boldsymbol{k}_0)+1 = \Phi n(\boldsymbol{k}_0). \end{cases} \tag{5.6}$$

Also, we will show, for N sufficiently large (depending only on a bound on $\varrho|\hat{f}(\boldsymbol{k})|^2$), that

$$\begin{aligned} \left|L_{n(\boldsymbol{k})}\left(\frac{\varrho}{2N}|\hat{f}(\boldsymbol{k})|^2\right)\right| &\leqq 1\,, \\ \left|L_{n(\boldsymbol{k})}\left(\frac{\varrho}{2N}|\hat{f}(\boldsymbol{k})|^2\right)-L_{n(\boldsymbol{k})+1}\left(\frac{\varrho}{2N}|\hat{f}(\boldsymbol{k})|^2\right)\right| &\leqq \frac{\text{const}}{N}\,. \end{aligned} \tag{5.7}$$

Using (5.6), we rewrite the left side of (5.4):

$$\left|\sum_{\left\{\substack{\Sigma n(\boldsymbol{k})=N \\ n(\boldsymbol{k})\leqq g(\boldsymbol{k})}\right\}} \prod_{\boldsymbol{k}\neq\boldsymbol{k}_0} L_{n(\boldsymbol{k})}\left(\frac{\varrho}{2N}|\hat{f}(\boldsymbol{k})|^2\right)\left[L_{n(\boldsymbol{k}_0)}\left(\frac{\varrho}{2N}|\hat{f}(\boldsymbol{k})|^2\right) - L_{n(\boldsymbol{k}_0)+1}\left(\frac{\varrho}{2N}|\hat{f}(\boldsymbol{k})|^2\right)\right]\right|.$$

By (5.7), this is dominated by the right side of (5.4) since $\varrho\geqq a>0$. So to complete the proof we only need to establish (5.6) and (5.7).

To prove (5.7) we notice that the Laguerre polynomials of order l,

$$L_l(z)=\sum_{m=0}^{l}\frac{l(l-1)\dots(l-m+1)}{m!^2}(-z)^m\,, \tag{5.8}$$

have the following properties:

$$L_l\left(\frac{z}{l}\right)\to J_0(2\sqrt{z})=\sum_{m=0}^{\infty}\frac{1}{m!^2}(-z)^m \tag{5.9}$$

as $l\to\infty$, as entire analytic functions.

$$\left|L_l\left(\frac{z}{n}\right)\right|\leqq 1\,,\qquad z\in[0,d] \tag{5.10}$$

for any $d\geqq 0$, $l\leqq n$, and $n\geqq n_0(d)$.

$$\frac{z}{l+1}L'_{l+1}(z)=L_{l+1}(z)-L_l(z)\,. \tag{5.11}$$

$$\left|\frac{1}{l+1}L'_{l+1}\left(\frac{z}{n}\right)\right|\leqq \text{const},\qquad z\in[0,d]\,, \tag{5.12}$$

for any $d>0$, $l\leqq n$, where the constant depends on d. The convergence (5.9) is clear and (5.12) follows from the convergence of the derivatives; also (5.11) is well known. We check (5.10): Since $|J_0(2\sqrt{z})|<1$ for $z>0$, we have by (5.9) for given $\varepsilon>0$ and l sufficiently large that $\left|L_l\left(\frac{z}{l}\right)\right|\leqq 1$

for $z \in [\varepsilon, d]$. Choose $\varepsilon > 0$ so that $L_l\left(\frac{z}{l}\right) > -1$ and $\frac{d}{dz} J_0(2\sqrt{z}) < -\frac{1}{4}$ for $z \in [0, \varepsilon]$. From the convergence of the derivatives of (5.9), we have for sufficiently large l that $\frac{d}{dz} L_l\left(\frac{z}{l}\right) < -\frac{1}{8}$, $z \in [0, \varepsilon]$. Therefore, since $L_l(0) = 1$, $\left|L_l\left(\frac{z}{l}\right)\right| \leqq 1$ for $z \in [0, d]$ and $l \geqq l_0(d)$. Finally, for the finitely many cases $l < l_0(d)$, we choose $n_0(d)$ sufficiently large so that $|L_l(z)| \leqq 1$ for $z \in \left[0, \frac{d}{n_0(d)}\right]$.

Finally, (5.6) is given by the following proposition. We make a trivial change in notation and consider sequences rather then functions on $\frac{2\pi}{L} \boldsymbol{Z}^3$.

Combinatorial Proposition. *Given a non-negative integer N and a sequence of non-negative integers $g = \{g_0, g_1, \ldots\}$ such that $\Sigma g_i = 2N + 1$, let*

$$S_g = \{\{n_0, n_1, \ldots\} : 0 \leqq n_i \leqq g_i,\ \Sigma n_i = N,\ n_i \in \boldsymbol{Z}\}$$

$$S'_g = \{\{m_0, m_1, \ldots\} : 0 \leqq m_i \leqq g_i,\ \Sigma m_i = N + 1,\ m_i \in \boldsymbol{Z}\}.$$

There exists a bijection $\Phi : S_g \to S'_g$ with the property that

$$\Sigma\, |n_i - (\Phi n)_i| = 1\,, \tag{$*$}$$

for all $\{n_0, n_1, \ldots\} \in S_g$.

Proof. The proposition is trivial if there are only one or two non-zero g_i's and it is easily checked if $N = 0, 1, 2$. We shall give a proof by induction on N and the number of non-zero g_i's.

For notational convenience suppose that the non-zero g_i's are $\{g_0, g_1, \ldots, g_j\}$. Let

$$k[h]\, l = \{\{k, a_1, a_2, \ldots, a_{j-1}, l\} : 0 \leqq a_i \leqq g_i, \Sigma\, a_i = h,\ a_i \in \boldsymbol{Z}\}\,,$$

given integers k, h, and l. Let $G \equiv \sum_{i=1}^{j-1} g_i = 2N + 1 - g_0 - g_j$. In this notation we have

$$S_g = \bigcup_{\substack{0 \leqq k \leqq g_0 \\ 0 \leqq l \leqq g_j \\ 0 \leqq N-k-l \leqq G}} k[N - k - l]\, l\,,$$

$$S'_g = \bigcup_{\substack{0 \leqq k \leqq g_0 \\ 0 \leqq l \leqq g_j \\ 0 \leqq N-k-l+1 \leqq G}} k[N - k - l + 1]\, l\,.$$

Let

$$S_g^{(1)} = \Bigg(\bigcup_{\substack{1 \leqq l \leqq g_j \\ 0 \leqq N - g_0 - l \leqq G}} g_0[N - g_0 - l]\, l \Bigg) \cup \Bigg(\bigcup_{\substack{0 \leqq k \leqq g_0 \\ 0 \leqq N - k \leqq G}} k[N-k]0 \Bigg)$$

$$S_g^{(2)} = \bigcup_{\substack{0 \leqq k \leqq g_0 - 1 \\ 1 \leqq l \leqq g_j \\ 0 \leqq N - k - l \leqq G}} k[N - k - l]\, l$$

$$S_g'^{(1)} = \Bigg(\bigcup_{\substack{1 \leqq l \leqq g_j \\ 0 \leqq n - g_0 - l + 1 \leqq G}} g_0[N - g_0 - l + 1]\, l \Bigg) \cup \Bigg(\bigcup_{\substack{0 \leqq k \leqq g_0 \\ 0 \leqq N - k + 1 \leqq G}} k[N-k+1]0 \Bigg)$$

$$S_g'^{(2)} = \bigcup_{\substack{0 \leqq k \leqq g_0 - 1 \\ 1 \leqq l \leqq g_j \\ 0 \leqq N - k - l + 1 \leqq G}} k[N - k - l + 1]\, l \,.$$

Then

$$S_g = S_g^{(1)} \cup S_g^{(2)}, \qquad S_g' = S_g'^{(1)} \cup S_g'^{(2)}.$$

All of the above unions are disjoint (i.e., the sets involved are pairwise non-intersecting). We shall use induction on j to show that there exists a bijection $\Phi : S_g^{(1)} \to S_g'^{(1)}$ with the property $(*)$ and induction on N to show that there exists a bijection $\Phi : S_g^{(2)} \to S_g'^{(2)}$ with the property $(*)$.

Let $\phi^{(1)}$ and $\phi^{(2)}$ be mappings of sequences given by

$$\{a_0, a_1, \ldots, a_j\} \;\begin{cases} \xrightarrow{\phi^{(1)}} \{a_0 + a_j, a_1, \ldots, a_{j-1}, 0\} \\ \xrightarrow{\phi^{(2)}} \{a_0, a_1, \ldots, a_{j-1}, a_j - 1\} . \end{cases}$$

Let

$$g^{(1)} = \phi^{(1)} g \,,$$
$$g^{(2)} = \{g_0 - 1, g_1, \ldots, g_{j-1}, g_j - 1\} .$$

So $g^{(1)}$ has only $j-1$ non-zero entries and $g^{(2)}$ corresponds to $N-1$.

Now, by using the above representations of $S_g^{(\alpha)}$ and $S_g'^{(\alpha)}$ as disjoint unions, it is easy to see that

$$\phi^{(\alpha)} : S_g^{(\alpha)} \to S_{g^{(\alpha)}}, \qquad \phi^{(\alpha)} : S_g'^{(\alpha)} \to S'_{g^{(\alpha)}}$$

are bijections ($\alpha = 1, 2$). For example, it is clear that

$$\phi^{(1)} : \Bigg(\bigcup_{\substack{1 \leqq l \leqq g_j \\ 0 \leqq N - g_0 - l \leqq G}} g_0[N - g_0 - l]\, l \Bigg) \to \Bigg(\bigcup_{\substack{1 \leqq l \leqq g_j \\ 0 \leqq N - g_0 - l \leqq G}} g_0 + l[N - g_0 - l]0 \Bigg)$$

and

$$\phi^{(1)} : \Bigg(\bigcup_{\substack{0 \leqq k \leqq g_0 \\ 0 \leqq N - k \leqq G}} k[N-k]0 \Bigg) \xrightarrow{\text{identity}} \Bigg(\bigcup_{\substack{0 \leqq k \leqq g_0 \\ 0 \leqq N - k \leqq G}} k[N-k]0 \Bigg)$$

are bijections while the images give the disjoint union

$$\left(\bigcup_{\substack{1\leqq l\leqq g_j \\ 0\leqq N-g_0-l\leqq G}} g_0+l[N-g_0-l]0\right)\cup\left(\bigcup_{\substack{0\leqq k\leqq g_0 \\ 0\leqq N-k\leqq G}} k[N-k]0\right)$$

$$=\bigcup_{\substack{0\leqq k\leqq g_0+g_j \\ 0\leqq N-k\leqq G}} k[N-k]0=S_{g^{(1)}}.$$

By the inductive hypothesis, there exist correspondences

$$S_{g^{(\alpha)}}\leftrightarrow S'_{g^{(\alpha)}}$$

with the property (*). We will show that the property (*) is preserved under $\phi^{(\alpha)-1}$; so we will obtain the desired correspondences

$$S_g^{(\alpha)}\leftrightarrow S_g'^{(\alpha)}$$

with the property (*). This is trivial in the case $\alpha=2$. For the $\alpha=1$ case we note that corresponding elements in $S_{g^{(1)}}$ and $S'_{g^{(1)}}$ can only belong respectively to

$$g_0+l[N-g_0-l]0 \quad \text{and} \quad g_0+l[N-g_0-l+1]0,$$

$$g_0+l-1[N-g_0-l+1]0 \quad \text{and} \quad g_0+l[N-g_0-l+1]0$$

or

$$\bigcup_{\substack{0\leqq k\leqq g_0 \\ 0\leqq N-k\leqq G}} k[N-k]0 \quad \text{and} \quad \bigcup_{\substack{0\leqq k\leqq g_0 \\ 0\leqq N-k+1\leqq G}} k[N-k+1]0.$$

In the first two cases, $\phi^{(1)-1}$ acts by removing l or $l-1$ from the 0^{th} entry and placing l or $l-1$ in the j^{th} entry, and one can see that the property (*) is preserved. In the third case, $\phi^{(1)-1}$ is just the identity operation.

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References

1. Araki, H., Woods, E. J.: Representations of the canonical commutation relations describing a non-relativistic infinite free Bose gas. J. Math. Phys. **4**, 637–662 (1963).
2. Kac, M.: Informal lecture notes.
3. Lewis, J. T., Pulé, J.: The equilibrium states of the free boson gas, to appear.

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