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Energy and Angular Momentum Flow into a Black Hole

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Abstract. The area of the event horizon round a rotating black hole will increase in the presence of a non-axisymmetric or time dependent perturbation. If the perturbation is a matter field, the area increase is related to the fluxes of energy and of angular momentum into the black hole in such a way as to maintain the formula for the area in the Kerr solution. For purely gravitational perturbations one cannot define angular momentum locally but one can use the area increase and the expression for area in terms of mass and angular momentum to calculate the slowing down of a black hole caused by a non-axisymmetric distribution of matter at a distance. It seems that the coupling between the rotation of a black hole and the orbit of a particle going round it can be significant if the angular momentum of the black hole is close to its maximum possible value and if the angular velocity of the particle is nearly equal to that of the black hole.

I. Introduction

It has recently been shown [1, 2] that a rotating black hole cannot be stationary unless the surrounding space-time is asymmetric about the axis of rotation. This seems to imply that a black hole formed in the presence of non-axisymmetric fields would gradually lose its angular momentum to the sources of these fields and would approach a static, non-rotating state. There is no requirement for a static black hole to be axisymmetric. An estimate of this rotational damping has already been made by Press [3] for the case of a scalar field. He calculated the torque produced on its sources by the scalar field and equated this to the rate of loss of angular momentum of the black hole. In this paper we shall describe a different approach which can be used to determine the rate of loss of angular momentum caused by any matter field such as a scalar or electromagnetic field and also by purely gravitational perturbations produced by a non-axisymmetric distribution of matter at a distance from the black hole. In this approach we take the black hole to be represented by a sequence of Kerr solutions with slowly varying para-

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meters m (the mass) and a (the angular momentum per unit mass). The area A of the event horizon round the black hole is then

$$A = 8\pi m(m + (m^2 - a^2)^{\frac{1}{2}}), \tag{1}$$

(units are such that $G = c = 1$). This area can never decrease with time and it will remain constant if and only if the black hole is in a stationary state [1]. One can regard $(A/16\pi)^{\frac{1}{2}}$ as the irreducible part of the total mass m [4] and the remainder as the rotational energy of the black hole. In the presence of a non-axisymmetric perturbation, the area of the horizon and hence the irreducible mass will increase. In the case that the perturbation is a matter field such as a scalar or electromagnetic field we obtain an expression for the rate of increase of area of the horizon in terms of the fluxes of angular momentum and of energy across the horizon. The area increase is just such as to maintain Eq. (1). For a stationary perturbation, such as that considered by Press, there is no energy flow across the horizon and so m remains constant. The angular momentum ma must therefore decrease as the area increases. What is happening is that the rotational energy of the black hole is being converted by dissipation into irreducible mass.

Encouraged by the consistency in the case of matter fields, we are led to apply our method to the case of purely gravitational perturbations produced by distant matter. There is no local energy-momentum tensor for the gravitational field and so we cannot calculate the flux of energy or angular momentum across the horizon as before. However, if the distant sources of the perturbation are initially stationary, the rotational energy that they acquire as a result of the interaction with the black hole will be proportional to the square of the angular momentum transfer. The energy flux in gravitational radiation at infinity will be of the same order or lower and so the mass of the black hole will be constant to first order in the angular momentum also. One can therefore use Eq. (1) to determine the rate of this loss to first order if one knows the rate of increase of area of the horizon. This is more difficult to calculate than in the case of matter field perturbations when the energy-momentum tensor of a matter field gives rise to a Ricci tensor component which enters directly in the equations for the rate of expansion of the horizon. A purely gravitational perturbation on the other hand influences the rate of expansion only indirectly by inducing shear in the horizon. The shear then appears squared in the expansion equation. What happens is that the perturbation produces tidal forces which distort the horizon. If the black hole is not rotating, the distortion of an element of the horizon will be constant in time and the shear will be zero. If the black hole is rotating, however, distortion will be periodic in time and so there will be shear. One can draw an analogy with the tidal drag of a satellite on ordinary rotating stars or planets. In these the rotational energy is dissipated at a rate given by the square of the shear times the coefficient of viscosity. The situation with black holes is similar: the dissipation is given by the square of the shear (only here it is a surface rather than a volume shear) and the dimensionless analogue of the coefficient of viscosity is of order one.

Another reason that purely gravitational perturbations are more difficult than matter ones is that the equations for gravitational perturbations probably do not admit separable solutions. We therefore use dimensional arguments to estimate the dependence of the rate of slowing down on the masses and distances of the distant matter. A calculation of the numerical constant involved will appear in another paper.

II. Matter Field Perturbations

We shall consider the effect of a weak matter field $\mu_{ab\ldots d}$ (such as a scalar or electromagnetic field) on a rotating black hole with mass m and specific angular momentum a. We expand the field in a perturbation series of the form

$$\mu_{ab\ldots d} = \lambda \mu^{(1)}{}_{ab\ldots d} + 0(\lambda^2) \tag{2}$$

where λ is a formal parameter which measures the strength of the field. The energy-momentum tensor of the field will then have the form

$$T_{ab} = \lambda^2 T^{(2)}{}_{ab} + 0(\lambda^3)\,. \tag{3}$$

We shall assume that the sources of the field are far enough away and have low enough mass to source-strength ratio that their gravitational effect on the black hole is negligible compared to the effect of the matter field (this will usually be the case). The metric can then be taken to have the form

$$g_{ab} = g^{(0)}{}_{ab} + \lambda^2 g^{(2)}{}_{ab} + 0(\lambda^3) \tag{4}$$

where $g^{(0)}{}_{ab}$ is the metric of a Kerr solution with parameters m_0 and a_0. We assume that the black hole formed in the past as a result of the collapse of a rotating body. We therefore express the metric in the form (4) only to the future of some spacelike surface $S(t_0)$ which intersects the future event horizon and future null infinity $\mathscr{I}^+$. The region of the Kerr solution to the past of $S(t_0)$ has no physical significance. The field $\mu^{(1)}{}_{ab\ldots d}$ which obeys the appropriate field equation in the background Kerr metric $g^{(0)}{}_{ab}$ need not therefore, be regular on the past event horizon of that metric though it must be so on the future horizon. The surface $S(t_0)$ can be chosen so that nowhere outside the horizon is it tangent to K^a where K^a is that Killing vector of the metric $g^{(0)}{}_{ab}$ which is timelike at infinity and so corresponds to time translations. One can then define a family of spacelike surfaces $S(t)$ $(t \geqq t_0)$ by moving each point of $S(t_0)$ a parameter distance $t - t_0$ up the K^a Killing vector lines. These surfaces can be regarded as surfaces of constant time.

To calculate the increase with time of $A(t)$, the area of the 2-surface $\partial B(t)$ of the intersection of $S(t)$ with the event horizon, we use two of the Newman-Penrose equations [5].

$$\frac{d\varrho}{dv} = \varrho^2 + \sigma\bar{\sigma} + (\varepsilon + \bar{\varepsilon})\,\varrho + \Phi_{00}\,, \tag{5}$$

$$\frac{d\sigma}{dv} = 2\varrho\sigma + (3\varepsilon - \bar{\varepsilon})\,\sigma + \psi_0\,. \tag{6}$$

Here $\varrho = l_{a;b} m^a \bar{m}^b$ is the convergence of the null geodesic generators of the event horizon with tangent vector $l^a = \dfrac{dx^a}{dv}$ and $\sigma = l_{a;b} m^a m^b$ is their complex shear. The complex conjugate null vectors m^a and $\bar{m}^a$ are orthogonal to l^a and satisfy $m^a \bar{m}_a = -1$. The other quantities involved are

$$\Phi_{00} = -\tfrac{1}{2} R_{ab} l^a l^b = 4\pi\, T_{ab} l^a l^b\,,$$

$$\psi_0 = C_{abcd}\, l^a m^b l^c m^d\,,$$

and

$$\varepsilon = \tfrac{1}{2}(l_{a;b} n^a l^b + m_{a;b} \bar{m}^a l^b)$$

where n^a is a real null vector orthogonal to m^a and $\bar{m}^a$ and satisfying $l^a n_a = 1$. We shall choose the vectors m^a and $\bar{m}^a$ to be parallel transported along the null geodesic generators of the horizon and then $\varepsilon = \bar{\varepsilon}$. The real part of ε would be zero if the parameter v along the generator were an affine parameter but instead we shall take it to equal our coordinate t. With this choice

$$\frac{dA(t)}{dt} = -2\int \varrho\, dA \tag{7}$$

where the integral is over the 2-surface $\partial B(t)$.

Since the perturbation in the metric is of order λ^2 from the background Kerr solution, the perturbations of the generators of the horizon will be of the same order. Thus, on the horizon

$$\varrho = \lambda^2 \varrho^{(2)} + 0(\lambda^3)\,, \tag{8}$$

$$\sigma = \lambda^2 \sigma^{(2)} + 0(\lambda^3)\,, \tag{9}$$

since the unperturbed values of the quantities are zero. On the other hand,

$$\varepsilon = \varepsilon^{(0)} + 0(\lambda^2) \tag{10}$$

where on the horizon

$$\varepsilon^{(0)} = \frac{(1 - x^2)^{\frac{1}{2}}}{4m_0(1 + (1 - x^2)^{\frac{1}{2}})}\,, \qquad x = a_0/m_0\,. \tag{11}$$

From Eq. (5) one has

$$\frac{d\varrho^{(2)}}{dt} = 2\varepsilon^{(0)}\varrho^{(2)} + 4\pi T^{(2)}{}_{ab} l^a l^b . \tag{12}$$

Consider as a perturbing field $\mu^{(1)}{}_{ab\ldots d}$ a wave packet which is zero before some time t_1 and which dies away after some time $t_2 (t_2 \gg t_1 \gg t_0)$. The solution will eventually settle down to a stationary state in which the area of the horizon is constant and $\varrho^{(2)}$ is zero. Thus

$$\varrho^{(2)}(t) = -4\pi \exp(2\varepsilon^{(0)} t) \left[\int_t^\infty \exp(-2\varepsilon^{(0)} t') \, T^{(2)}{}_{ab} l^a l^b \, dt' \right] . \tag{13}$$

Note that for $t < t_1, \varrho^{(2)}$ will be non-zero but exponentially small. Substituting this in Eq. (7) and interchanging orders of integration, one obtains an expression for δA, the increase in area caused by the perturbation:

$$\begin{aligned} \delta A = & -\frac{4\pi}{\varepsilon^{(0)}} \exp(2\varepsilon^{(0)} t_0) \int \exp(-2\varepsilon^{(0)} t) \, T^{(2)}{}_{ab} l^a d\Sigma^b \\ & + \frac{4\pi}{\varepsilon^{(0)}} \int T^{(2)}{}_{ab} l^a d\Sigma^b \end{aligned} \tag{14}$$

where $d\Sigma^b = l^b dA\, dt$ is the 3-surface element of the event horizon and the integrals are taken over the portion of the event horizon to the future of $S(t_0)$. The first term in Eq. (14) can be neglected provided that $2\varepsilon^{(0)}(t_1 - t_0) \gg 1$. The null vector l^a tangent to the horizon can be expressed in terms of K^a and $\tilde{K}^a$, the Killing vectors of the background Kerr metric which correspond to time translations and rotations about the axis of symmetry respectively.

$$l^a = K^a - w\tilde{K}^a + 0(\lambda^2) \tag{15}$$

where

$$w = \frac{x}{2m_0[1 + (1 - x^2)^{\frac{1}{2}}]^2} \tag{16}$$

is the angular velocity of the black hole. The vectors $T_{ab}{}^{(2)} K^a$ and $T_{ab}{}^{(2)} \tilde{K}^a$ represent the flow of energy and angular momentum respectively in the matter fields. They are conserved in the background Kerr metric and their fluxes across the horizon give the rates of change of mass and angular momentum of the black hole. Thus

$$\delta A = \frac{4\pi}{\varepsilon^{(0)}} [(1 - w a_0)\, \delta m - w m_0\, \delta a] . \tag{17}$$

Using Eqs. (11) and (16), this becomes

$$\delta A = \frac{8\pi m_0}{(1 - x^2)^{\frac{1}{2}}} [(2 - x^2 + 2(1 - x^2)^{\frac{1}{2}})\, \delta m - x\, \delta a] \tag{18}$$

which is just the change in area needed to maintain Eq. (1). The effect of the wave packet perturbation is to cause the original Kerr solution to evolve to a new Kerr solution with parameters $m_0 + \delta m$, $a_0 + \delta a$. It is therefore reasonable to use a slowly varying family of Kerr solutions to treat the effect of a perturbing matter field $\mu_{ab\ldots d}$ which does not necessarily die away after a certain time. The metric can be expressed as

$$g_{ab} = g^{(0)}{}_{ab}(t) + \lambda^2 g^{(2)}{}_{ab} + 0(\lambda^3) \tag{19}$$

where $g^{(0)}{}_{ab}(t)$ is the Kerr metric with parameters $m(t)$ and $a(t)$ which are the mass and specific angular momentum of the black hole as measured from infinity at the retarded time defined by the intersection of the surface $S(t)$ with $\mathscr{I}^+$. To first order, the matter field at time t will obey the relevant field equation in the metric $g^{(0)}{}_{ab}(t)$ and the rate of charge at $m(t)$ and $a(t)$ can be found by calculating the fluxes of energy and of angular momentum across the horizon. Alternatively, if one is known, the other can be determined from the rate of increase of area of the horizon. A particular case of interest is that in which the sources of the matter field are at rest at a time t_1. As the matter field interacts with the black hole, angular momentum will be transferred to the sources which will start to rotate. However this transfer will be of order λ^2 and so the matter field will be stationary to first order with respect to the Killing vector K^a of the metric $g^{(0)}{}_{ab}(t_1)$. For a stationary scalar or electromagnetic field, one can show directly from the field equations that the flux of energy across the horizon is zero. In fact this must be the case for any perturbation field whose sources are stationary and conserved to first order, for the energy in the field will be constant to order λ^2, the rotational energy of the sources will be of order λ^4 and the flux of gravitational waves at infinity will be of order λ^4 or lower as the perturbation in the metric is of order λ^2. The mass of the black hole will therefore be constant to order λ^2 and the rate of loss of angular momentum will be given by

$$\frac{da}{dt} = -\frac{(m^2 - a^2)^{\frac{1}{2}}}{8\pi ma} \frac{dA}{dt}. \tag{20}$$

Note that as $\frac{dA}{dt} \geqq 0$, a black hole can never gain angular momentum from a stationary field.

III. Purely Gravitational Perturbations

Eq. (20) is not necessary for calculating the slowing down of a black hole by a matter field perturbation since one can calculate the flow of angular momentum across the event horizon from the energy-momentum tensor. It comes into its own for purely gravitational perturbations, however, as these have no well-defined energy-momentum tensor. We shall show how it can be used to determine the rate of slowing down of a black hole by a non-axisymmetric gravitational field produced by distant masses of order λ which are stationary at time t_1. We express the metric as

$$g_{ab} = g^{(0)}{}_{ab} + \lambda g^{(1)}{}_{ab} + 0(\lambda^2)\,. \tag{21}$$

Here $g^{(1)}{}_{ab}$ is the stationary perturbation produced by the stationary masses. The first order stationary perturbation causes a perturbation in the horizon which is stationary in first order. The convergence and shear of the generators of the horizon therefore have the form

$$\varrho = \lambda\varrho^{(1)} + \lambda^2\varrho^{(2)} + 0(\lambda^3)\,, \tag{22}$$

$$\sigma = \lambda\sigma^{(1)} + 0(\lambda^2)\,, \tag{23}$$

where $\varrho^{(1)}$ and $\sigma^{(1)}$ are constant along the lines of K^a. The generators of the horizon intersect a given K^a line repeatedly as the horizon rotates. Thus $\varrho^{(1)}$ and $\sigma^{(1)}$ will be periodic along a generator. Eq. (5) gives the rate of charge of $\varrho^{(1)}$ along a generator as

$$\frac{d\varrho^{(1)}}{dt} = 2\varepsilon^{(0)}\varrho^{(1)}\,. \tag{24}$$

The only periodic solution is $\varrho^{(1)} = 0$. Then

$$\frac{d\varrho^{(2)}}{dt} = \sigma^{(1)}\bar{\sigma}^{(1)} + 2\varepsilon^{(0)}\varrho^{(2)}\,.$$

Integrating this over the 2-surface of intersection of $S(t)$ with the horizon and putting the formal parameter $\lambda = 1$, one obtains

$$\frac{dA}{dt} = \frac{1}{\varepsilon^{(0)}}\left[\int \sigma^{(1)}\bar{\sigma}^{(1)}\,dA - \frac{d}{dt}\int \varrho^{(2)}\,dA\right]. \tag{25}$$

For a stationary perturbation, the second term will be zero and so one obtains finally

$$\frac{da}{dt} = -\frac{A}{4am}\int \sigma^{(1)}\bar{\sigma}^{(1)}\,dA\,. \tag{26}$$

The perturbation in the shear can be determined from equation (6),

$$\sigma^{(1)} = -\exp(2\varepsilon^{(0)}t)\int_t^\infty \exp(-2\varepsilon^{(0)}t')\,\psi_0{}^{(1)}\,dt'\,. \tag{27}$$

Alternatively, one can determine $\sigma^{(1)}$ directly by finding the closed stationary null surface in the stationary metric $g^{(0)}{}_{ab} + \lambda g^{(1)}{}_{ab}$. This will be done in a later paper.

While no precise calculation of the rate of decrease of angular momentum will be given here, Eqs. (25) and (27) can be used to make simple dimensional estimates of the result in various situations. A mass

m' at a distance r will produce a tidal force of order m'/r^3. The component $\psi_0^{(1)}$ of the Weyl tensor will be zero if the black hole is not rotating, as in this case the solution will be stationary. Therefore it seems reasonable on dimensional grounds to guess that $\psi_0^{(1)}$ will be of the order of $(mm'w/r^3)\exp(2iwt)$. The rate of loss of angular momentum will then be of the order of $am^4 m'^2/r^6$.

If the particle is orbiting about the black hole at a distance r large compared to m, it will lose angular momentum in the form of gravitational radiation at a rate of the order of $m^{5/2} m'^2/r^{7/2}$. This is always greater than the rate at which it acquires angular momentum from the black hole. However, the above calculation is valid only if a particle is nearly stationary, i.e. if the angular velocity w' of the particle is small compared to w. If this is not the case one can apply a similar argument in a frame which is rotating at angular velocity w' and estimate that the rate of loss of angular momentum by the black hole is of order

$$(w - w')\, m^8 m'^2/[r^6(m^2 - a^2 + 16\, m^4(w - w')^2)]\,.$$

This will be large if a is nearly equal to m and w' is nearly equal to w.

Another situation in which there could be an appreciable effect is if, instead of a single particle, there was a ring of particles orbiting the black hole. In this case the loss of angular momentum in the form of gravitational radiation would be very small since the ring would be nearly axisymmetric. If the ring was inclined to the equatorial plane of the black hole, it would tend to slow the black hole down. One would expect the axis of rotation of the black hole to change its orientation so as to become perpendicular to the plane of the ring. To show this, however, will require an extension of the present analysis.

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References

1. Hawking, S. W.: Commun. math. Phys. **25**, 152 (1972).
2. — Ellis, G. F. R.: The large scale structure of spacetime. Cambridge University Press, to be published.
3. Press, W.: Astrophys. J., **175**, 243 (1972).
4. Christodoulou, D.: Phys. Rev. Letters **25**, 1596 (1970).
5. Newman, E., Penrose, R.: J. Math. Phys. **3**, 566 (1962).

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