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Some Results in Non-Commutative Ergodic Theory

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Abstract. We study some properties of invariant states on a C^*-algebra $\mathscr{A}$ with a group G of automorphisms. Using the concept of G-factorial state, which is a "non-commutative" generalization of the concept of ergodic measure, in general wider in scope than G-ergodic state, we show that under a certain abelianity condition on $(\mathscr{A}, G)$, which in particular holds for the quasi-local algebras used in statistical mechanics, two different G-ergodic states are disjoint. We also define the concept of G-factorial linear functional, and show that under the same abelianity condition such a functional is proportional to a G-ergodic state. This generalizes an earlier result for complex ergodic measures.

1. Introduction

In a recent paper [1] we studied a possible extension of the concept of ergodic measure from the classical case of a positive measure to an arbitrary complex measure, requiring that for every G-invariant ($|m|$-a.e.) measurable subset E of the space X we have either $m(E)=0$ or $m(X-E)=0$. Here G is the group of transformations of X, and $|m|$ is the total variation of m. It turned out that this extension is essentially trivial, in the sense that such an ergodic measure m is of the form $k|m|$, with k a complex constant ("ergodicity implies positivity"). A related result – which, although it can be considered to be a direct corollary of the above result, is as easily proved directly from the extremality property of positive ergodic measures – is that two positive measures on the same space, ergodic under the same group, are either orthogonal (i.e. their supports are disjoint), or proportional. Namely, if m_1 and m_2 are two non-proportional positive ergodic measures, form $m=m_1+m_2$. Unless there is a measurable set E, G-invariant (m-a.e.), such that $m_1(E)=0$, $m_2(X-E)=0$, m is ergodic, which contradicts the non-trivial decomposition $m=m_1+m_2$. – Expressed in the C^*-algebra language, with $\mathscr{A}$ a C^*-algebra, acted on by a group G of automorphisms, this means that two different G-ergodic states on a commutative C^*-algebra are disjoint, i.e. the corresponding cyclic representations of $\mathscr{A}$ are disjoint. This follows if we realize $\mathscr{A}$ by the Gelfand isomorphism as the C^*-algebra $C(X)$ of continuous functions on the space X of irreducible representations of $\mathscr{A}$.

The present paper is devoted to the extension of these results to non-commutative C^*-algebras. In this connection the natural generalization of ergodic positive measure seems to be not G-ergodic state, but the – in the general case – wider concept of G-factorial state (Definition 3.1). This concept can be defined in a way which carries over to a general G-invariant continuous linear functional on $\mathscr{A}$ (Definition 4.3). In the commutative case this definition coincides with the definition above of an ergodic complex measure (Section 5).

If the set of G-factorial states coincides with the set of G-ergodic states – this is true if $(\mathscr{A}, G)$ satisfies a certain abelianity condition (A) (Definition 2.6), weaker than the requirement that G is a "large" group of automorphisms – then any two different G-ergodic states are disjoint. This applies in particular to the G-ergodic states of the quasi-local algebras used in the description of infinite systems in statistical mechanics. In this description the G-ergodic states are often taken to correspond to equilibrium states of pure phases; two different such states then give rise to disjoint representations of the algebra, i.e. the two sets of pure (irreducible) states into which the two equilibrium states can be decomposed, are disjoint. This disjointness property is of course closely connected with the fact that we have to do with an infinite system; for a finite system it cannot be expected to hold strictly. A very simple example of this is a classical gas in a container, with equilibrium states described by the canonical distribution. Two equilibrium states corresponding to different temperatures certainly overlap in phase space, but in the infinite volume limit, when the canonical distribution approaches the micro-canonical one, the overlap goes to zero.

Still assuming the validity of condition (A) we show that every G-factorial linear functional is proportional to a G-factorial (hence in this case G-ergodic) state, thus generalizing the result in [1]. It should be stressed that in general ergodicity does not imply positivity, i.e. one can find systems $(\mathscr{A}, G)$ – even G-abelian ones – having G-factorial functionals not proportional to G-factorial states, and where different G-ergodic states are not disjoint (they can even be equivalent). Although indefinite linear functionals have no obvious physical interpretation in the ordinary scheme, it should be remarked that an indefinite hermitian functional defines a representation of the algebra in a space with indefinite metric, if one carries through the GNS-construction separately for its positive and negative parts. The concept of G-factorial functional could thus be of interest in this wider framework. Of course, since the resulting representation decomposes into separate representations in the positive and negative signature spaces, one should have, as is generally the case in theories with indefinite metric, a richer structure with additional operators outside the algebra of observables, mixing the representations.

2. Basic Notations and Auxiliary Results

$(\mathscr{A}, G)$ denotes a C^*-algebra $\mathscr{A}$ with a group G of automorphisms: $A \to \tau_g(A)$. $\mathscr{A}^*$ is the dual of $\mathscr{A}$, $\mathscr{A}_G^*$ the set of G-invariant elements in the dual, $\mathscr{E}_G$ the set of G-invariant states on $\mathscr{A}$. All representations we shall consider will be covariant representations (π, U) of $(\mathscr{A}, G)$ in a Hilbert space $\mathscr{H}: \pi(\tau_g(A)) = U(g)\,\pi(A)\,U(g)^{-1}$. $\pi'(U')$ is the commutant of $\pi(U)$, and π'' the von Neumann algebra generated by π (weak closure of π). $\mathscr{C}_G = \pi' \cap \pi'' \cap U'$ is the commutative von Neumann algebra of G-invariant central elements in π''. For $\varrho \in \mathscr{E}_G(\pi_\varrho, U_\varrho, \mathscr{H}_\varrho, x_\varrho)$ denote the corresponding cyclic representation (π_ϱ, U_ϱ) in $\mathscr{H}_\varrho$, with normalized cyclic G-invariant vector x_ϱ, so that $\varrho(A) = (x_\varrho, \pi_\varrho(A)x_\varrho)$. $\pi \geqq \pi_\varrho = \pi P_\varrho$ means that π_ϱ is a subrepresentation of π, with P_ϱ the projector on π_ϱ. Since in this case $\varrho(A) = (x_\varrho, \pi(A)x_\varrho)$ and π'' is the weak closure of π, ϱ extends by continuity to π''; for simplicity we keep the same notation and put $\varrho(B) = (x_\varrho, Bx_\varrho)$, $B \in \pi''$.

We start by stating and proving some simple Lemmas, which will be used in Sections 3 and 4.

Lemma 2.1. *Assume* $\varrho \in \mathscr{E}_G$ *and* $\pi \geqq \pi_\varrho$. *We associate with* ϱ *three projectors acting in* $\mathscr{H}$:

1. *ϱ's projector P_ϱ. We have* $P_\varrho = \mathrm{Proj}[\pi'' x_\varrho]$ *(denotes projector on the subspace generated by* $\{Bx_\varrho : B \in \pi''\}$*). Evidently* $P_\varrho \in \pi' \cap U'$.

2. *ϱ's support E_ϱ. $I - E_\varrho$ is the largest projector in π'' such that* $\varrho(I - E_\varrho) = 0$ ([2], A 26, p. 337). *If $x \in \mathscr{H}$ has the property that $\varrho(A) = (x, \pi(A)x)$, $A \in \mathscr{A}$, then* $E_\varrho = \mathrm{Proj}[\pi' x]$. *We have* $E_\varrho \in \pi'' \cap U'$.

3. *ϱ's central support F_ϱ. F_ϱ is the central support of both P_ϱ and E_ϱ, i.e. is the smallest projector in $\pi' \cap \pi''$ such that $F_\varrho \geqq P_\varrho$ (or $F_\varrho \geqq E_\varrho$). F_ϱ is G-invariant, i.e. $F_\varrho \in \mathscr{C}_G$, and can be expressed in the form* $F_\varrho = \mathrm{Proj}[\pi'\pi'' x_\varrho]$.

Proof. The results in 1. are obvious. – 2. We show that $E_\varrho = \mathrm{Proj}[\pi' x] \equiv E'_\varrho$. Firstly, $E'_\varrho \in \pi''$, and leaves x invariant, so $\varrho(I - E'_\varrho) = 0$. Since $\varrho(E_\varrho) = (x, E_\varrho x) = 1$ for the support E_ϱ, we have $E_\varrho x = x$, i.e. E_ϱ leaves x invariant. Since by definition E_ϱ commutes with π', E_ϱ must contain E'_ϱ. But then $E_\varrho = E'_\varrho$. To show that $E_\varrho \in U'$ we observe that $x = x_\varrho$ is a possible choice; as $U(g)x_\varrho = x_\varrho$, and $U(g)\pi' U(g)^{-1} = \pi'$, it follows from $E_\varrho = \mathrm{Proj}[\pi' x_\varrho]$ that $E_\varrho \in U'$. –

3. The central supports of P_ϱ and E_ϱ are the projectors on $\pi' P_\varrho \mathscr{H}$ and $\pi'' E_\varrho \mathscr{H}$, respectively, i.e. they are both $\mathrm{Proj}[\pi'\pi'' x_\varrho]$; this representation also shows that $F_\varrho \in U'$.

Lemma 2.2. *Assume $(\mathscr{B}, U)$ is a covariant von Neumann algebra (i.e. $\tau_g(\mathscr{B}) \equiv U(g)\mathscr{B}U(g)^{-1} = \mathscr{B}$) in $\mathscr{H}$, and Q a projector in $\mathscr{B}' \cap U'$, with central support I (unit operator). $\mathscr{B}_Q = \mathscr{B}Q$ is the induced von Neumann algebra in $Q\mathscr{H}$; it is evidently covariant. Define $\mathscr{C}_G = \mathscr{B} \cap \mathscr{B}' \cap U'$, $\mathscr{C}_{GQ} = \mathscr{B}_Q \cap \mathscr{B}'_Q \cap U'$. Then the mapping $B \to BQ$ is an isomorphism from $\mathscr{C}_G$ onto $\mathscr{C}_{GQ}$.*

Proof. According to ([2], A 15, A 20, pp. 335–336) $\mathscr{B} \ni B \to BQ \in \mathscr{B}_Q$ is an isomorphism, hence so is $\mathscr{B} \cap \mathscr{B}' \ni B \to BQ \in \mathscr{B}_Q \cap \mathscr{B}'_Q$. As $\tau_g(Q) = Q$, all $g \in G$, this isomorphism commutes with all τ_g, so the result follows.

Lemma 2.3. *Assume $(\mathscr{B}, U)$ as in Lemma 2.2, and put*

$$f(B) = \sum_{\nu=1}^{n} a_\nu (x_\nu, B x_\nu),$$

all $B \in \mathscr{B}$; here $(a_1, \dots a_n) \in C^n$, and $x_1, \dots x_n$ are G-invariant vectors in $\mathscr{H}$, defining a subspace $\mathscr{H}_n$. Define $Q = \mathrm{Proj}[\mathscr{B}\mathscr{H}_n]$; evidently $Q \in \mathscr{B}' \cap U'$. F is the central support of Q, and $\mathscr{C}_G = \mathscr{B} \cap \mathscr{B}' \cap U'$, $\mathscr{C}_{GF} = \mathscr{B}_F \cap \mathscr{B}'_F \cap U'$, $\mathscr{C}_{GQ} = \mathscr{B}_Q \cap \mathscr{B}'_Q \cap U'$.

Then (a), (b), *and* (c) *are equivalent:*

(a) $f(P)f(I-P) = 0$, *every projector* $P \in \mathscr{C}_G$.

(b) $f(P)f(F-P) = 0$, *every projector* $P \in \mathscr{C}_{GF}$.

(c) $f(P)f(Q-P) = 0$, *every projector* $P \in \mathscr{C}_{GQ}$.

In particular, if $\varrho \in \mathscr{E}_G$, and $\pi \geqq \pi_\varrho$, then $\varrho(P) = 0$ or 1 for every projector $P \in \mathscr{C}_G$ if and only if $\varrho(P) = 0$ or 1 for every projector $P \in \mathscr{C}_{\varrho G} = \pi'_\varrho \cap \pi''_\varrho \cap U'_\varrho$.

Proof. (b)⇔(c), since from Lemma 2.2 $\mathscr{C}_{GQ}$ and $\mathscr{C}_{GF}$ are isomorphic, and $f(B) = f(BQ)$, all $B \in \mathscr{B}_F$. (b)⇒(a) follows from $f(B) = f(BF)$, all $B \in \mathscr{B}$. (a)⇒(b) is a consequence of the fact that $f(F-P) = f(I-P)$, every projector $P \in \mathscr{C}_{GF} \subset \mathscr{C}_G$.

We recall the following definitions and criteria ([2], 5.2 and 5.3):

Definition 2.4. *Two states ϱ_1, ϱ_2 are said to be equivalent ($\simeq$), quasi-equivalent ($\approx$), or disjoint ($\between$), if the corresponding relation holds between π_{ϱ_1} and π_{ϱ_2}.*

Criteria 2.5. *for quasi-equivalence and disjointness: Assume $\pi \geqq \pi_{\varrho_1}$, $\pi \geqq \pi_{\varrho_2}$, F_1 and F_2 the central supports of ϱ_1 and ϱ_2, respectively. Then we have*

$$\varrho_1 \approx \varrho_2 \Leftrightarrow F_1 = F_2$$

$$\varrho_1 \between \varrho_2 \Leftrightarrow F_1 F_2 = 0\,.$$

Finally we introduce and discuss an abelianity condition for $(\mathscr{A}, G)$, which will turn out to be a sufficient condition under which we can generalize to the non-commutative case the results mentioned in the beginning of the introduction for the commutative case.

Definition 2.6. *$(\mathscr{A}, G)$ is said to satisfy condition (A), if $\pi'_\varrho \cap U'_\varrho \subset \pi''_\varrho$ for every $\varrho \in \mathscr{E}_G$.*

The inclusion relation $\pi'_\varrho \cap U'_\varrho \subset \pi''_\varrho$ is trivially fulfilled for every G-ergodic ϱ. If the group G is trivial, $G=\{e\}$, then condition (A) is equivalent to requiring $\mathscr{A}$ to be commutative. This is perhaps most directly seen using the result (Theorem 3.4) that if condition (A) holds, any two different G-ergodic states are disjoint. Assume that $\mathscr{A}$ had an irreducible representation of dimension $\geqq 2$; then two different unit vectors in the space of this representation would define two different, equivalent pure states on $\mathscr{A}$, which contradicts Theorem 3.4. So all irreducible representations of $\mathscr{A}$ are one-dimensional, and this implies that $\mathscr{A}$ is commutative. – That $\mathscr{A}$ commutative implies condition (A) for any G is evident, since for any cyclic representation of a commutative $\mathscr{A}$ we have $\pi' = \pi''$ (π'' is a maximal commutative von Neumann algebra).

If G is a "large" group of automorphisms of $\mathscr{A}$ (hence a fortiori if $(\mathscr{A}, G)$ is weakly asymptotically abelian; see [3], p. 430), condition (A) holds ([4], Ex. 6 C, p. 164). Condition (A) is strictly weaker than G being a "large" group, as is shown by example "$1 \neq 2$" ([3], p. 431) of the algebra of compact operators in a Hilbert space with a certain group G, which is not a "large" group; in this case there is only one G-invariant state, which is then G-ergodic, so condition (A) holds. The G-abelian C^*-algebra of complex 2×2-matrices, with G the group of diagonal unitary matrices acting by conjugation (example "$0 \neq 1$" in [3], p. 431) does not fulfil condition (A): the set of G-invariant states is given by $\varrho_\alpha = \alpha\varrho_1 + (1-\alpha)\varrho_0$, $0 \leqq \alpha \leqq 1$, where the extremal states (which are even pure states) ϱ_0 and ϱ_1 are equivalent; by direct construction one easily shows that the non-extremal states don't fulfil the requirement of condition (A). Alternatively, it follows from Theorem 3.4 that condition (A) cannot hold. Thus G-abelianness does not imply condition (A). We conjecture that the inverse implication is also not true, so that G-abelianness and condition (A) are independent abelianity conditions.

3. G-Factorial States

We recall that a state ϱ on $\mathscr{A}$ is *factorial*, if $\pi'_\varrho \cap \pi''_\varrho = \{\lambda I\}$; $\varrho \in \mathscr{E}_G$ is *G-ergodic*, if it is extremal in $\mathscr{E}_G$, or, equivalently, if $\pi'_\varrho \cap U'_\varrho = \{\lambda I\}$.

Definition 3.1. *$\varrho \in \mathscr{E}_G$ is G-factorial, if $\pi'_\varrho \cap \pi''_\varrho \cap U'_\varrho = \{\lambda I\}$, i.e. if 0 and I are the only projectors in $\mathscr{C}_{\varrho G}$.*

In terms of the decomposition theory of states (see [5], Chapter 3, for a recent summary of the present status of this theory) Definition 3.1 is equivalent to requiring that the central measure μ_ϱ of ϱ on the state space $\mathscr{E}$ is ergodic, since under the isomorphism from $\pi'_\varrho \cap \pi''_\varrho$ onto $L^\infty(\mathscr{E}, \mu_\varrho)$ G-invariant projectors in $\pi'_\varrho \cap \pi''_\varrho$ evidently correspond to characteristic functions of G-invariant measurable sets (modulo μ_ϱ) in $\mathscr{E}$. We recall that the measure μ_ϱ is concentrated on the subset $\mathscr{E}_f$ of factorial states, and, at least if $\mathscr{A}$ is separable, intersects each quasi-equivalence class of factorial states at most in one point (loosely speaking). Since the decomposition theory, although very "anschaulich", is complicated by measure-theoretical intricacies, which often force one to make restrictive separability assumptions, we shall use it for purpose of illustration only.

If $\varrho \in \mathscr{E}_G$ is factorial, or G-ergodic, then ϱ is G-factorial. Thus we have $\mathscr{E}_{Ge} \subset \mathscr{E}_{Gf}$, where $\mathscr{E}_{Ge}(\mathscr{E}_{Gf})$ is the set of G-ergodic (G-factorial) states on $\mathscr{A}$.

Alternative characterizations of $\mathscr{E}_{Gf}$ are given by

Theorem 3.2. *Assume $\varrho \in \mathscr{E}_G$, $\pi \geqq \pi_\varrho$, $F \in \mathscr{C}_G$ the central support of ϱ. Then* (a), (b), *and* (c) *are equivalent.*

(a) $\varrho \in \mathscr{E}_{Gf}$.

(b) *F is minimal non-trivial projector in $\mathscr{C}_G$.*

(c) *For every projector $P \in \mathscr{C}_G$ we have $\varrho(P) = 0$ or 1.*

Proof. Follows directly from Definition 3.1 and Lemmas 2.2 and 2.3.

Next we give a theorem, which states that two G-factorial states are either quasi-equivalent or disjoint. Furthermore, the set of G-factorial states consists of quasi-equivalence classes of G-invariant states, where each such class forms a convex set, the extremal points of which (if they exist) are also extremal in $\mathscr{E}_G$, i.e. are G-ergodic.

Theorem 3.3. *Assume $\varrho_1, \varrho_2 \in \mathscr{E}_G$. Then we have*

(a) *If $\varrho_1, \varrho_2 \in \mathscr{E}_{Gf}$, then either $\varrho_1 \approx \varrho_2$, or $\varrho_1 \between \varrho_2$.*

(b) *If $\varrho_1 \in \mathscr{E}_{Gf}$, and $\varrho_1 \approx \varrho_2$, then for every $\alpha \in [0,1]$ $\varrho = \alpha\varrho_1 + (1-\alpha) \cdot \varrho_2 \in \mathscr{E}_{Gf}$, and $\varrho \approx \varrho_1$; in particular $\varrho_2 \in \mathscr{E}_{Gf}$.*

(c) *If $\varrho = \alpha\varrho_1 + (1-\alpha)\varrho_2 \in \mathscr{E}_{Gf}$, some $\alpha \in (0,1)$, then $\varrho_1, \varrho_2 \in \mathscr{E}_{Gf}$, and $\varrho_1 \approx \varrho_2$.*

In terms of the ergodic central measures of ϱ_1 and ϱ_2 (a) means that if one equivalence class of factorial states meets the supports of both measures, then every class either meets both supports, or none. We can also remark that if – for a given $(\mathscr{A}, G)$ – two different quasi-equivalent G-factorial states exist (example "$0 \neq 1$" at the end of Section 2 is an example of this), then the map $\mathscr{E}_G \ni \varrho \to \mu_\varrho$ cannot be affine, since this would lead to the absurd result that a convex combination of two different ergodic measures is ergodic.

Proof of Theorem 3.3. (a) Apply Criteria 2.5 to $\pi = \pi_{\varrho_1} \oplus \pi_{\varrho_2}$; if neither $\varrho_1 \approx \varrho_2$ nor $\varrho_1 \between \varrho_2$, then $F = F_1 F_2$ would be a non-trivial projector in

$\mathscr{C}_G$, strictly smaller than at least one of F_1 and F_2, thus contradicting Theorem 3.2 (b).

(b) Take ϱ, $0<\alpha<1$, and form $\pi=\pi_\varrho \oplus \pi_{\varrho_1} \oplus \pi_{\varrho_2}$; since $\varrho_1 \approx \varrho_2$, they have common central support F, which is minimal, since $\varrho_1 \in \mathscr{E}_{Gf}$; hence also $\varrho_2 \in \mathscr{E}_{Gf}$. From $\varrho_1(F)=\varrho_2(F)=1$ follows that $\varrho_1(P)=0$ or 1 and $\varrho_2(P)=0$ or 1 simultaneously for a projector $P \in \mathscr{C}_G$, and then we have also $\varrho(P)=0$ or 1, so that $\varrho \in \mathscr{E}_{Gf}$. In particular F is also the central support of ϱ, so $\varrho \approx \varrho_1$.

(c) π is defined as under (b). If $\varrho \in \mathscr{E}_{Gf}$, some α, $0<\alpha<1$, we have for every projector $P \in \mathscr{C}_G$ either $\varrho(P)=0$ or 1, and then necessarily $\varrho_1(P)=0$ or 1, $\varrho_2(P)=0$ or 1, simultaneously, hence $\varrho_1, \varrho_2 \in \mathscr{E}_{Gf}$, and $\varrho_1 \approx \varrho_2$.

From Theorem 3.3 follows easily

Theorem 3.4. *Consider the following statements about* $(\mathscr{A}, G)$*:*

(a) $(\mathscr{A}, G)$ *fulfils condition* (A), *Definition* 2.6.

(b) $\mathscr{E}_{Ge}=\mathscr{E}_{Gf}$, *i.e. every G-factorial state is G-ergodic.*

(c) *Any two different G-factorial states are disjoint.*

(d) *Any two different G-ergodic states are disjoint.*

We have (a)$\Rightarrow$(b)$\Leftrightarrow$(c)$\Rightarrow$(d).

Proof. (a)$\Rightarrow$(b) and (c)$\Rightarrow$(d) are trivial. (b)$\Rightarrow$(c): According to Theorem 3.3 (a) two different G-factorial states are either quasi-equivalent or disjoint; if they were quasi-equivalent, then every convex combination would by Theorem 3.3 (b) be G-factorial, hence G-ergodic, which is absurd. (c)$\Rightarrow$(b): if there were a G-factorial state which is not G-ergodic, then it is not extremal, and we could write it as a non-trivial convex combination of G-factorial states, which would then be quasi-equivalent, by Theorem 3.3 (c), contradicting (c).

Put in words Theorem 3.4 expresses the obvious fact that if every G-factorial state is G-ergodic – which follows trivially from condition (A) – then every G-factorial state is extremal, so the quasi-equivalence classes building up $\mathscr{E}_{Gf}$ contain only one element each. Combined with Theorem 3.3 (a) this gives the disjointness of different G-factorial states. An illustration is given by the fact – which we shall not prove here – that if condition (A) holds, then the map $\mathscr{E}_G \ni \varrho \rightarrow \mu_\varrho$ from G-invariant states to the corresponding central measures is actually affine; cf. remark after Theorem 3.3.

The weaker result that condition (A), combined with G-abelianness, implies that two different G-ergodic states are not quasi-equivalent, is well-known (see e.g. [4], Ex. 6 D, p. 165; [6], Theorem 3.9, p. 129). – (c) does not follow from G-abelianness alone, as we already remarked in the discussion following Definition 2.6. The G-ergodic states ϱ_0 and ϱ_1 in example "$0 \neq 1$" discussed there are even equivalent.

4. G-Factorial Linear Functionals

We recall that every $f \in \mathscr{A}^*$ has a unique decomposition $f = g + ih$ into hermitian elements g and h; further g and h have unique decompositions into orthogonal positive functionals:

$$g = g_+ - g_-, \quad \|g\| = \|g_+\| + \|g_-\|, \quad h = h_+ - h_-, \quad \|h\| = \|h_+\| + \|h_-\| .$$

If f is G-invariant, it follows from the uniqueness of the decompositions that also g_+ etc. are G-invariant. We put $\hat{f} = (g_+ + g_- + h_+ + h_-)/N$, $N = \|g\| + \|h\|$; if $f \in \mathscr{A}_G^*$, then evidently $\hat{f} \in \mathscr{E}_G$.

To generalize the concept "G-factorial" from states to general linear functionals we must define a class of representations carrying enough information about the functional. We also prove a Lemma exhibiting explicitly representations of this class, and showing that the definition of G-factorial linear functional introduced later is independent of the representative of the class one chooses for the definition.

Definition 4.1. *Assume $f \in \mathscr{A}_G^*$. A covariant representation (π, U) in $\mathscr{H}$ is called a f-representation, if there are G-invariant vectors $x_1, \dots x_n$ in $\mathscr{H}$, and complex numbers $a_1, \dots a_n$ such that*

$$f(A) = \sum_{\nu=1}^{n} a_\nu (x_\nu, \pi(A) x_\nu), \quad \textit{all} \quad A \in \mathscr{A} .$$

Lemma 4.2. *Assume $f \in \mathscr{A}_G^*$, $\hat{f} \in \mathscr{E}_G$ as above. Then*

(a) *If $\pi \geqq \pi_{\hat{f}}$, then π is a f-representation.*

(b) *Assume π_1 and π_2 are two f-representations, $\mathscr{C}_{1G}$ and $\mathscr{C}_{2G}$ the corresponding commutative von Neumann algebras. If for every projector $P_1 \in \mathscr{C}_{1G}$ either $f(P_1) = 0$ or $f(I - P_1) = 0$, then the same is true also for every projector $P_2 \in \mathscr{C}_{2G}$.*

Proof. (a): Assume x_0 is the G-invariant cyclic vector of $\pi_{\hat{f}}$, so that $\hat{f}(A) = (x_0, \pi_{\hat{f}}(A) x_0) = (x_0, \pi(A) x_0)$. As g_+/N is dominated by $\hat{f}$, there exists ([2], 2.5.1) a unique self-adjoint $T \in \pi'_{\hat{f}}$, $0 \leqq T \leqq 1$, such that $g_+(A) = N(Tx_0, \pi_{\hat{f}}(A) T x_0)$. The uniqueness of T and the fact that $U(g) x_0 = x_0$ implies that $T \in U'_{\hat{f}}$, thus $U(g) T x_0 = T x_0$. Put $x_1 = T x_0$, $a_1 = N$, and proceed similarly for g_-, h_+, and h_-; it follows that $\pi_{\hat{f}}$, and hence π, satisfies, with $n = 4$, the requirement for a f-representation.

(b): We form $\pi = \pi_1 \oplus \pi_2$ and apply Lemma 2.3 with $\mathscr{B} = \pi''$. f is defined on $\mathscr{B}$ either by vectors $(x_1, \dots x_m) \in \mathscr{H}_1$ or by $(y_1, \dots y_n) \in \mathscr{H}_2$; the corresponding projectors Q_1 and Q_2 are evidently projectors in $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. With $\mathscr{C}_{iG} = \pi'_i \cap \pi''_i \cap U'$, $\mathscr{C}_{GQ_i} = \mathscr{B}_{Q_i} \cap \mathscr{B}'_{Q_i} \cap U'$, $i = 1, 2$, we can use Lemma 2.3, (a)$\Leftrightarrow$(c), to run through the chain $\mathscr{C}_{1G} \to \mathscr{C}_{GQ_1} \to \mathscr{C}_G \to \mathscr{C}_{GQ_2} \to \mathscr{C}_{2G}$.

In view of Lemma 4.2 (b) and Theorem 3.2 (c) it is natural to introduce the following generalization of G-factorial state:

Definition 4.3. *$f \in \mathscr{A}_G^*$ is said to be G-factorial, if for some (and then for every) f-representation π we have the relation $f(P)f(I-P)=0$ for every projector $P \in \mathscr{C}_G = \pi' \cap \pi'' \cap U'$.*

If we study the C^*-algebra of 2×2-matrices, example "$0 \neq 1$" at the end of Section 2, which is a G-abelian algebra not fulfilling condition (A), and put $f = a\varrho_0 + b\varrho_1$, where a and b are complex non-zero numbers, such that a/b is not a real positive number, we get a G-factorial linear functional, which is not a complex multiple of a G-factorial state. However, the following theorem holds:

Theorem 4.4. *If $(\mathscr{A}, G)$ satisfies condition (A), then every G-factorial f is of the form $k\hat{f}$, where k is a complex number $(|k| = \|f\|)$, and $\hat{f}$ is a G-ergodic state.*

Theorem 4.4 is a direct consequence of the following central Lemma:

Lemma 4.5. *Given two non-zero positive G-invariant functionals g_1, g_2, which are orthogonal, i.e. $\|g_1 - g_2\| = \|g_1\| + \|g_2\|$. Put $\hat{g}_i = g_i/\|g_i\|$. If condition (A) holds, $\hat{g}_1$ and $\hat{g}_2$ are disjoint.*

Before proving this Lemma, we use it to prove Theorem 4.4: If $f = g + ih$ is G-factorial, it follows that $g = g_+ - g_-$ and $h = h_+ - h_-$ are also G-factorial. Assume condition (A) holds; if g_+ and g_- were both different from zero, they would have non-trivial orthogonal central supports $F_\pm$ in a f-representation. Evidently this means e.g. $g(F_+)\,g(I-F_+) \neq 0$, which is a contradiction. Hence g is definite, say positive, $g = \|g\|\,\hat{g}$, where $\hat{g}$ is a G-factorial, i.e. from Theorem 3.4 a G-ergodic state. Similarly for h. Finally, if g and h are both non-zero, it follows from the fact that f is G-factorial that $\hat{g}$ and $\hat{h}$ have the same central support, i.e. by Theorem 3.4 they are the same state, so Theorem 4.4 follows.

Proof of Lemma 4.5. The orthogonality of g_1 and g_2 is equivalent to the property that in a representation π containing the cyclic representations π_1 and π_2 corresponding to $\hat{g}_1$ and $\hat{g}_2$, the supports E_1 and E_2 are orthogonal ([2], 12.3.1). To show that $E_1 E_2 = 0$ implies $\hat{g}_1 \between \hat{g}_2$, we have to show that the only intertwining operator between π_1 and π_2 is trivial: $T\pi_1(A) = \pi_2(A)\,T$, all $A \in \mathscr{A}$, implies $T = 0$.

Form $\varrho = \frac{1}{2}\hat{g}_1 + \frac{1}{2}\hat{g}_2$, $\varrho(A) = (x_\varrho, \pi_\varrho(A)x_\varrho)$. We can find unique self-adjoint positive $T_i \in \pi'_\varrho \cap U'_\varrho$, such that $\hat{g}_i(A) = 2(T_i x_\varrho, \pi_\varrho(A)\,T_i x_\varrho)$, $T_1^2 + T_2^2 = I$. From Lemma 2.1 we can write the supports

$E_i = \mathrm{Proj}[\pi'_\varrho T_i x_\varrho]$. Since $E_1 E_2 = 0$, we have $(T_1 x_\varrho, B' T_2 x_\varrho) = 0$, all $B' \in \pi'_\varrho$. Condition (A) implies that $T_i \in \pi''_\varrho$, so we get $(T_2 T_1 x_\varrho, B' x_\varrho) = 0$. In particular we can take $B' = T_2 T_1$, and from $T_2 T_1 x_\varrho = 0$ and the fact that x_ϱ is separating for π'_ϱ we get $T_2 T_1 = 0$. Hence $(T_1 + T_2)^2 = I$, and T_1 and T_2 are orthogonal projectors on two complementary subspaces of $\mathscr{H}_\varrho$. We have $\pi_i \simeq \pi_\varrho / T_i \mathscr{H}_\varrho$, $i = 1, 2$. An intertwining operator is then a $T \in \pi'_\varrho$ with $T T_2 = 0$ and range in $T_2 \mathscr{H}_\varrho$, i.e. $T_1 T = 0$. But since condition (A) implies $T T_1 = T_1 T$, we get $T = 0$.

Let us note that a modification of the proof gives an alternative proof of (a)$\Rightarrow$(d) in Theorem 3.4: if $\hat{g}_1$ and $\hat{g}_2$ are G-ergodic and different, one can conclude – independently of condition (A) – that $T_i^2 = T_i$, so that T_i are projectors with $T_1 + T_2 = I$; then $\hat{g}_1 \mathbin{\natural} \hat{g}_2$ follows as above.

The conclusion of Lemma 4.5 does not hold if we assume $(\mathscr{A}, G)$ to be G-abelian instead of satisfying condition (A), as is shown by the orthogonal, equivalent states ϱ_0 and ϱ_1 of example "0 ∓ 1".

5. The Commutative Case

We want to show that if $\mathscr{A}$ is commutative, the definition of G-factorial linear functional is equivalent to the definition of complex ergodic measure, in the case that this measure has a bounded total variation. We shall also study the relation between Theorem 4.4 and the result in [1] that a complex ergodic measure is proportional to a positive ergodic measure.

(X, B, m, G) is a set X with a σ-algebra B of subsets, m a complex measure on (X, B); $|m|$ is its total variation, and we assume $|m|(X) < \infty$. G is a flow on X, a group of B-measurable ($E \in B \Rightarrow g(E) \in B$, all $g \in G$) and m-measure-preserving ($m(E) = m(g(E))$, all $E \in B$) transformations of X. By Lemma 1 in [1] m-measure-preserving implies $|m|$-measure-preserving. $E \in B$ is G-invariant ($|m|$ – a.e.) if $|m|(g(E) \triangle E) = 0$, all $g \in G$; here $F \triangle E = F \cup E - F \cap E$. m is said to be *ergodic*, if for every G-invariant ($|m|$ – a.e.) $E \in B$ we have $m(E)\, m(X - E) = 0$.

Introducing the space $L^\infty(X, B, |m|)$ of $|m|$-equivalence classes of $|m|$-essentially bounded measurable functions on X, which with the norm $\|\varphi\| = \operatorname{ess\,sup}\{|\varphi(x)|; x \in X\}$ and natural definitions of product and involution is a commutative C^*-algebra, m defines in the obvious way a continuous linear functional f_m on L^∞. Evidently there is a one-to-one correspondence $E \leftrightarrow \chi_E$ between $|m|$-equivalence classes of sets in B and projectors (characteristic functions) in L^∞, and we have $m(E) = f_m(\chi_E)$. G-invariant sets correspond to G-invariant projectors. L^∞ has an isomorphic representation as a concrete C^*-algebra of operators (multiplication by the L^∞ function) in the Hilbert space $L^2(X, B, |m|/|m|(X))$; this is just the f_m-representation $\pi_{f_m}(\pi$, for short). π is evidently already weakly closed, i.e. a von Neumann algebra; this corresponds to the fact that L^∞, as the dual of L^1, is a W^*-algebra, an "abstract" von Neumann algebra ([5], p. 1, p. 45). Since furthermore π is cyclic and commutative, we have $\pi = \pi' = \pi''$. So the G-invariant projectors in L^∞ coincide with the projectors in $\mathscr{C}_G = \pi' \cap \pi'' \cap U'$, and evidently the condition that m is ergodic corresponds to the condition that f_m is G-factorial. Theorem 4.4 then implies that m is a complex multiple of a positive ergodic measure, which is Theorem 1 in [1].

Conversely we show that if $\mathscr{A}$ in $(\mathscr{A}, G)$ is commutative, Theorem 1 in [1] implies Theorem 4.4 in this paper. A commutative $\mathscr{A}$ can be considered, by the Gelfand isomorphism, as a C^*-algebra $C_0(X)$ of continuous functions, vanishing at infinity, on a locally compact space X. If B is the family of Borel sets on X, generated by the open sets of X, the action of G on the elements of $\mathscr{A}$ is transformed into homeomorphisms of X, mapping B onto itself. An element $f \in \mathscr{A}_G^*$ corresponds to a G-invariant complex measure m_f on (X, B), of bounded total variation $|m_f|$. The f-representation $\pi_f(=\pi)$ of $\mathscr{A} = C_0(X)$ is evidently given by the functions in $C_0(X)$ acting by multiplication in $L^2(X, B, |m_f|/|m_f|(X))$. In this case we find $\pi' = \pi'' = L^\infty(X, B, |m_f|)$. As before we conclude that f G-factorial is equivalent to m_f ergodic. So Theorem 1 in [1] shows that $m_f = k|m_f|$, where of course $|m_f|$ is positive and ergodic; this gives Theorem 4.4.

In a certain sense Theorem 1 in [1] implies Theorem 4.4 also in the general non-commutative case. If condition (A) holds, the mapping from G-invariant states to G-invariant central measures on the state space is affine, as we remarked at the end of Section 3. (The proof of this result runs largely parallel to the proof of Lemma 4.5). This mapping can then be extended by linearity to a mapping from $\mathscr{A}_G^*$ to complex bounded G-invariant central measures. Under this mapping a G-factorial linear functional corresponds to a complex ergodic measure, so an application of Theorem 1 in [1] gives Theorem 4.4.

A final remark: In one respect the result in [1] is more general, since it holds also if $|m|$ is supposed to be only σ-finite instead of finite, as we have assumed here. This would correspond to unbounded linear functionals on the C^*-algebra $\mathscr{A}$, a concept which has been studied by Pedersen in a series of papers [7].

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