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Higher Order Perturbation Theory for Exponential Lagrangians: Third Order*

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Abstract. We define the vacuum expectation value of the time-ordered product of three exponentials of free massless fields as a continuous linear functional over a suitable test function space using minimal singularity as a criterion.

I. Introduction

The present paper is an extension of an earlier work [1] devoted to the analysis of the structure of exponential interactions as given by the Lagrangian $\mathscr{L}_{\text{int}}(f\phi)$

$$\mathscr{L}_{\text{int}}(f\phi(x)) = :e^{f\phi(x)} - 1 := L_{\text{int}}(x) \tag{1}$$

where ϕ is a free scalar field of mass m.

In [1] we discussed the second order contribution to the Green's functions in an expansion in powers of $\mathscr{L}_{\text{int}}(f\phi)$. To achieve uniqueness we introduced a minimality principle. We argued that with the least singular choice of the time-ordered product $TL_{\text{int}}(x_1)\,L_{\text{int}}(x_2)$ the Green's functions correspond most closely to the given classical Lagrangian (in second order).

Here we go one step beyond the results of Ref. [1] and show that the minimality principle can be generalized to third order, at least for the case of a massless field. The generalized minimality principle leads to a unique, least singular definition of the time-ordered product $TL_{\text{int}}(x_1)\dots L_{\text{int}}(x_3)$. Because of the simple relation between time- and normal-ordered products of exponential Lagrangians it is sufficient to analyze the structure of the vacuum expectation values

$$\begin{aligned}\langle 0|\,TL_{\text{int}}(x_1)\dots L_{\text{int}}(x_3)\,|0\rangle &= \prod_{1\leqq i<j\leqq 3} [e^{f^2 iD_F(x_i-x_j)}-1] \\ &\quad+\tfrac{1}{2}\sum_{\sigma\in\mathfrak{S}_3}[e^{f^2 iD_F(x_i-x_j)}-1]\,[e^{f^2 iD_F(x_j-x_k)}-1] \\ &= \prod_{1\leqq i<j\leqq 3} iE_F(x_i-x_j)+\tfrac{1}{2}\sum_{\sigma\in\mathfrak{S}_3}[iE_F(x_i-x_j)]\,[iE_F(x_j-x_k)]\,.\end{aligned} \tag{2}$$

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Here $\mathfrak{S}_3$ denotes the group of permutations of three objects.

$$\sigma(1,2,3)=(i,j,k)\,. \tag{3}$$

The ambiguity in defining $\langle 0|TL_{\text{int}}(x_1)\ldots L_{\text{int}}(x_3)|0\rangle$ consists in a translation and Lorentz invariant real symmetric distribution $\in \mathscr{C}'(R^{12})$ [2] the support of which is confined to $x_1=x_2=x_3$. Hence the second term on the r.h.s. of (2) although already defined may be replaced by

$$\begin{aligned}\frac{1}{2}\sum_{\sigma\in\mathfrak{S}_3}\Big\{&[iE_F(x_i-x_j)]\,[iE_F(x_j-x_k)]-\pi^6\lambda^4\left[g\left(-\frac{\lambda}{4}\Box\right)\delta(x_i-x_j)\right]\\ &\cdot\left[g\left(-\frac{\lambda}{4}\Box\right)\delta(x_j-x_k)\right]\Big\}\end{aligned}\tag{4}$$

with the least singular superpropagator $iE_F(x)$, $\lambda=f^2/4\pi^2$ and

$$g(Z)=\sum_{m=0}^{\infty}\frac{Z^m}{m!(m+1)!(m+2)!}\,. \tag{5}$$

The real part of the Fourier transform of expression (4) is given by

$$\begin{aligned}\frac{-\pi^2\lambda^4}{8}\delta\left(\sum_1^3 p_n\right)\sum_{\sigma\in\mathfrak{S}_3}\Big\{&\operatorname{Re}\tilde{E}_F(p_i)\cdot\operatorname{Re}\tilde{E}_F(p_k)-\pi^2\left[\Theta(-p_i^2)g\left(\frac{\lambda}{4}p_i^2\right)-\delta(p_i^2)\right]\\ &\cdot\left[\Theta(-p_k^2)g\left(\frac{\lambda}{4}p_k^2\right)-\delta(p_k^2)\right]+\pi^2\left(\left[\Theta(-p_i^2)g\left(\frac{\lambda}{4}p_i^2\right)-\delta(p_i^2)\right]\cdot g\left(\frac{\lambda}{4}p_k^2\right)\right.\\ &\left.+[i\leftrightarrow k]\right)\Big\}.\end{aligned}\tag{6}$$

Suppose that (6) is smeared in the spatial variables with a testfunction $\tilde{\phi}(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3)\in\mathscr{D}(R^9)$ [3] such that there exists a constant K with

$$0\leqq \boldsymbol{p}_n^2<K^2 \qquad n=1,2,3 \quad \text{for} \quad (\boldsymbol{p}_1,\cdot,\boldsymbol{p}_3)\in\operatorname{supp}\tilde{\phi}\,. \tag{7}$$

Hence

$$p_{n,0}^2-K^2<p_n^2 \qquad n=1,2,3\,. \tag{8}$$

The various terms of (6) smeared with $\tilde{\phi}$ are of the following type for large $|p_{i,0}|$ or $|p_{k,0}|$:

$$\begin{aligned}\delta(\Sigma p_{n,0})\Big\{&O(\operatorname{Max}\{|p_{i,0}|,|p_{k,0}|\}^{-N})+(O(|p_{i,0}|^{-N})\times\text{entire function}\\ &\text{of order } 2/3 \text{ in } p_{k,0}+[i\leftrightarrow k])\Big\}\end{aligned}\tag{9}$$

for any $N\in\mathbb{N}$.

With the help of this information one obtains the following structure for

$$\frac{1}{2}\sum_{\sigma\in\mathfrak{S}_3}\left\langle \mathrm{Re}\left\{[iE_F(x_i-x_j)]\,[iE_F(x_j-x_k)]-\pi^6\lambda^4\left[g\left(-\frac{\lambda}{4}\square\right)\delta(x_i-x_j)\right]\right.\right.$$
$$\left.\left.\cdot\left[g\left(-\frac{\lambda}{4}\square\right)\delta(x_j-x_k)\right]\right\},\ \phi(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)\right\rangle \tag{10}$$

as a function of the time differences

$$\sum_{\sigma\in\mathfrak{S}_3}\left\{h^\sigma(x_i^0-x_j^0,x_j^0-x_k^0)+\sum_{\nu=0}^{\infty}\delta^{(\nu)}(x_i^0-x_j^0)\,h_\nu^\sigma(x_j^0-x_k^0)\right\} \tag{11}$$

with

$$h^\sigma\in C^\infty(R^2),\qquad h_\nu^\sigma\in C^\infty(R^1)\qquad \nu=0,1,2,\ldots$$

and

$$\sum_{\nu=0}^{\infty}\delta^{(\nu)}(x_i^0-x_j^0)\,h_\nu^\sigma(x_j^0-x_k^0)\in\mathscr{C}'(R^2)\quad\text{for all}\quad \sigma\in\mathfrak{S}_3\,.$$

The singularities are contained in the second term of (11). They are attached to the lines $x_1^0=x_2^0$, $x_2^0=x_3^0$ and $x_3^0=x_1^0$. In particular, for no spatial testfunction ϕ does the expression (10) involve singularities belonging to the point $\boldsymbol{x}_1=\boldsymbol{x}_2=\boldsymbol{x}_3$.

The crux of the problem consists of showing that in the weighted space average of the real part of the first term on the r.h.s. of (2) background, line singularities and point singularity can be separated from each other. Once this has been proved one particular definition of $\prod_{1\leqq i<j\leqq 3}[iE_F(x_i-x_j)]$ can be singled out by the absence of point singularities in the weighted space averages of its real part or in other words by its being least singular. Then the vacuum expectation value of the time ordered product $TL_{\mathrm{int}}(x_1)\ldots L_{\mathrm{int}}(x_3)$ is defined by the sum of that particular definition of

$$\prod_{1\leq i<j\leq 3}[iE_F(x_i-x_j)]\ \text{plus}\ \frac{1}{2}\sum_{\sigma\in\mathfrak{S}_3}\left\{[iE_F(x_i-x_j)]\,[iE_F(x_j-x_k)]\right.$$
$$\left.-\pi^6\lambda^4\left[g\left(-\frac{\lambda}{4}\square\right)\delta(x_i-x_j)\right]\left[g\left(-\frac{\lambda}{4}\square\right)\delta(x_j-x_k)\right]\right\}.$$

Clearly, the weighted space averages of the real part of $\langle 0|TL_{\mathrm{int}}(x_1)\ldots$ $\ldots L_{\mathrm{int}}(x_3)|0\rangle$ again do not contain point singularities. Hence $\langle 0|TL_{\mathrm{int}}(x_1)\ldots L_{\mathrm{int}}(x_3)|0\rangle$ so defined is least singular.

In Section II we give a particular definition for $\prod_{1\leqq i<j\leqq 3}[iE_F(x_i-x_j)]$ and prove that it satisfies unitarity and locality. In Section III we show that this particular definition is already suggested by the behavior of $\prod_{1\leqq i<j\leqq 3}[iE_F(x_i-x_j)]$ in the neighborhood of the point $x_1=x_2=x_3$ and

consequently is the least singular one. We conclude with a remark concerning the singularity structure in the invariant momenta of the corresponding amplitude in momentum space.

It will not escape the reader's attention that all of the preceding and subsequent considerations carry easily over to Lagrangians of the form

$$\mathcal{L}_{\mathrm{int}}(f\phi) = \int_0^\infty d\mu(q) : e^{qf\phi} - 1 : \tag{12}$$

where $[\exp q^\alpha]\, d\mu(q)$ is a bounded real measure for some $\alpha > 2$.

II. Definition of the Time-Ordered Product, Unitarity and Locality

To prepare the ground for the subsequent discussion, let us investigate powers

$$[Q(p_1, p_2; t_1, t_2) \pm i0]^\mu \tag{13}$$

and

$$[Q(p_1, p_2; t_1, t_2) \pm i\varepsilon]^\mu \tag{13'}$$

of the parameter depending quadratic form

$$Q(p_1, p_2; t_1, t_2) = -\frac{p_1^2 + t_1 p_2^2 + t_1 t_2 (p_1 + p_2)^2}{1 + t_2 + t_2 t_1} \tag{14}$$

for $\mathrm{Re}\,\mu > -4$, $t_j \in I = [0, 1]\, j = 1, 2$ and $\varepsilon > 0$ [3].

As a distribution-valued function $[Q(p_1, p_2; t_1, t_2) \pm i0]^\mu$ is infinitely differentiable with respect to t_1 and t_2 as long as the quadratic form $Q(p_1, p_2; t_1, t_2)$ is not degenerate, i.e. away from $t_1 = 0$. For $\mathrm{Re}\,\mu > -4$ the behavior of (13) when t_1 approaches the left end of I is given by

$$[Q(p_1, p_2; t_1, t_2) \pm i0]^\mu = G_\mu(p_1, p_2; t_1, t_2) + t_1^{\mu+2}\, G'_\mu(p_1, p_2; t_1, t_2) \tag{15}$$

where the distribution-valued function G_μ is infinitely differentiable with respect to t_1 and t_2 and where G'_μ is a bounded function of t_1 and t_2 for $t_j \in I, j = 1, 2$ (bounded in the sense of $\mathcal{S}'$)[1]. Moreover, we have

$$\lim_{\varepsilon \downarrow 0} [Q(p_1, p_2; t_1, t_2) \pm i\varepsilon]^\mu = [Q(p_1, p_2; t_1, t_2) \pm i0]^\mu \quad \text{for} \quad \mathrm{Re}\,\mu > -4 \quad \text{and} \quad t_1 \neq 0\,. \tag{16}$$

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[1] Qualitatively, this behavior can be seen from the partial Fourier transform with respect to the variable p_1. It is a special case of Lemma 2.2.20 in E. R. Speer and M. J. Westwater, Generic Feyman Amplitudes, IAS preprint (1970). However, also in the general situation, an alternative, more direct proof of the corresponding behavior when the parameter depending quadratic form Q degenerates can be given which does not involve analytic continuation from the Symanzik region.

Thus the distribution

$$\begin{aligned}\delta\left(\sum_{l=1}^{3} p_l\right) & T_0^{\sigma\pm}(\{p\}; s_1, s_2, s_3) \\ & = \delta\left(\sum_{l=1}^{3} p_l\right) \int_0^1 dt_2\, t_2^{s_3+1} \int_0^1 dt_1\, t_1^{-s_1-1} [1+t_2+t_2 t_1]^{-2} \\ & \cdot [Q(p_i, p_j; t_1, t_2) \pm i0]^{s_3+s_2+s_1+2},\end{aligned} \tag{17}$$

unambiguously defined in

$$\left\{s_1, s_2, s_3 / \mathrm{Re}\, s_1 < 0, \mathrm{Re}\, s_3 > -2, \sum_{i=1}^{3} \mathrm{Re}\, s_i > -2\right\},$$

can be analytically continued in s_1, s_2, s_3 to a function meromorphic in Ω

$$\Omega = \left\{s_1, s_2, s_3 \middle/ \sum_{k=l+1}^{3} \mathrm{Re}\, s_k > -2(3-l) \quad \text{for all} \quad 0 \leqq l \leqq 2\right\}. \tag{18}$$

If we use the same symbol for the continued function, then

$$\delta\left(\sum_{l=1}^{3} p_l\right) \Gamma(-s_1)^{-1}\, T_0^{\sigma\pm}(\{p\}; s_1, s_2, s_3) \tag{19}$$

is analytic in Ω. We define

$$\begin{aligned}\delta\left(\sum_{l=1}^{3} p_l\right) & T_\varepsilon^{\sigma\pm}(\{p\}; s_1, s_2, s_3) \\ & = \delta\left(\sum_{l=1}^{3} p_l\right) \int_0^1 dt_2\, t_2^{s_3+1} \int_0^1 dt_1\, t_1^{-s_1-1} [1+t_2+t_2 t_1]^{-2} \\ & \cdot [Q(p_i, p_j; t_1, t_2) \pm i\varepsilon]^{s_3+s_2+s_1+2}.\end{aligned} \tag{20}$$

Then for $\{s\} \in \Omega$, there exists the limit in the sense of $\mathscr{S}'(R^{12})$

$$\lim_{\varepsilon \downarrow 0} \delta\left(\sum_{l=1}^{3} p_l\right) \Gamma(-s_1)^{-1}\, T_\varepsilon^{\sigma\pm}(\{p\}; s_1, s_2, s_3) \tag{21}$$

and is equal to the corresponding distribution (19).

For the moments p_1, p_2, p_3 in a compact set K, i.e. in $\mathscr{D}'(K)$ the distributions (20) are equal to

$$\begin{aligned}-\delta\left(\sum_{l=1}^{3} p_l\right) \frac{\pi}{\sin \pi s_1} \int_0^1 dt_2\, t_2^{s_3+1} \frac{1}{2\pi i} \int_{C_{K,\varepsilon}} dt_1 (-t_1)^{-s_1-1} [1+t_2+t_2 t_1]^{-2} \\ \cdot [Q(p_i, p_j; t_1, t_2) \pm i\varepsilon]^{\Sigma s_l+2}.\end{aligned} \tag{22}$$

Here the contour $C_{K,\varepsilon}$ depending on K and ε starts and ends at $+1$ and encircles the origin once counter-clockwise. It is so close to the real interval $[0, 1]$ that $Q(p_i, p_j; t_1, t_2) \pm i\varepsilon$ does not vanish in

$$\{t_1/t_1 \in \text{region encircled by } C_{K,\varepsilon}\} \times \{t_2/t_2 \in I\}\,. \tag{23}$$

The function $(-t)^{-s_1-1}$ is defined such that $\arg(-t) = 0$ for $t < 0$.

Next, we turn to the asymptotic behavior of the distributions

$$\delta\left(\sum_{l=1}^{3} p_l\right) \Gamma(-s_1)^{-1}\, T_\varepsilon^{\sigma\pm}(\{p\}; s_1, s_2, s_3) \quad \text{for} \quad \varepsilon \geqq 0\,. \tag{24}$$

We expand

$$\left[-(1+t_2)^{-1} \cdot p_1^2 - \tau(1+t_2)(1+t_2+t_2 t_1)^{-1} \cdot \left(\frac{t_2}{1+t_2} p_1 + p_2\right)^2 \pm i\varepsilon\right]^{\sum_1^3 s_l + 2} \quad \text{for} \quad \varepsilon \geqq 0 \tag{25}$$

with respect to τ using the formula

$$f(x) = \sum_{n=0}^{N} \frac{x^n}{n!} f^{(n)}(0) + \frac{1}{N!} \int_0^1 dv (1-v)^N \frac{\partial^{N+1}}{\partial v^{N+1}} f(vx) \quad \text{for} \quad f \in C^{N+1}(R^1)\,, \tag{26}$$

set τ equal to t_1 and choose N such that $3 + \sum_1^3 \operatorname{Re} s_l \leqq N < 4 + \sum_1^3 \operatorname{Re} s_l$. This expansion leads to a corresponding expansion for (24) a term by term estimate of which gives the following result:

There exists a positive constant M such that the set of distributions

$$\left\{ \left(M \left[1 + \sum_{l=1}^{3} \sum_{\nu=0}^{3} |p_{l,\nu}|^2\right] \left[1 + \frac{\left[\sum_1^3 \operatorname{Im} s_l\right]^2}{\left[\operatorname{Max}\left\{1, \sum_1^3 \operatorname{Re} s_l + 2\right\}\right]^2}\right]^{\frac{1}{2}} \right)^{-\sum_1^3 \operatorname{Re} s_l - 6} \cdot e^{-\pi |\operatorname{Im} s_1|} |\Gamma(1+s_1)|^{-1} e^{-\pi \left|\sum_1^3 \operatorname{Im} s_l\right|} \delta\left(\sum_1^3 p_l\right) \Gamma(-s_1)^{-1}\, T_\varepsilon^{\sigma\pm}(\{p\}; s_1, s_2, s_3) \Bigg/ (s_1, s_2, s_3) \in \Omega,\ 0 \leqq \varepsilon \leqq 1 \right\} = B \subset \mathscr{S}'(R^{12}) \tag{27}$$

is bounded.

We are now in a position to define

$$\mathscr{T}(x_1, x_2, x_3) = \prod_{1 \leqq j < k \leqq 3} [iE_F(x_j - x_k)]$$

and

$$\bar{\mathscr{T}}(x_1, x_2, x_3) = \prod_{1 \leqq j < k \leqq 3} [-iE_{\bar{F}}(x_j - x_k)] \tag{28}$$

in momentum space where

$$\tilde{E}_{F(\bar{F})}(k) = \frac{\lambda^2}{4} \frac{1}{2\pi i} \int_L ds \frac{\Gamma(-1-s)\,\Gamma(-s)\cos\pi s}{\Gamma(3+s)} \left[\frac{\lambda}{4}(-k^2_{\mp (+)} i0)\right]^s . \tag{29}$$

The loop L starts and ends at $+\infty$ and encircles the poles $-1, 0, +1, \ldots$ once clockwise. $\lambda = \frac{f^2}{4\pi^2}$ is assumed to be positive.

$$[\tilde{E}_F(k)]^* = \tilde{E}_{\bar{F}}(k), \quad \text{i.e.} \quad [E_F(x)]^* = E_{\bar{F}}(x) . \tag{30}$$

Definition.

$$\begin{aligned} \mathscr{F}_{x_1,x_2,x_3}\{\mathscr{T}(x_1, x_2, x_3)\}\,(p_1, p_2, p_3) &= \tilde{\mathscr{T}}(p_1, p_2, p_3) \\ &= \frac{\pi^2\lambda^4}{4} \sum_0^\infty \frac{1}{m_3!(m_3+1)!} \sum_0^\infty \frac{1}{m_2!(m_2+1)!} \\ &\cdot \frac{1}{2\pi i} \int_C ds_1 \frac{\Gamma(-1-s_1)}{\Gamma(3+s_1)} \cos\pi s_1 [1+\sin^2\pi s_1] \\ &\cdot \Gamma(-m_3-m_2-s_1)\left(\frac{\lambda}{4}\right)^{m_3+m_2+s_1} \\ &\cdot \sum_{\sigma\in\mathfrak{S}_3} \delta\left(\sum_1^3 p_n\right) T_0^{\sigma-}(\{p\}; s_1, m_2-1, m_3-1) + \tilde{\mathfrak{R}}^- \end{aligned} \tag{31}$$

where the loop C starts and ends at $+\infty$ and encircles the points $-2, -1, 0, +1, \ldots$ once clockwise and where $\tilde{\mathfrak{R}}^-$ is given by

$$\begin{aligned} \tilde{\mathfrak{R}}^- &= \frac{\pi^2\lambda^4}{4} \sum_0^\infty \left(\frac{\lambda}{4}\right)^l \frac{1}{l!} \frac{1}{(2\pi i)^2} \int_{S_2-i\infty}^{S_2+i\infty} ds_2 \int_{S_1-i\infty}^{S_1+i\infty} ds_1 \prod_{n=1}^2 \\ &\cdot \left[\frac{\Gamma(-1-s_n)}{\Gamma(3+s_n)}\right] \frac{\Gamma(s_2+s_1+1-l)}{\Gamma(l+1-s_1-s_2)} \sum_{\sigma\in\mathfrak{S}_3} \delta\left(\sum_1^3 p_n\right) T_0^{\sigma-}(\{p\}; s_1, s_2, l-2-s_1-s_2), \end{aligned} \tag{32}$$

$S_2 < -1,\ S_1 < -2,\ -7/2 < S_2 + S_1 < -3.$

$\tilde{\bar{\mathscr{T}}}(p_1, p_2, p_3)$ is obtained from $\tilde{\mathscr{T}}(p_1, p_2, p_3)$ by replacing $T_0^{\sigma-}$ by $T_0^{\sigma+}$.

We note the relation

$$[\tilde{\mathcal{T}}(p_1,p_2,p_3)]^* = \tilde{\bar{\mathcal{T}}}(p_1,p_2,p_3), \text{ i.e. } [\mathcal{T}(x_1,x_2,x_3)]^* = \bar{\mathcal{T}}(x_1,x_2,x_3). \quad (33)$$

Using (27) and standard estimates on gamma and beta functions one verifies that

$$\tilde{\mathcal{T}}(p_1,p_2,p_3) \in \mathfrak{M}'_{1/3}(R^{12}) \quad \text{and} \quad \tilde{\bar{\mathcal{T}}}(p_1,p_2,p_3) \in \mathfrak{M}'_{1/3}(R^{12}) \quad (34)$$

where $\mathfrak{M}'_{1/3}(R^l)$ denotes the space of all linear continuous functionals over the test function space

$$\mathfrak{M}_{1/3}(R^l) = \{\tilde{f}(q_1,\dots,q_l)/\tilde{f} \in C^\infty(R^l), \langle \tilde{f} \rangle_d^{(\alpha)} = \sup_q |g(d\,\|q\|^2)$$

$$D^{(\alpha)}\tilde{f}(q_1,\dots,q_l)| < \infty \text{ for all } d = 1,2,\dots \text{ and all } (\alpha) \in \mathbb{N}^l\} \quad (35)$$

$$D^{(\alpha)} = \prod_{j=1}^{l} \left(\frac{\partial}{\partial q_j}\right)^{\alpha j}, \qquad \|q\|^2 = \sum_{j=1}^{l} q_j^2 \,.$$

$\mathfrak{M}_{1/3}(R^l)$ is equipped with the topology given by the norms $\langle \tilde{f} \rangle_d^{(\alpha)}$ [2].

In order to prove that the above definitions of $\mathcal{T}$ and $\bar{\mathcal{T}}$ are permissible in the sense that they satisfy unitarity and locality, we consider the following functions of an auxiliary parameter γ, first for γ real and larger than nine [2]

$$\overset{(-)}{\mathcal{T}}_r(p_1,p_2,p_3;\gamma) = \frac{\pi^2\lambda^4}{4}\frac{1}{(2\pi i)^3}\iiint_{S-i\infty}^{S+i\infty} ds_3\,ds_2\,ds_1 \prod_{j=1}^{3}\left[\frac{\Gamma(-\gamma(1+s_j))}{\Gamma(3+s_j)}\right]$$

$$\cdot\left[\frac{\lambda}{4}e^{i\pi r}\right]^{\sum_1^3 s_j+2} \Gamma\left(-\sum_1^3 s_j - 2\right)\sum_{\sigma\in\mathfrak{S}_3}\delta\left(\sum_1^3 p_n\right) T_0^{\sigma(\overset{-}{+})}(\{p\};s_1,s_2,s_3)\,, \quad (36)$$

$$r = \pm 1,\ \pm 3;\ -\tfrac{3}{2} < S < -1\,.$$

With the help of (27) and standard estimates the integrals can be shown to exist for these values of γ. However, for values of γ close to $+1$ they will not exist as they stand. We swing the s_3- and s_2-contours around the real axis from -1 to $+\infty$ which will push the s_1-contour to the far left. Subdividing this deformed s_1-contour into 2 parts: one parallel to the imaginary axis: $\mathrm{Re}\,s_1 = S$, the other one being a loop which encircles the poles of $\Gamma\left(-\sum_1^3 s_j - 2\right)$ with $\mathrm{Re}\,s_1 \leqq S$ once clockwise (a singularity on $\mathrm{Re}\,s_1 = S$ is to be accounted for by either the first or the second part, but

[2] This type of regularization was first introduced by M. K. Volkov in Ann. Phys. (N.Y.) **49**, 202 (1968).

not by both) we derive the following identity

$$\overset{(\tilde{-})}{\mathcal{T}}_r(p_1,p_2,p_3;\gamma)=\frac{\pi^2\lambda^4}{4}\frac{1}{\gamma^2}\sum_0^\infty\frac{(-1)^{m_3}}{m_3!\,\Gamma(2+m_3/\gamma)}\sum_0^\infty\frac{(-1)^{m_2}}{m_2!\,\Gamma(2+m_2/\gamma)}$$
$$\cdot\frac{1}{2\pi i}\int_{S-i\infty}^{S+i\infty}ds_1\,\frac{\Gamma(-\gamma(1+s_1))}{\Gamma(3+s_1)}\left[\frac{\lambda}{4}e^{i\pi r}\right]^{(m_3/\gamma)+(m_2/\gamma)+s_1}$$
$$\cdot\Gamma\left(-\frac{m_3}{\gamma}-\frac{m_2}{\gamma}-s_1\right)\Gamma(-s_1)\sum_{\sigma\in\mathfrak{S}_3}\delta\left(\sum_1^3 p_n\right)\Gamma(-s_1)^{-1} \tag{37}$$
$$\cdot T_0^{\sigma(\bar{+})}\left(\{p\};s_1,\frac{m_2}{\gamma}-1,\frac{m_3}{\gamma}-1\right)+\tilde{\mathfrak{R}}^{(\bar{+})}(\gamma)$$

where

$$\tilde{\mathfrak{R}}^{(\bar{+})}(\gamma)=\frac{\pi^2\lambda^4}{4}\sum_0^\infty\frac{\left(\frac{\lambda}{4}\right)^l}{l!}\left\{\frac{1}{(2\pi i)^2}\iint_{S-i\infty}^{S+i\infty}ds_2\,ds_1\prod_{j=1}^2\frac{\Gamma(-\gamma(1+s_j))}{\Gamma(3+s_j)}\right.$$
$$\cdot\frac{\Gamma(\gamma[s_2+s_1+1-l])}{\Gamma(l+1-s_1-s_2)}\sum_{\sigma\in\mathfrak{S}_3}\delta\left(\sum_1^3 p_n\right)T_0^{\sigma(\bar{+})}(\{p\};s_1,s_2,l-2-s_1-s_2)$$
$$+\frac{1}{2\pi i}\int_Z ds_3\frac{\Gamma(-\gamma(1+s_3))\cos\pi\gamma(1+s_3)}{\Gamma(3+s_3)}\frac{1}{2\pi i}\int_{S-i\infty}^{S+i\infty}ds_1\frac{\Gamma(-\gamma(1+s_1))}{\Gamma(3+s_1)}$$
$$\cdot\frac{\pi}{\sin\pi\gamma(s_1-l)}\cdot\Gamma(1+\gamma[l-1-s_1-s_3])^{-1}\,\Gamma(1+l-s_1-s_3)^{-1} \tag{38}$$
$$\left.\cdot\sum_{\sigma\in\mathfrak{S}_3}\delta\left(\sum_1^3 p_n\right)T_0^{\sigma(\bar{+})}(\{p\};s_1,l-2-s_1-s_3,s_3)\right\}.$$

S is the same in both formulae (37) and (38) and $-\frac{3}{2}<S<-1$. The contour Z encircles those of the poles $s_3=-1+\frac{m_3}{\gamma}$, $m_3=0,1,\dots$ once clockwise whose real part is less or equal to $l-2-2S$.

In much the same way as before, one shows from this form of $\overset{(\tilde{-})}{\mathcal{T}}_r$ that

$$\overset{(\tilde{-})}{\mathcal{T}}_r(p_1,p_2,p_3;\gamma)\in\mathfrak{M}'_{1/3}(R^{12}) \tag{39}$$

and that $\tilde{\mathcal{T}}_r(p_1,p_2,p_3;\gamma)$ and $\tilde{\tilde{\mathcal{T}}}_r(p_1,p_2,p_3;\gamma)$ are analytic functions of γ in the chisel shaped region W_δ

$$W_\delta=\left\{\gamma=\gamma_1+i\gamma_2/\gamma_1>1,\ |\gamma_2|<\mathrm{Min}\left[\delta,\frac{\pi}{4}(\gamma_1-1)\right]\right\},\qquad\delta>0. \tag{40}$$

Furthermore, the following limits exist in $\mathfrak{M}'_{1/3}$:

$$\lim_{\substack{\gamma\to1\\ \gamma\in W_\delta}}\left\{\frac{5}{8}\sum_{r=\pm1}\overset{(\tilde{-})}{\mathcal{T}}_r(p_1,p_2,p_3;\gamma)-\frac{1}{8}\sum_{r=\pm3}\overset{(\tilde{-})}{\mathcal{T}}_r(p_1,p_2,p_3;\gamma)\right\} \tag{41}$$

and are equal to $\tilde{\mathscr{T}}(p_1, p_2, p_3)$ and $\tilde{\tilde{\mathscr{T}}}(p_1, p_2, p_3)$ respectively. Next, we introduce the parameter depending superpropagators

$$\tilde{E}_{F(\bar{F}),r}(k;\gamma) = \frac{\lambda^2}{4}\frac{1}{2\pi i}\int_{L_\gamma} ds \frac{\Gamma(-\gamma(1+s))\,\Gamma(-s)}{\Gamma(3+s)}\, e^{i\pi rs}\left[\frac{\lambda}{4}(\boldsymbol{k}^2 - k^2_{0(\mp)}i0)\right]^s$$
$$r = \pm 1,\ \pm 3 \tag{42}$$

where the loop L_γ encircles the poles of the gamma functions once clockwise. Using Stirling's formula and Jensen's inequality one shows:

a) $\tilde{E}_{F(\bar{F}),r}(k;\gamma) \in \mathfrak{M}'_{1/3}(R^4)$ for $\operatorname{Re}\gamma \geqq 1$, (43)

b) $\tilde{E}_{F(\bar{F}),r}(k;\gamma)$ are analytic functions of γ in W_δ,

c) the following limits exist in $\mathfrak{M}'_{1/3}(R^4)$

$$\lim_{\substack{\gamma\to+1\\ \gamma\in W_\delta}} \tilde{E}_{F(\bar{F}),r}(k;\gamma)$$

and are equal to

$$\tilde{E}_{F(\bar{F}),r}(k) = \frac{\lambda^2}{4}\frac{1}{2\pi i}\int_{L} ds \frac{\Gamma(-1-s)\,\Gamma(-s)}{\Gamma(3+s)}\, e^{i\pi rs}\left[\frac{\lambda}{4}(-k^2_{(\mp)}i0)\right]^s. \tag{44}$$

For the difference $E_{F,r}$ and $E_{\bar{F},r}$ one finds

$$\tilde{E}_{F,r}(k;\gamma) - \tilde{E}_{\bar{F},r}(k;\gamma) = -2\pi i\frac{\lambda}{\gamma}\delta(k^2)$$
$$-2\pi i\frac{\lambda^2}{4\gamma}\sum_0^\infty \frac{(-1)^m\left[\frac{\lambda}{4}e^{i\pi r}\right]^{(m+1)/\gamma-1}(k^2)_+^{(m+1)/\gamma-1}}{\Gamma\left(\frac{m+1}{\gamma}\right)\Gamma(m+2)\,\Gamma\left(\frac{m+1}{\gamma}+2\right)} \tag{45}$$
$$= \tilde{E}_r^{(+)}(k;\gamma) + \tilde{E}_r^{(-)}(k;\gamma)$$

with

$$\tilde{E}_r^{(\pm)}(k;\gamma) = -2\pi i\frac{\lambda}{\gamma}\delta_\pm(k^2) + \frac{\lambda^2}{4}\Theta(\pm k_0)\int_{L'_\gamma} ds \frac{\Gamma(-\gamma(1+s))}{\Gamma(1+s)\,\Gamma(3+s)}$$
$$\cdot\left[\frac{\lambda}{4}e^{i\pi r}\right]^s (k^2)_+^s \tag{46}$$

where the loop L'_γ encircles the poles $s = -1 + \frac{m}{\gamma}$ $m = 1, 2, \ldots$ once clockwise. $E^{(+)}$ and $E^{(-)}$ are related by

$$E_r^{(+)}(x;\gamma) = E_r^{(-)}(-x;\gamma). \tag{47}$$

Going through the same routine as before, one shows that

a) $\tilde{E}_r^{(\pm)}(k;\gamma) \in \mathfrak{M}'_{1/3}(R^4)$ for $\operatorname{Re}\gamma \geqq 1$. (43′)

b) $\tilde{E}_r^{(\pm)}(k;\gamma)$ are analytic functions of γ in W_δ.

c) the following limits exist in $\mathfrak{M}'_{1/3}(R^4)$

$$\lim_{\substack{\gamma \to +1 \\ \gamma \in W_\delta}} \tilde{E}_r^{(\pm)}(k;\gamma),$$

are independent of $r = \pm 1, \pm 3$ and are equal to

$$\tilde{E}^{(\pm)}(k) = -2\pi i \lambda \delta_\pm(k^2) - 2\pi i \frac{\lambda^2}{4} \Theta(\pm k_0)\, \Theta(k^2) \sum_0^\infty \frac{\left(\frac{\lambda}{4} k^2\right)^m}{m!(m+1)!(m+2)!} \tag{44′}$$

$iE_{F,r}(x;\gamma)$ and $-iE_{\bar{F},r}(x;\gamma)$ are time-ordered functions in the following sense:

$$iE_{F,r}(x;\gamma) = \begin{cases} iE_r^{(+)}(x;\gamma) & \text{for} \quad x^0 > 0 \\ iE_r^{(+)}(-x;\gamma) & \text{for} \quad x^0 < 0, \end{cases} \tag{48}$$

$$-iE_{\bar{F},r}(x;\gamma) = \begin{cases} iE_r^{(+)}(-x;\gamma) & \text{for} \quad x^0 > 0 \\ iE_r^{(+)}(x;\gamma) & \text{for} \quad x^0 < 0 \end{cases} \tag{48′}$$

where for instance the first line of Eq. (48) is to be understood as follows: given a test function f with $\tilde{f} \in \mathfrak{M}_{1/3}(R^4)$ and with $\operatorname{supp} f \in \{x/x^0 > 0\}$; then

$$\int dx f(x) iE_{F,r}(x;\gamma) = \int dx f(x) iE_r^{(+)}(x;\gamma).$$

For γ real and larger than nine the contours L_γ and L'_γ may be opened and pushed to the left such that $-2 < \operatorname{Re} s = S_0 < -\frac{3}{2}$ whence it follows that for these values of γ $E_{F(\bar{F}),r}(x;\gamma)$ and $E_r^{(\pm)}(x;\gamma)$ are locally L^2 integrable functions of x for which products as for example

$$\prod_{1 \leqq j < k \leqq 3} [iE_{F,r}(x_j - x_k;\gamma)], \quad \prod_{1 \leqq j < k \leqq 3} [-iE_{\bar{F},r}(x_j - x_k;\gamma)],$$

$$iE_{F,r}(x_i - x_j;\gamma)\, iE_r^{(+)}(x_j - x_k;\gamma)\, iE_r^{(+)}(x_i - x_k;\gamma) \tag{49}$$

$$\text{and} \quad -iE_{\bar{F},r}(x_i - x_j;\gamma)\, iE_r^{(+)}(x_k - x_j;\gamma)\, iE_r^{(+)}(x_k - x_i;\gamma)$$

are unambiguously defined. Formal manipulations which are correct for locally L^2 integrable functions yield the following relations:

For $\tilde{f} \in \mathfrak{M}_{1/3}(R^{12})$

$$\int \cdots \int \prod_{n=1}^{3} d^4x_n f(x_1, x_2, x_3) \cdot \left\{ \prod_{1 \leqq j<k \leqq 3} [iE_{F,r}(x_j - x_k; \gamma)] - \prod_{1 \leqq j<k \leqq 3} [-iE_{\bar{F},r}(x_j - x_k; \gamma)] \right\}$$
$$= \tfrac{1}{2} \sum_{\sigma \in \mathfrak{S}_3} \int \cdots \int \prod_{n=1}^{3} d^4x_n f(x_1, x_2, x_3) \tag{50}$$
$$\cdot \{[iE_{F,r}(x_i - x_j; \gamma)]\, [iE_r^{(+)}(x_j - x_k; \gamma)]\, [iE_r^{(+)}(x_i - x_k; \gamma)]$$
$$- [-iE_{\bar{F},r}(x_i - x_j; \gamma]\, [iE_r^{(+)}(x_k - x_j; \gamma)]\, [iE_r^{(+)}(x_k - x_i; \gamma]\} .$$

For $\tilde{f} \in \mathfrak{M}_{1/3}(R^{12})$ with $x_i^0 - x_k^0 > 0$ and $x_j^0 - x_k^0 > 0$ for all $x_i, x_j, x_k \in \operatorname{supp} f$

$$\int \cdots \int \prod_{n=1}^{3} d^4x_n f(x_1, x_2, x_3) \prod_{1 \leqq j<k \leqq 3} [iE_{F,r}(x_j - x_k; \gamma)]$$
$$= \int \cdots \int \prod_{n=1}^{3} d^4x_n f(x_1, x_2, x_3)\, [iE_{F,r}(x_i - x_j; \gamma)]\, [iE_r^{(+)}(x_j - x_k; \gamma)] \tag{51}$$
$$\cdot [iE_r^{(+)}(x_i - x_k; \gamma)]$$

$$\int \cdots \int \prod_{n=1}^{3} d^4x_n f(x_1, x_2, x_3) \prod_{1 \leqq j<k \leqq 3} [-iE_{\bar{F},r}(x_j - x_k; \gamma)]$$
$$= \int \cdots \int \prod_{n=1}^{3} d^4x_n f(x_1, x_2, x_3)\, [-iE_{\bar{F},r}(x_i - x_j; \gamma)]\, [iE_r^{(+)}(x_k - x_j; \gamma)] \tag{51'}$$
$$\cdot [iE_r^{(+)}(x_k - x_i; \gamma)] .$$

In order to establish the connection between

$$\prod_{1 \leqq j<k \leqq 3} [iE_{F,r}(x_j - x_k; \gamma)] \quad \text{and} \quad \mathcal{T}_r(x_1, x_2, x_3; \gamma), \tag{52}$$

$$\prod_{1 \leqq j<k \leqq 3} [-iE_{\bar{F},r}(x_j - x_k; \gamma)] \quad \text{and} \quad \bar{\bar{\mathcal{T}}}_r(x_1, x_2, x_3; \gamma) \tag{52'}$$

for γ real and larger than nine we introduce with Speer [4] the functions $E_{F,r,\varepsilon}(x; \gamma)$ and $E_{\bar{F},r,\varepsilon}(x; \gamma)$ given by

$$E_{F(\bar{F}),r,\varepsilon}(k; \gamma) = \frac{\lambda^2}{4} \frac{1}{2\pi i} \int_{L_\gamma} ds \frac{\Gamma(-\gamma(1+s))\Gamma(-s)}{\Gamma(3+s)} e^{i\pi rs} \left[\frac{\lambda}{4}(-k^2_{(\mp)} i\varepsilon)\right]^s, \varepsilon > 0 . \tag{53}$$

Then, the following relations hold in L^2_{loc} and L^1_{loc} respectively:

$$\lim_{\varepsilon\downarrow 0} E_{F(\bar{F}),r,\varepsilon}(x;\gamma) = E_{F(\bar{F}),r}(x;\gamma) \tag{54}$$

$$\lim_{\varepsilon\downarrow 0} \prod_{1\leqq j<k\leqq 3} [(\overset{+}{-})iE_{F(\bar{F}),r,\varepsilon}(x_j-x_k;\gamma)] = \prod_{1\leqq j<k\leqq 3} [(\overset{+}{-})iE_{F(\bar{F}),r}(x_j-x_k;\gamma)]. \tag{55}$$

According to Speer we obtain

$$\begin{aligned}
&\mathscr{F}_{x_1,x_2,x_3}\left\{\prod_{1\leqq j<k\leqq 3}\left[(\overset{+}{-})iE_{F(\bar{F}),r,\varepsilon}(x_j-x_k;\gamma)\right]\right\}(p_1,p_2,p_3) = \frac{\pi^2\lambda^4}{4}\frac{1}{(2\pi i)^3}\\
&\cdot \iiint_{S_0-i\infty}^{S_0+i\infty} ds_3\, ds_2\, ds_1 \prod_{n=1}^{3}\left[\frac{\Gamma(-\gamma(1+s_n))}{\Gamma(3+s_n)}\right]\\
&\cdot\left[\frac{\lambda}{4}e^{i\pi r}\right]^{\Sigma s_n+2}\Gamma\left(-\sum_1^3 s_n-2\right)\sum_{\sigma\in\mathfrak{S}_3}\delta\left(\sum_1^3 p_n\right)\\
&\cdot\int_0^1 dt_2\, t_2^{s_3+1}\int_0^1 dt_1\, t_1^{-s_1-1}\,[1+t_2+t_2t_1]^{-2}\\
&\cdot\left[\frac{-p_i^2-t_1p_j^2-t_2t_1p_k^2}{1+t_2+t_2t_1}(\overset{-}{+})i\varepsilon(t_2^{-1}+1+t_1)\right]^{\Sigma s_n+2}.
\end{aligned} \tag{56}$$

As a consequence of the fact that $-4<\sum_1^3 \operatorname{Re} s_n+2<0$, $\operatorname{Re} s_3+1>-1$, uniform convergence of the s-integration and Eq. (16), the limit $\varepsilon\downarrow 0$ of the r.h.s. exists (bounded convergence) and is equal to

$$\begin{aligned}
&\frac{\pi^2\lambda^4}{4}\frac{1}{(2\pi i)^3}\iiint_{S_0-i\infty}^{S_0+i\infty} ds_3\, ds_2\, ds_1\prod_{j=1}^{3}\left[\frac{\Gamma(-\gamma(1+s_j))}{\Gamma(3+s_j)}\right]\\
&\cdot\left[\frac{\lambda}{4}e^{i\pi r}\right]^{\Sigma s_n+2}\Gamma\left(-\sum_1^3 s_n-2\right)\sum_{\sigma\in\mathfrak{S}_3}\delta\left(\sum_1^3 p_n\right)\int_0^1 dt_2\, t_2^{s_3+1}\int_0^1 dt_1\, t_1^{-s_1-1}\\
&\cdot[1+t_2+t_2t_1]^{-2}[Q(p_i,p_j;t_1,t_2)(\overset{-}{+})i0]^{\Sigma s_n+2}.
\end{aligned} \tag{57}$$

The limit $\varepsilon\downarrow 0$ of the l.h.s. of Eq. (56) is given by (55). Thus we find

$$\prod_{1\leqq j<k\leqq 3}[(\overset{+}{-})iE_{F(\bar{F}),r}(x_j-x_k;\gamma)] = \overset{(-)}{\mathscr{T}_r}(x_1,x_2,x_3;\gamma). \tag{58}$$

Eq. (58) implies that the left-hand sides of the Eqs. (50), (51) and (51′) are analytic functions of γ in W_δ and that their limits exist as γ tends to $+1$ from W_δ. Also, the products of the right-hand sides of (50), (51) and (51′) viewed as convolution integrals in momentum space are analytic functions of γ in W_δ. This follows since the integrands are analytic in γ for $\gamma\in W_\delta$ and the integrations are uniformly convergent for $\tilde{f}\in\mathfrak{M}_{1/3}(R^{12})$ (for $\tilde{f}\in\mathscr{D}(R^{12})$ the regions of integration are even compact). The limits of the r.h.s. exist as γ tends to $+1$ from W_δ. By the uniqueness of analytic continuation in simply connected regions we obtain the desired unitarity and locality relations

$$2i\int\cdots\int\prod_{n=1}^{3} d^4x_n f(x_1,x_2,x_3)\,\mathrm{Im}\,\mathscr{T}(x_1,x_2,x_3)=\left(\tfrac{5}{8}\sum_{r=\pm1}-\tfrac{1}{8}\sum_{r=\pm3}\right)\tfrac{1}{2}\sum_{\sigma\in\mathfrak{S}_3}$$

$$\cdot\int\cdots\int\prod_{1}^{3} d^4x_n f(x_1,x_2,x_3)\,\{[iE_{F,r}(x_i-x_j)]\,[iE_r^{(+)}(x_j-x_k)] \tag{59}$$

$$\cdot\,[iE_r^{(+)}(x_i-x_k)]-[-iE_{\bar F,r}(x_i-x_j)]\,[iE_r^{(+)}(x_k-x_j)]\,[iE_r^{(+)}(x_k-x_i)]\}$$

$$=\tfrac{1}{2}\sum_{\sigma\in\mathfrak{S}_3}\int\cdots\int\prod_{1}^{3} d^4x_n f(x_1,x_2,x_3)\,\{[iE_F(x_i-x_j)]\,[iE^{(+)}(x_j-x_k)]$$

$$\cdot\,[iE^{(+)}(x_i-x_k)]-[-iE_{\bar F}(x_i-x_j)]\,[iE^{(+)}(x_k-x_j)]\,[iE^{(+)}(x_k-x_i)]\}$$

etc. Here the Eqs. (33), (41), (44) and (44′) have been used. This completes the proof that the above definitions of $\mathscr{T}$ and $\bar{\mathscr{T}}$ are permissible.

Finally, we want to point out a special property of these definitions. For $p_l^2>0$, $l=1,2,3$, the real part of $\tilde{\mathscr{T}}(p_1,p_2,p_3)$ can be brought into the form

$$\frac{\pi^2\lambda^4}{4}\frac{1}{(2\pi i)^3}\iiint_{S-i\infty}^{S+i\infty} ds_3\,ds_2\,ds_1\prod_{n=1}^{3}\left[\frac{\Gamma(-1-s_n)}{\Gamma(3+s_n)}\right]\left(\frac{\lambda}{4}\right)^{\Sigma s_n+2}\Gamma\left(-\sum_1^3 s_n-2\right)$$

$$\cdot\sum_{\sigma\in\mathfrak{S}_3}\delta\left(\sum_1^3 p_n\right)\int_0^1 dt_2\,t_2^{s_3+1}\int_0^1 dt_1\,t_1^{-s_1-1}\,[1+t_2+t_2t_1]^{-\Sigma s_n-4} \tag{60}$$

$$\cdot\,[p_i^2+t_1p_j^2+t_1t_2p_k^2]^{\Sigma s_n+2}$$

or into the form

$$\frac{\pi^2\lambda^4}{4}\frac{1}{2\pi i}\int_{T-i\infty}^{T+i\infty} ds\left(\frac{\lambda}{4}\right)^s\Gamma(-s)\frac{1}{(2\pi i)^2}\iint_{U-i\infty}^{U+i\infty} ds_3\,ds_1\prod_{j=1,3}\left[\frac{\Gamma(-1-s_j)}{\Gamma(3+s_j)}\right]$$

$$\cdot\frac{\Gamma(s_3+s_1+1-s)}{\Gamma(1+s-s_1-s_3)}\sum_{\sigma\in\mathfrak{S}_3}\delta\left(\sum_1^3 p_n\right)\int_0^1 dt_2\,t_2^{s_3+1}\int_0^1 dt_1\,t_1^{-s_1-1} \tag{61}$$

$$\cdot\,[1+t_2+t_2t_1]^{-s-2}\,[p_i^2+t_1p_j^2+t_1t_2p_k^2]^s$$

where $U=-\frac{3}{2}$, $T>-3$.

Simple estimates using Stirling's formula show that $\mathrm{Re}\,\tilde{\mathscr{T}}$ tends to zero in the region $\mathcal{O}$

$$\mathcal{O}=\{p_1,p_2,p_3/p_j^2>0\quad j=1,2,3\} \tag{62}$$

as the invariant momenta p_l^2 tend to $+\infty$ with a decrease of type

$$\delta\left(\sum_1^3 p_n\right)\cdot O\left(\underset{l=1,2,3}{\mathrm{Max}}^{-5/2}\,p_l^2\right)$$

and

$$\delta\left(\sum_1^3 p_n\right)\cdot O\left(\operatorname*{Min}_{l=1,2,3}^{-3} p_l^2\right).$$

This fall-off property fixes the time-ordered function completely. For, any other choice would differ from our definition in momentum space by a real entire function of p_1^2, p_2^2, p_3^2 of order $<\frac{1}{2}$. These entire functions do not decrease in any direction in $\mathbb{C}^3$. Thus the real part of any other choice of the time-ordered function does not decrease in momentum space in any direction inside the region $\mathcal{O}$.

III. Naturalness of the Definition

In this section we want to illustrate that the choice (31) of the product of the three superpropagators is the most natural one.

We note that the imaginary part of this product is completely fixed by unitarity. Its real part is fixed by locality away from the point where all coordinates coincide, the difference between two admissible choices being a real distribution concentrated in that point. We confine our attention to the real part of the product since it is this part that contains all the ambiguities and needs precise definition.

We consider the function $\tau^f(t_1, t_2, t_3)$ of the time differences $t_j - t_k$ that arises from $\operatorname{Re}\mathscr{T}(x_1, x_2, x_3)$, given by (31), after smearing over the spatial variables with test functions $f: \tilde{f}\in\mathfrak{M}_{1/3}(R^9)$. In order to determine the detailed structure of this function we need more information about $\operatorname{Re}\tilde{\mathscr{T}}$ than the previously established decrease $O\left(\operatorname*{Min}_{j=1,2,3}^{-3} p_j^2\right)$ and $O\left(\operatorname*{Max}_{j=1,2,3}^{-5/2} p_j^2\right)$ in $\mathcal{O}=\{p_1, p_2, p_3/p_j^2>0\; j=1,2,3\}$. We have to know the behavior of $\operatorname{Re}\tilde{\mathscr{T}}$ in $\mathscr{A}_{K^2}^l$

$$\mathscr{A}_{K^2}^l=\{p_1, p_2, p_3/-K^2<p_l^2<0, p_m^2>0, p_n^2>0\}, \qquad 0<K^2<\infty \tag{63}$$

as p_m^2 (and p_n^2) tend to $+\infty$. To this end, we study the following function

$$\begin{aligned}F(x_1, x_2, x_3)={}&\sum_0^\infty \frac{1}{m_3!(m_3+1)!}\sum_0^\infty \frac{1}{m_2!(m_2+1)!}\,\frac{1}{2\pi i}\int_C ds_1\,\frac{\Gamma(-1-s_1)}{\Gamma(3+s_1)}\\&\cdot\cos\pi s_1[1+\sin^2\pi s_1]\,\Gamma(-m_3-m_2-s_1)\,\Gamma(-s_1)\\&\cdot\text{analyt.cont.}\,\Gamma(-s_1)^{-1}\int_0^1 dt_2\,t_2^{m_3}\int_0^1 dt_1\,t_1^{-s_1-1}[1+t_2+t_2t_1]^{-m_3-m_2-s_1-2}\\&\cdot\tfrac{1}{2}\{[x_1-t_1(x_2+t_2x_3)-i0]^{m_3+m_2+s_1}+[x_1-t_1(x_2+t_2x_3)+i0]^{m_3+m_2+s_1}\}\\&+\int_0^1 dt_2\int_0^1 dt_1\,\mathfrak{R}(x_1, x_2, x_3; t_1, t_2)\end{aligned}\tag{64}$$

where

$$\mathfrak{R}(x_1,x_2,x_3;t_1,t_2)=\sum_{l=0}^{\infty}\frac{[x_1-t_1(x_2+t_2x_3)]^l}{l![1+t_2+t_2t_1]^{l+2}}\frac{1}{(2\pi i)^2}\int_{S_2-i\infty}^{S_2+i\infty}ds_2$$

$$\cdot\int_{S_1-i\infty}^{S_1+i\infty}ds_1\prod_{j=1}^{2}\left[\frac{\Gamma(-1-s_j)}{\Gamma(3+s_j)}\right]\frac{\Gamma(s_2+s_1+1-l)}{\Gamma(1+l-s_1-s_2)}t_2^{l-1-s_1-s_2}t_1^{-s_1-1}, \tag{65}$$

$$S_2<-1,\ S_1<-2,\ -7/2<S_2+S_1<-3.$$

We are interested in the behavior of $F(x_1,x_2,x_3)$ for $x_j>0$ $j=1,2,3$, x_2 (and x_3) tending to $+\infty$ and x_1 staying finite $0<x_1<K^2$. We split the t_1 integrations into three parts: 1) from 0 to $\xi(t_2)$, 2) from $\xi(t_2)$ to $2\xi(t_2)$ and 3) from $2\xi(t_2)$ to $+1$ where $\xi(t_2)=\dfrac{x_1}{x_2+t_2x_3}$. Accordingly, we write $F(x_1,x_2,x_3)$ as a sum of three terms $F_j(x_1,x_2,x_3)$ corresponding to the respective intervals. It is not difficult to determine the behavior of the terms F_3 and F_2 which can be cast into the form

$$F_3(x_1,x_2,x_3)=\int_0^1 dt_2\int_0^{\xi(t_2)^{-1}-2}dt_1\frac{1}{(2\pi i)^3}\iiint_{S-i\infty}^{S+i\infty}ds_3ds_2ds_1\prod_1^3\left[\frac{\Gamma(-1-s_j)}{\Gamma(3+s_j)}\right]$$

$$\cdot\Gamma\left(-\sum_1^3 s_j-2\right)t_2^{s_3+1}(2+t_1)^{-s_1-1}(1+t_1)^{\Sigma s_j+2} \tag{66}$$

$$\cdot[1+t_2+t_2(2+t_1)\cdot\xi(t_2)]^{-\Sigma s_j-4}x_1^{s_3+s_2+2}[x_2+t_2x_3]^{s_1}.$$

$$F_2(x_1,x_2,x_3)=\int_0^1 dt_2\sum_0^{\infty}\frac{(-x_1t_2)^{m_3}}{m_3!(m_3+1)!}\sum_0^{\infty}\frac{(-x_1)^{m_2}}{m_2!(m_2+1)!}$$

$$\cdot\frac{1}{2\pi i}\int_{S_1-i\infty}^{S_1+i\infty}ds_1\frac{\Gamma(-1-s_1)}{\Gamma(3+s_1)}\frac{\pi}{\sin\pi s_1}\Gamma(-m_3-m_2-s_1)[x_2+t_2x_3]^{s_1} \tag{67}$$

$$\cdot\frac{1}{2\pi i}\int_1^{0+}dt_1(-t_1)^{m_3+m_2+s_1}[1+t_1]^{-s_1-1}[1+t_2+t_2(1+t_1)\cdot\xi(t_2)]^{-m_3-m_2-s_1-2}$$

$$+\int_0^1 dt_2\int_{\xi(t_2)}^{2\xi(t_2)}dt_1\,\mathfrak{R}(x_1,x_2,x_3;t_1,t_2).$$

Clearly, F_3 is of type $O(x_2^{-5/2})$. It is an immediate consequence of the definition of $\mathfrak{R}(x_1,\dots,t_2)$ that the second term of the r.h.s. of (67) is of type $O(x_2^{-5/2})$. Hence, also F_2 is of type $O(x_2^{-5/2})$. F_1 is given by

$$F_1(x_1,x_2,x_3)=\int_0^1 dt_2 \sum_0^\infty \frac{(x_1t_2)^{m_3}}{m_3!(m_3+1)!}\sum_0^\infty \frac{x_1^{m_2}}{m_2!(m_2+1)!}\frac{1}{2\pi i}$$

$$\cdot\int_C ds_1 \frac{\Gamma(-1-s_1)}{\Gamma(3+s_1)}\frac{\pi\cos\pi s_1}{\sin\pi s_1}[1+\sin^2\pi s_1]\Gamma(-m_3-m_2-s_1)[x_2+t_2x_3]^{s_1}$$

$$\cdot\left\{\frac{1}{2\pi i}\int_{Z_r} dt_1(-t_1)^{-s_1-1}(1-t_1)^{m_3+m_2+s_1}[1+t_2+t_2t_1\cdot\xi(t_2)]^{-m_3-m_2-s_1-2}\right\}$$

$$+\int_0^1 dt_2\int_0^{\xi(t_2)} dt_1\,\Re(x_1,x_2,x_3;t_1,t_2) \tag{68}$$

where Z_r is the circle of radius r around the origin with positive orientation, r larger than one and fixed.

Again, the second term of the r.h.s. is of type $O(x_2^{-5/2})$. We note that the curled bracket is bounded by $2^{m_3+m_2+\mathrm{Re}\,s_1}\cdot\exp\left[\frac{\pi}{3}|\mathrm{Im}\,s_1|\right]$. The first term of the r.h.s. of (68) may be written as the sum of two terms according to the factor $[1+\sin^2\pi s_1]$. In the term corresponding to 1 the contour C may be straightened out which leads to a behavior of type $O(x_2^{-5/2})$ for this term. In the term corresponding to $\sin^2\pi s_1$ the s_1 and t_1 integrations may be replaced by summations over the respective residues which leads to an entire function of x_2 and x_3 of order 1/3. Summarizing, the behavior of $F(x_1,x_2,x_3)$ for $x_j>0$ $j=1,2,3$ as x_2 (and x_3) tends to $+\infty$ and x_1 stays finite: $0<x_1<K^2$ is given by a sum of an entire function of x_2 and x_3 of order 1/3 plus a function of type $O(x_2^{-5/2})$.

Now we are in a position to determine the behavior of $\mathrm{Re}\,\tilde{\mathscr{T}}(p_1,p_2,p_3)$ when some of the time components of the momenta tend to $\pm\infty$ while the space components are fixed in a compact set. Due to the presence of the energy conservation δ-function there are only two alternatives

1) $p_{j,0}^2\to+\infty$ for $j=1,2,3$.
2) $p_{l,0}^2$ stays finite, $p_{m,0}^2$ and $p_{n,0}^2$ tend to $+\infty$.

In the first case all p_j^2 tend to $+\infty$ which is a situation we have already dealt with in the preceding section where we found that $\mathrm{Re}\,\tilde{\mathscr{T}}(p_1,p_2,p_3)=\delta\left(\sum_1^3 p_n\right)O\left(\mathrm{Min}^{-3}_{j=1,2,3}\,p_j^2\right)$. In the second case p_m^2 and p_n^2 tend to $+\infty$ while p_l^2 stays finite. If $p_l^2>0$ we invoke once again a result of the preceding section and obtain

$$\mathrm{Re}\,\tilde{\mathscr{T}}(p_1,p_2,p_3)=\delta\left(\sum_1^3 p_n\right)O(\mathrm{Max}^{-5/2}\{p_m^2,p_n^2\}).$$

If $p_l^2 < 0$ we go back to the definition of $\tilde{\mathscr{T}}$, formula (31), and split the sum $\sum_{\sigma\in\mathfrak{S}_3}$ into two terms according to whether $i \neq l$ or $i = l$. The term with $i \neq l$ can be brought into a form analogous to (60) leading again to a behavior of type $\delta\left(\sum_1^3 p_n\right) \cdot O(\mathrm{Max}^{-5/2}\{p_m^2, p_n^2\})$ whereas the term corresponding to $i = l$ can be expressed by means of $F(x_1, x_2, x_3)$ and is equal to

$$\delta\left(\sum_1^3 p_n\right)\frac{\pi^2\lambda^4}{4}\left\{F\left(-\frac{\lambda}{4}p_l^2, \frac{\lambda}{4}p_m^2, \frac{\lambda}{4}p_n^2\right) + F\left(-\frac{\lambda}{4}p_l^2, \frac{\lambda}{4}p_n^2, \frac{\lambda}{4}p_m^2\right)\right\}.$$

The behavior of $F(x_1, x_2, x_3)$ in the region of interest has been established above. Thus we see that the behavior of $\mathrm{Re}\,\tilde{\mathscr{T}}(p_1, p_2, p_3)$ for $p_l^2 < 0$, p_m^2 and p_n^2 tending to $+\infty$ is given by

$$\delta\left(\sum_1^3 p_r\right)\{g(p_l^2, p_m^2, p_n^2) + O((p_m^2)^{-5/2}) + O((p_n^2)^{-5/2})\}$$

where g is an entire function of p_m^2 and p_n^2 of order 1/3.

Combining the various pieces of information, we obtain for $\tilde{f} \in \mathscr{D}(R^9)$

$$\begin{aligned}\hat{\tau}^f(p_{1,0}, p_{2,0}, p_{3,0}) &= \int \cdots \int \prod_1^3 d^3\boldsymbol{p}_j \tilde{f}(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3)\,\mathrm{Re}\,\tilde{\mathscr{T}}(p_1, p_2, p_3)\\ &= \delta\left(\sum_1^3 p_{n,0}\right)\left\{O\left(\mathrm{Min}\left\{\mathrm{Min}_j^{-6}|p_{j,0}|, \mathrm{Max}_j^{-5}|p_{j,0}|\right\}\right) + \sum_{\sigma\in\mathfrak{S}_3}\Theta(E - |p_{i,0}|)\cdot\right.\\ &\quad\left.\cdot\left[O(|p_{j,0}|^{-5}) + O(|p_{k,0}|^{-5}) + \hat{g}(p_{i,0}, p_{j,0}, p_{k,0})\right]\right\}. \qquad (69)\end{aligned}$$

Here E is some constant depending on the support of $\tilde{f}$ and $\hat{g}(p_{i,0}, p_{j,0}, p_{k,0})$ is an entire function of $p_{j,0}$ and $p_{k,0}$.

The statement (69) implies the following structure of $\tau^f(t_1, t_2, t_3)$:

$$\tau^f(t_1, t_2, t_3) = h^f(t_1 - t_2, t_2 - t_3, t_3 - t_1) + \sum_{\sigma\in\mathfrak{S}_3}\sum_{\nu=0}^{\infty}\delta^{(\nu)}(t_i - t_j)h^f_{\nu,\sigma}(t_j - t_k) \quad (70)$$

for $\tilde{f} \in \mathscr{D}(R^9)$ where h^f is a three times continuously differentiable function and $h^f_{\nu,\sigma}$ are infinitely differentiable functions such that

$$\sum_{\nu=0}^{\infty}\delta^{(\nu)}(t_i - t_j)\,h^f_{\nu,\sigma}(t_j - t_k) \in \mathscr{F}\mathfrak{M}'_{1/3}(R^2).$$

This result does not only hold for $\tilde{f} \in \mathscr{D}(R^9)$. Due to uniform convergence of the $\boldsymbol{p}_j$-integrations it also holds $\tilde{f} \in \mathfrak{M}_{1/3}(R^9)$.

The singularities of τ^f, dictated by locality, are contained in the second term of the r.h.s. of (70). They are attached to the lines $t_1 = t_2$,

$t_2 = t_3$ and $t_3 = t_1$. For no $\tilde{f} \in \mathfrak{M}_{1/3}(R^9)$ does τ^f possess any singularity attached exclusively to the point $t_1 = t_2 = t_3$. Any other choice of the time-ordered function would lead to a corresponding τ^f possessing such a point singularity for at least one f with $\tilde{f} \in \mathfrak{M}_{1/3}(R^9)$.

IV. Conclusions

The crucial property of a certain class of non-polynomial Lagrangians to which the exponential Lagrangian belongs is that in the weighted space averages of the time-ordered two- and three-point functions background (line singularities) and point singularity can be separated from each other. The special feature of our particular choices of the time-ordered two- and three-point functions consist in the absence of point singularities or in other words in their being least singular. However, it should be mentioned that the triangle graph considered here is a relatively simple object and enjoys special properties that need not be true for the general perturbation theoretic term.

Finally, it is worth noting that in spite of the extremely singular nature of the exponential interaction the analyticity structure of the triangle graph in the invariant momenta on all sheets of the Riemann surface is just the same as in renormalizable models. This is so because $\tilde{\mathfrak{R}}^-$ is an entire function and the sums and the integral in the first term of the r.h.s. of (31) converge uniformly in a neighborhood of any point of analyticity of $T_0^{\sigma -}(\{p\}; s_1, m_2 - 1, m_3 - 1)/_{\Sigma p_j = 0}$ regarded as a funtion on the complex invariant momenta. Thus the singularities are determined by $T_0^{\sigma -}(\{p\}; s_1, m_2 - 1, m_3 - 1)/_{\Sigma p_j = 0}$, their location is independent of s_1, m_2 and m_3 and hence the same as in conventional theories.

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