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Dispersion Relations for the Vertex Function from Local Commutativity

I. One-Dimensional Dispersion Relations

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Abstract. Dispersion relations for the vertex function are derived which are valid when two of the scalar variables are arbitrary complex inside certain domains of the product of the complex planes and the third scalar variable is evaluated just below or just above the physical region-cut.

The domains of validity of the dispersion relations for the complex variables are domains with three real dimensions and can be described as neighbourhoods of the boundaries of the "axiomatic" analyticity region of Källén and Wightman.

The discontinuity of the vertex function across the cut-surface in the third variable for such values of the remaining variables is expressed only in terms of the dynamical on-mass-shell matrix elements of the locally commuting field operators.

1. Introduction

In a series of earlier papers, hereafter called I [1], II [2], and III [3], we have derived a set of relations of the kind usually called sum rules for the vertex function both in momentum space [2, 3] and in coordinate-space [1].

The basic assumptions behind the results are the analyticity properties proved by Källén and Wightman [4] from some very general assumptions, which ought to be fulfilled in all "interesting" field theories. These authors assumed essentially that the field theory should admit

(i) *covariance* under *Lorentz transformations* and *translations*, i.e., among other things the existence of energy-momentum-operators

(ii) *"reasonable" mass spectrum* of the energy-momentum-operators, i.e., that the occurring energies and masses should be positive, and that the Hilbert space is spanned by eigenstates of these operators

(iii) *causality* in the form of *local commutativity* properties of the occurring field operators.

For the results of paper I–III we further need assumptions on the particle interpretation of the theory along the conventional lines of reduction formalism [7, 8] and some moderate boundedness and integrability properties of the vertex function when the boundaries of the holomorphy domain of Källén and Wightman is approached "from inside".

In this and a further paper we will use methods similar to the ones employed in I–III in order to derive representation formulas in terms of contour-integrals in the complex plane, of the kind called dispersion relations, for the vertex function. Here we will write down "one-dimensional" relations in the sense that only one-dimensional integral relations are employed. By means of the formalism developed here it is possible to express the vertex function for values of the arguments close to the boundary of the Källén-Wightman domain. Thus, the relations are valid when two of the scalar variables are arbitrary complex numbers inside certain domains (of three real dimensions) in the product of the complex planes, while the third variable is in the physical region in the sense that it is evaluated just below or just above the corresponding cut along the positive real axis.

The main results of the paper are relations which express the discontinuity across the above-mentioned physical-region-cut for complex values of the two remaining variables only in terms of the dynamical quantities of the field theory, i.e., the onmass-shell matrix elements of the field operators. These results closely resemble the well-known Källén-Lehmann [7–9, 11] representation formulas for the two-point function.

The formalism is applicable if the vertex function is at most polynomially increasing when one (or sometimes two) of the arguments is (are) allowed to approach infinity in directions inside the Källén-Wightman domain. These conditions are fulfilled for perturbation theoretical examples based upon polynomial interaction Hamiltonians.

In Section 2 the results of the paper of Källén and Wightman are briefly surveyed, in Section 3 the representation formulas are derived. In Section 4 we give some definitions for the vertex function in order to relate the boundary values in the dispersion relations to the physical quantities of the theory, i.e., the causal and time-ordered boundary values. In Section 5 the dispersion relations are expressed in terms of the matrix elements. Section 6 contains a few further remarks and extensions.

For simplicity, we will consider a scalar field theory in this paper and we will only write the results in terms of the momentum-space quantities. The results can certainly be generalized both to higher-spin field theories and to coordinate-space quantities.

2. The Källén-Wightman Domain

In this section we will for reference briefly review a few of the results of the paper by Källén and Wightman [4] on the analyticity properties of the vertex function. We will throughout this paper consider a scalar field theory and in that case the existence of a unique Lorentz invariant vertex function both in momentum-space (here called G) and in coordinate-space (F) can be proved [4]. The well-known connections between matrix elements between particle states of different operators and the different boundary values of the vertex function is briefly discussed in Section 4.

The vertex function will depend upon three scalar variables, which can be chosen as the Lorentz squares of the coordinate-differences between the field points in coordinate space and as the Lorentz squares of the external energy-momentum vectors in momentum space (here called Z_i, $i = 1, 2, 3$). Due only to the very general assumptions mentioned in Section 1, the vertex function exhibits analyticity properties in a rather large domain of the three-dimensional complex variable-space. This domain is explicitly constructed in the classical paper of Källén and Wightman [4].

The simplest way to describe the domain of holomorphy of the vertex function, which due to the complete symmetry between momentum-space and coordinate space is the same in both cases, is to describe the boundary surfaces. To that end we will divide the three-dimensional complex variable space into eight disjoint sections, corresponding to the eight different possibilities of choosing the signs of the imaginary parts of the three variables. In each one of these "octants" the relevant boundaries of the Källén-Wightman domain are given by the cut-surfaces along the positive real axes of the variables defined above, as well as a more complicated hypersurface. The "cuts" are actually reminiscences of the corresponding two-point function boundaries. There are, however, four different more complicated surfaces and each one of them constitutes the relevant boundary of the holomorphy domain in two of the octants, which are "opposite" in the sense that to go from one to the other one has to change the signs of the imaginary parts of all three variables. Thus, in case the quantities $\operatorname{Im} Z_i$ and $\operatorname{Im} Z_j$ have the same sign, while $\operatorname{Im} Z_k$ $(i \neq j \neq k \neq i)$ has the opposite sign, the relevant boundary is the F'_{ij}-surface:

$$(r - Z_i)(r - Z_j) + r Z_k = 0\,, \qquad r > 0\,. \tag{1}$$

The three different F'_{ij}-surfaces are known from perturbation theoretical examples [4]. In the remaining two octants, i.e., in case all the signs of the imaginary parts are equal, perturbation theoretical functions do not exhibit any other singularities except the cut-surfaces [5], but the axiomatic approach does not exclude singularities outside the completely symmetrical boundary-surface called $\mathscr{F}$:

$$r^2 - r(Z_1 + Z_2 + Z_3) + Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3 = 0\,, \qquad r > 0\,. \tag{2}$$

Due to the large symmetry of the above-described Källén-Wightman domain it is for our purposes only necessary to investigate a few special cases in detail and then the general results can be found by obvious permutations of indices.

In this paper we will make use of the analyticity properties in the neighbourhood of the boundaries of the holomorphy domain. To be specific, we will in the same spirit as in papers I–III consider the vertex function just "above" the surface of intersection between one of the cut-surfaces and one of the F'_{ij}- or $\mathscr{F}$-surfaces. We will be satisfied to discuss the case when the third argument is chosen just below or just above the corresponding cut-surface, i.e., the positive real axis, while the other two arguments are general complex numbers but restricted to a neighbourhood of the boundary of the domain of holomorphy. As this boundary is different in the different octants as described above, we will have to differ between the two cases when

$$\operatorname{Im} Z_1 \cdot \operatorname{Im} Z_2 > 0 \quad \text{and} \quad \operatorname{Im} Z_1 \cdot \operatorname{Im} Z_2 < 0\,.$$

In Section 3A the first case is discussed and in Section 3B the second one. We will in both cases write the relations in terms of the momentum space function G and only in the end comment upon the differences in case the coordinate-space function F is used instead of G.

3. Dispersion Relations for the Vertex Function along the Boundary of the Källén-Wightman Domain

3A. The Case $\operatorname{Im} Z_1 \cdot \operatorname{Im} Z_2 > 0$

We will in this section discuss in some detail the properties of the vertex function close to the intersection between the F'_{12}-surface, respectively the $\mathscr{F}$-surface of Eqs. (1) and (2) and the positive real Z_3-axis.

To that end we will first consider the function $\Gamma^{+(3)}$ defined by

$$\Gamma^{+(3)}(\zeta_1; r, \zeta_3) = G\left(\zeta_1, r + \frac{r\zeta_3}{r - \zeta_1}, \zeta_3 + i\varepsilon\right). \tag{3}$$

The parameters r and ζ_3 as well as the infinitesimal quantity ε will be restricted to the positive real numbers.

From the "axiomatic" analyticity properties we deduce just as in the similar examples of paper I that the function $\Gamma^{(+)(3)}$ is analytic in the lower half-complex ζ_1-plane. This is due to the fact that with the restrictions above on the parameters r and ζ_3 the second argument ζ_2 fulfils

$$\begin{aligned} \zeta_2 &= r + \frac{r\zeta_3}{r-\zeta_1} \\ \operatorname{Im}\zeta_2 &= r\cdot\zeta_3 \frac{\operatorname{Im}\zeta_1}{|r-\zeta_1|^2} \end{aligned} \tag{4}$$

and thus the sign of $\operatorname{Im}\zeta_2$ is the *same* as the sign of $\operatorname{Im}\zeta_1$. The first line of Eq. (4) can be seen to be the expression for the F'_{12}-surface. Because of the (infinitesimal) translation ($i\varepsilon$) in the third argument of Eq. (3), the vertex function is as a matter of fact evaluated just inside the axiomatic domain. In this octant the only remaining axiomatic singularities are the Z_1- and Z_2-cuts according to what has been said in Section 2 and these surfaces are easily seen to cover the real ζ_1-axis.

By the same argument the function

$$\Gamma^{-(3)}(\zeta_1; r, \zeta_3) = G\left(\zeta_1, r + \frac{r\zeta_3}{r-\zeta_1}, \zeta_3 - i\varepsilon\right) \qquad r>0,\ \zeta_3>0,\ \varepsilon>0\,, \tag{5}$$

i.e. in which the third argument of the vertex functions is evaluated just below the Z_3-cut, exhibits no "axiomatic" singularities in the upper half complex ζ_1-plane.

We further note that by the exchange

$$r + \zeta_3 = \varrho > 0 \tag{6}$$

we get from Eq. (4)

$$\zeta_2 \equiv \varrho + \frac{\zeta_1\zeta_3}{\varrho - \zeta_1 - \zeta_3}\,, \tag{7}$$

i.e., the analytic expression for the $\mathscr{F}$-surface of Eq. (2). Note that in this case we have the restriction

$$0 < \zeta_3 < \varrho\,. \tag{8}$$

By the same argument as above we can then deduce that the function $\Gamma^{+(3)}$ is actually axiomatically analytic also in the upper half complex ζ_1-plane and $\Gamma^{-(3)}$ correspondingly in the lower half complex ζ_1-plane. To that end we make use of the analyticity properties proved by Källén and Wightman in the octants where all the imaginary parts of the variables are equal, i.e. the octants where only the $\mathscr{F}$-surface and the cuts constitute relevant boundaries. In this way we have, consequently, derived the results that the two functions $\Gamma^{\pm(3)}$ are both of them analytic in the *whole complex ζ_1-plane* except for cuts along the real axis stemming from the corresponding two-point function boundaries, the Z_1-cut and the Z_2-cut.

These results immediately imply that Cauchy's theorem can be used to derive what is usually called dispersion relations, i.e., representation formulas for the function $\Gamma^{\pm(3)}$ in terms of contour integrals in the complex plane.

To that end it is necessary to make assumptions on the behaviour of the functions $\Gamma^{\pm(3)}$ in the neighbourhood of the real ζ_1-axis as well as for large values of the variable ζ_1. Such assumptions are seen to be equivalent to assumptions on the behaviour of the vertex function G in the neighbourhood of the intersection between two or sometimes three of the cut-surfaces, as well as assumptions on the behaviour when one of the arguments tends to infinity in directions inside the domain of holomorphy while the remaining variables are close to positive real values.

We will for our purpose be satisfied to assume moderate integrability properties (satisfied e.g. for tempered distributions) and at most a polynomial increase of $\Gamma^{\pm(3)}$ in different neighbourhoods of infinity (inside the domain of analyticity). These assumptions are "natural" in the sense that this behaviour is found in perturbation theoretical examples in theories with polynomial interaction Hamiltonians.

To be specific we will assume the limiting relation

$$L_n^{\pm} = \lim_{R\to\infty} \int_0^{\pi} d\Theta \frac{1}{R^n} G\left(R e^{i\Theta}, r - \frac{r\zeta_3}{R} e^{-i\Theta}, \zeta_3 \pm i\varepsilon\right) = 0 \qquad (9)$$

to be fulfilled for some finite integer n.

For simplicity we will start by assuming that the integer n in Eq. (9) can be chosen to be $n=0$. We then use Cauchy's theorem to deduce that if the pole-position Z_1 is chosen in the upper half complex plane then for an integration curve C that encircles Z_1 in a positive sense we have:

$$\Gamma^{\pm(3)}(Z_1; r, \zeta_3) = \frac{1}{2\pi i} \oint_C \frac{d\zeta_1}{\zeta_1 - Z_1} \Gamma^{\pm(3)}(\zeta_1; r, \zeta_3). \qquad (10)$$

By choosing the integration curve as a large semi-circle around the origin with radius R and a straight line along the real ζ_1-axis we get from Eq. (9) for $n=0$ in the limit when R tends to infinity and we, consequently, can neglect the contribution from the semi-circle:

$$\Gamma^{\pm(3)}(Z_1; r, \zeta_3) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{d\zeta_1}{\zeta_1 - Z_1} G\left(\zeta_1 + i\varepsilon', r + \frac{r\zeta_3}{r-\zeta_1} + i\varepsilon'', \zeta_3 \pm i\varepsilon\right), \quad \operatorname{Im} Z_1 > 0. \qquad (11)$$

We further note that the integral of Eq. (11) vanishes if the pole-position Z_1 is chosen in the lower half complex plane, because in that case there is no pole "inside" the limiting curve C. A completely similar discussion can now be performed for the lower half ζ_1-plane and we get

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{d\zeta_1}{\zeta_1-Z_1}G\left(\zeta_1-i\varepsilon', r+\frac{r\zeta_3}{r-\zeta_1}-i\varepsilon'', \zeta_3\pm i\varepsilon\right) \tag{12}$$
$$\begin{aligned} &= -\Gamma^{\pm(3)}(Z_1; r, \zeta_3) \qquad \operatorname{Im} Z_1 < 0 \\ &= 0 \qquad\qquad\qquad\qquad\quad \operatorname{Im} Z_1 > 0 . \end{aligned}$$

We note especially the occurring minus sign in Eq. (12) which corresponds to the fact that all complex integrals must be performed in the "positive sense".

By combining the two Eqs. (11) and (12) and remembering the remark made after Eq. (11), we get representation formulas for the functions $\Gamma^{\pm(3)}$ which are valid in the whole complex Z_1-plane except for the real axis:

$$\Gamma^{\pm(3)}(Z_1; r, \zeta_3) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{d\zeta_1}{\zeta_1-Z_1}\int d\zeta_2\, \delta\left(\zeta_2-r-\frac{r\zeta_3}{r-\zeta_1}\right) \tag{13}$$
$$\cdot\{G(\zeta_1+i\varepsilon', \zeta_2+i\varepsilon'', \zeta_3\pm i\varepsilon)-G(\zeta_1-i\varepsilon', \zeta_2-i\varepsilon'', \zeta_3\pm i\varepsilon)\} .$$

By some straightforward algebraic manipulations (note that the quantities (ζ_1-r) and (ζ_2-r) have opposite signs due to the restrictions on the parameters r and ζ_3) we get the following result:

$$\begin{aligned} G(Z_1, Z_2, \zeta_3\pm i\varepsilon) &= \frac{1}{2\pi i}\int d\zeta_1\, d\zeta_2\, \delta[(r-\zeta_1)(r-\zeta_2)+r\zeta_3] \\ &\cdot\{G(\zeta_1+i\varepsilon', \zeta_2+i\varepsilon'', \zeta_3\pm i\varepsilon)-G(\zeta_1-i\varepsilon', \zeta_2-i\varepsilon'', \zeta_3\pm i\varepsilon)\} \\ &\cdot\left\{(\zeta_1-r)\frac{\Theta(\zeta_1-r)\Theta(r-\zeta_2)}{\zeta_1-Z_1}+(r-\zeta_1)\frac{\Theta(r-\zeta_1)\Theta(\zeta_2-r)}{\zeta_1-Z_1}\right\} \\ &(r-Z_1)(r-Z_2)+r\zeta_3=0 , \qquad r>0,\ \zeta_3>0 . \end{aligned} \tag{14}$$

To spell out the actual symmetry of the integrand of Eq. (14) we note the following equality which is useful for the second term in the last parenthesis of the integrand:

$$\frac{r-\zeta_1}{\zeta_1-Z_1}=\frac{r-\zeta_1}{\zeta_1-r-\dfrac{r\zeta_3}{r-Z_2}}=\frac{(r-Z_2)(r-\zeta_1)}{(\zeta_1-r)(\zeta_2-Z_2)}=\frac{\zeta_2-r}{\zeta_2-Z_2}-1 . \tag{15}$$

In Eq. (15) we have used the relation between the variables Z_1 and Z_2 from the last line of Eq. (14) as well as the corresponding relation between ζ_1 and ζ_2 from the δ-function. Except maybe for the last term, (-1), in Eq. (15), we thus get obviously symmetrical contributions from the integration ranges $(\zeta_1>r>0;\ \zeta_2<r)$ and $(\zeta_1<r;\ \zeta_2>r>0)$, i.e., from the parts of the integration ranges corresponding to respectively the Z_1-cut and the Z_2-cut.

The last term can, however, also be made into a symmetrical contribution in the same sense, if we make use of the sum rules of paper I–III. The rum rules of Eq. (14) in paper I – in that paper expressed in terms of the coordinate-space vertex function F – and repeated in terms of the momentum-space function G in Eq. (A1) of appendix A of paper II as well as the sum rules of Eq. (26) in paper III imply under the limiting condition of Eq. (9)

$$\begin{aligned}&\int d\zeta_1\, d\zeta_2\, \delta[r\zeta_3+(r-\zeta_1)(r-\zeta_2)]\,\{G(\zeta_1+i\varepsilon',\zeta_2+i\varepsilon'',\zeta_3\pm i\varepsilon)\\ &-G(\zeta_1-i\varepsilon',\zeta_2-i\varepsilon'',\zeta_3\pm i\varepsilon)\}\,\{\Theta(\zeta_1-r)\,\Theta(r-\zeta_2)-\Theta(r-\zeta_1)\,\Theta(\zeta_2-r)\}=0\\ &\qquad r>0,\ \zeta_3>0 \qquad (16)\end{aligned}$$

and we can then by using Eqs. (15) and (16) finally write for the case $\operatorname{Im} Z_1$, $\operatorname{Im} Z_2>0$

$$\begin{aligned}&G(Z_1,Z_2,\zeta_3\pm i\varepsilon)\\ &=\frac{1}{4\pi i}\int d\zeta_1 d\zeta_2 \delta[r\zeta_3+(r-\zeta_1)(r-\zeta_2)]\,\{G(\zeta_1+i\varepsilon',\zeta_2+i\varepsilon'',\zeta_3\pm i\varepsilon)\\ &\quad-G(\zeta_1-i\varepsilon',\zeta_2-i\varepsilon'',\zeta_3\pm i\varepsilon)\}\left\{\frac{\zeta_1+Z_1-2r}{\zeta_1-Z_1}\Theta(\zeta_1-r)\,\Theta(r-\zeta_2)\right. \qquad (17)\\ &\quad\left.+\frac{\zeta_2+Z_2-2r}{\zeta_2-Z_2}\,\Theta(r-\zeta_1)\,\Theta(\zeta_2-r)\right\}\\ &(Z_1-r)(Z_2-r)+r\zeta_3=0\,, \qquad r>0,\ \zeta_3>0\,.\end{aligned}$$

We will actually consider the difference

$$(G(Z_1,Z_2,\zeta_3+i\varepsilon)-G(Z_1,Z_2,\zeta_3-i\varepsilon))\,,$$

i.e., the dispersion relation for the discontinuity across the Z_3-cut in the applications of Section 5.

Eq. (17) can also be generalised to the case when Eq. (9) is not satisfied for $n=0$ but is satisfied for a finite positive integer n (it is of course evident that if Eq. (9) is satisfied not only for $n=0$ but also for negative integers, then Eq. (17) needs no further modification).

In that case we consider instead of the function $\Gamma^{\pm(3)}(Z_1,r,\zeta_3)$ in Eq. (13) the following integral for an arbitrary but fixed "subtraction point χ_1":

$$\begin{aligned}I_n=\frac{1}{2\pi i}\int\frac{d\zeta_1}{(\zeta_1-Z_1)}\frac{1}{(\zeta_1-\xi_1)^n}&\left\{G\left(\zeta_1+i\varepsilon',r+\frac{r\zeta_3}{r-\zeta_1}+i\varepsilon'',\zeta_3\pm i\varepsilon\right)\right.\\ &\left.-G\left(\zeta_1-i\varepsilon',r+\frac{r\zeta_3}{r-\zeta_1}-i\varepsilon'',\zeta_1\pm i\varepsilon\right)\right\}. \qquad (18)\end{aligned}$$

It is easily seen that the integral I_n equals

$$I_n = \left[\frac{G\left(Z_1, r+\frac{r\zeta_3}{r-Z_1}, \zeta_3 \pm i\varepsilon\right) - \sum_{m=0}^{n-1} (Z_1-\xi_1)^m \frac{1}{m!}\left(\frac{\partial}{\partial \xi_1}\right)^m G\left(\xi_1, r+\frac{r\zeta_3}{r-\xi_1}, \zeta_3 \pm i\varepsilon\right)}{(Z_1-\xi_1)^n}\right]. \tag{19}$$

The expressions of Eq. (18) and Eq. (19) can be "symmetrised" in a way similar to the discussion above in connection with Eqs. (15)–(17) but we will not write out the details of such expressions.

3B. The Case $\operatorname{Im} Z_1 \cdot \operatorname{Im} Z_2 < 0$

There is a further set of dispersion relations similar to the ones in Section 3A and also based upon the analyticity properties which can be derived from the Källén-Wightman results. In these cases we will consider the remaining four octants of the three-dimensional complex variable space, i.e., those in which $\operatorname{Im} Z_1 \cdot \operatorname{Im} Z_2 < 0$. The third variable will once again be evaluate just above or below the real positive axis.

To that end we will consider the functions $\gamma^{\pm(3)}$ defined by

$$\gamma^{\pm(3)}(\zeta_1; r, \zeta_3) = G\left(\zeta_1, \frac{\zeta_3 - r}{r}(r-\zeta_1), \zeta_3 \pm i\varepsilon\right) \tag{20}$$

with the following restrictions on the real, positive parameters r and ζ_3:

$$0 < r < \zeta_3, \tag{21}$$

In this case the second argument of the vertex function G fulfils

$$\begin{aligned} \zeta_2 &= \frac{\zeta_3 - r}{r}(r-\zeta_1) \\ \operatorname{Im}\zeta_2 &= -\frac{\zeta_3 - r}{r}\operatorname{Im}\zeta_1 . \end{aligned} \tag{22}$$

The first line of Eq. (22) is seen to be the analytic expression for the F'_{13}-surface (cf. Eq. (1)), the second line tells that the sign of $\operatorname{Im}\zeta_2$ is *opposite* to the sign of $\operatorname{Im}\zeta_1$. We are, because of the (infinitesimal) translation $\pm i\varepsilon$ in the third argument, consequently once again investigating the vertex function just above the surface of intersection between one of the complicated analytic hypersurface boundaries of Källén and Wightman, in this case the F'_{13}-surface, and one of the cut-surfaces.

In a way that is very similar to the one used in Section 3A we may then deduce that the function $\gamma^{+(3)}(\zeta_1)$ and $\gamma^{-(3)}(\zeta_1)$ are "axiomatically" analytic in the upper respectively lower half complex ζ_1-plane. Further the Z_1-cut and Z_2-cut once again cover the real ζ_1-axis.

By making the change of parameter

$$\varrho = \zeta_3 - r > 0 \tag{23}$$

(note that we have $\zeta_3 - \varrho = r > 0$) we find, however, from the first line of Eq. (22) that the second argument ζ_2 can also be written as

$$\zeta_2 = \varrho + \frac{\varrho \zeta_1}{\varrho - \zeta_3}. \tag{24}$$

But this is the analytic expression for the F'_{23}-curve (cf. Eq. (1)). This surface constitutes the relevant boundary of the Källén-Wightman domain when the signs of the imaginary parts of the second and third argument are the same, while the imaginary part of the first argument of the vertex function has the opposite sign. From the second line of Eq. (22) which written in terms of the new parameter ϱ (note the remark after Eq. (23)) is

$$\operatorname{Im} \zeta_2 = -\frac{\varrho}{\zeta_3 - \varrho} \operatorname{Im} \zeta_1 \tag{25}$$

we may deduce that the functions $\gamma^{+(3)}$ and $\gamma^{-(3)}$ represent the vertex function also along the intersection of the F'_{23}-surface and the Z_3-cut in the lower respectively upper ζ_1-plane. Due to the infinitesimal translation $\pm i\varepsilon$ in the third argument we also deduce that we are inside the holomorphy domain, and that the functions $\gamma^{+(3)}$ and $\gamma^{-(3)}$ consequently are analytic also in the lower respectively upper half complex ζ_1-plane.

Thus from the Källén-Wightman results we deduce that the functions $\gamma^{\pm(3)}$ are (just as the functions $\Gamma^{\pm(3)}$) analytic in the whole complex ζ_1-plane with the exception of the real ζ_1-axis where there may be "axiomatic" singularities corresponding to the two-point function boundaries, the Z_1-cut and the Z_2-cut. Due to the similarity between the analyticity properties of the functions $\Gamma^{\pm(3)}$ of Section 3A and the functions $\gamma^{\pm(3)}$ discussed above we can use the same arguments to derive representation formulas in terms of contour-integrals. The relations corresponding to Eq. (13) of Section 3A are

$$\begin{aligned} \gamma^{\pm(3)}(Z_1; r, \zeta_3) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{d\zeta_1}{\zeta_1 - Z_1} \int d\zeta_2\, \delta\left[\zeta_2 - \frac{\zeta_3 - r}{r}(r - \zeta_1)\right] \\ \cdot \{G(\zeta_1 + i\varepsilon', \zeta_2 - i\varepsilon'', \zeta_3 \pm i\varepsilon) - G(\zeta_1 - i\varepsilon', \zeta_2 + i\varepsilon'', \zeta_3 \pm i\varepsilon)\}. \end{aligned} \tag{26}$$

We have then for simplicity implicitly assumed the limiting relations

$$I_n^{\pm} = \lim_{R\to\infty} \int_0^{\pi} d\Theta \frac{1}{R^n} G\left(Re^{i\Theta}, -\frac{\zeta_3 - r}{r} Re^{i\Theta}, \zeta_3 \pm i\varepsilon\right) = 0 \tag{27}$$

to be fulfilled for the case $n = 0$ (cf. Eq. (9)).

It is further possible to "symmetrise" the relations of Eq. (26) in a similar way as in connection with Eq. (17) of Section 3A. To that end we introduce as new parameters α and β with

$$\begin{aligned} r &= \zeta_3 \alpha \\ \alpha + \beta &= 1 \end{aligned} \tag{28}$$

(note that $\alpha > 0$ and $\beta > 0$, cf. Eq. (23)).

In terms of them we can write both the F'_{13}- and F'_{23}-surfaces in this case as

$$\alpha\zeta_2 + \beta\zeta_1 = \alpha\beta\zeta_3 \,. \tag{29}$$

Then we find from Eq. (26) by a few obvious algebraic manipulations

$$\begin{aligned} &G(Z_1, Z_2, \zeta_3 \pm i\varepsilon) \\ &= \frac{1}{2\pi i} \int d\zeta_1\, d\zeta_2\, \delta(\alpha\zeta_2 + \beta\zeta_1 - \alpha\beta\zeta_3) \{G(\zeta_1 + i\varepsilon', \zeta_2 - i\varepsilon'', \zeta_3 \pm i\varepsilon) \\ &\quad - G(\zeta_1 - i\varepsilon', \zeta_2 + i\varepsilon'', \zeta_3 \pm i\varepsilon)\} \left\{\frac{\alpha}{\zeta_1 - Z_1} [\Theta(\zeta_1)\Theta(-\zeta_2) + \tfrac{1}{2}\Theta(\zeta_1)\Theta(\zeta_2)] \right. \\ &\quad \left. - \frac{\beta}{\zeta_2 - Z_2} [\Theta(\zeta_2)\,\Theta(-\zeta_1) + \tfrac{1}{2}\Theta(\zeta_1)\,\Theta(\zeta_2)]\right\} \\ &\beta Z_1 + \alpha Z_2 = \alpha\beta\zeta_3 \,. \end{aligned} \tag{30}$$

We have here used the equalities (cf. the corresponding Eq. (15) of Section 3A)

$$\frac{\alpha}{\zeta_1 - Z_1} = \frac{\alpha}{\zeta_1 - \alpha\zeta_3 + \dfrac{\alpha}{\beta} Z_2} = -\frac{\beta}{\zeta_2 - Z_2} \tag{31}$$

and then referred half of the contribution from the part of the integration range when both of the integration variables ζ_1 and ζ_2 are positive, to each one of the terms. Finally, in case Eq. (27) needs a "convergence power" $n > 0$ we can in the same way as in connection with Eq. (18) of

Section 3 A consider the integral I'_n instead of Eq. (26):

$$I'_n = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{d\zeta_1}{(\zeta_1 - Z_1)} \cdot \frac{1}{(\zeta_1 - \xi_1)^n} \left\{ G(\zeta_1 + i\varepsilon', \beta\zeta_3 - \frac{\beta\zeta_1}{\alpha} - i\varepsilon'', \zeta_3 \pm i\varepsilon) - G(\zeta_1 - i\varepsilon', \beta\zeta_3 - \frac{\beta\zeta_1}{\alpha} + i\varepsilon'', \zeta_3 \pm i\varepsilon) \right\}. \tag{32}$$

The value of the integral I'_n is easily seen to be

$$I'_n = \frac{1}{(Z_1 - \xi_1)^n} \left\{ G\left(Z_1, \beta\zeta_3 - \frac{\beta Z_1}{\alpha} i\varepsilon, \zeta_3 \pm i\varepsilon\right) - \sum_{m=0}^{n-1} (Z_1 - \xi_1)^m \frac{1}{m!} \left(\frac{\partial}{\partial \xi_1}\right)^m G\left(\xi_1, \beta\zeta_3 - \frac{\beta\xi_1}{\alpha}, \zeta_3 \pm i\varepsilon\right) \right\}. \tag{33}$$

For the cases $n = 1$ and $n = 2$ we can, e.g., write for this expression

$$I'_1 = \frac{1}{Z_1 - \xi_1} \{ G(Z_1, Z_2, \zeta_3 \pm i\varepsilon) - G(\xi_1, \xi_2, \zeta_3 \pm i\varepsilon) \}$$

$$I'_2 = \frac{1}{(Z_1 - \xi_1)^2} \left\{ G(Z_1, Z_2, \zeta_3 \pm i\varepsilon) - G(\xi_1, \xi_2, \zeta_3 \pm i\varepsilon) - (Z_1 - \xi_1) \frac{\partial}{\partial \xi_1} G(\xi_1, \xi_2, \zeta_3 \pm i\varepsilon) - (Z_2 - \xi_2) \frac{\partial}{\partial \xi_2} G(\xi_1, \xi_2, \zeta_3 \pm i\varepsilon) \right\} \tag{34}$$

$$\beta Z_1 + \alpha Z_2 = \alpha\beta\zeta_3$$

$$\beta\xi_1 + \alpha\xi_2 = \alpha\beta\zeta_3 .$$

We will end this section with a few remarks.

1. Because of the symmetry of the Källén-Wightman domain we can write completely similar relations as we have done for the functions $\Gamma^{\pm(3)}$ and $\gamma^{\pm(3)}$ above also for the functions $\Gamma^{\pm(j)}$ and $\gamma^{\pm(j)}$ $j = 1, 2$ which are defined in obvious ways. We will, however, not give explicit expressions since such formulas can be obtained by straightforward permutations of indices from Eqs. (17) and (30).

2. We have in all the formulas above used momentum-space quantities.·Because of the symmetry between momentum-space and coordinate space, all that has been said above, apart from the boundedness properties, i.e., the assumed limiting properties in Eqs. (9) and (27) can word for word be repeated in terms of the coordinate-space function F. In coordinate-space, however, perturbation theoretical examples in general imply exponential damping when one or more variables are allowed to tend to infinity in different directions *inside* the holomorphy domain ("cluster properties"), and therefore the "convergence factors" of Eqs. (18) and (32) might be unnecessary.

There is, however, a further and more deep-lying difference between momentum-space and coordinate space in the fact that even in theories containing no particles with vanishing mass, the two-point boundary surfaces, the Z_1- and Z_2-cuts are believed to start in the origin of coordinate-space ("lightcone singularities"). In momentum-space, however, the corresponding situation is different. We would expect, in general, to find isolated poles corresponding to the one-particle states situated at the square of the corresponding masses (cf. Section 4 and 5), and only further "up", i.e., further along the positive real axes, the thresholds from the scattering states. This "mass gap" will become evident when we express the boundary values $[G(\zeta_1 + i\varepsilon, \zeta_2 + i\varepsilon, \zeta_3 + i\varepsilon) - G(\zeta_1 - i\varepsilon, \zeta_2 - i\varepsilon, \zeta_3 + i\varepsilon)]$ etc. in terms of the on-mass-shell matrix elements in Section 5.

3. The integration ranges occurring in the representation formulas are actually parts of what is called the "distinguished boundary" of the Källén-Wightman domain of holomorphy. This concept plays in general an important role in representation theory for functions of several complex variables [6]. The distinguished boundary can in an intuitive way be described as the "utmost corners" of the domain of holomorphy.

4. We would like to stress that the boundary values of the vertex function which occurs in the integrands of the dispersion relations above are "physical" in the sense that they can be expressed in terms of the physical quantities of the field theory, i.e., the causal and time-ordered boundary values etc. of the vertex functions. These quantities are in a well-known way via reduction formalism related to the matrix elements of different operators. We will give the details of such expressions in Section 5 after having given a few relevant definitions for this program in the next section.

4. The Boundary Values of the Vertex Function

In this section we will give some relevant definitions for the vertex function and briefly touch upon the problem of expressing the boundary values occurring in the integrals of Section 3 in terms of the physical quantities of the theory.

We will rely heavily upon the formalism developed in papers I–III where similar problems have been treated in more detail. Just as in that case we will consider a field theory with three scalar fields A, B and C. Due to the assumed Lorentz-covariance of the theory, the vertex function in momentum space G is a Lorentz invariant function only depending upon the (complex) scalar variables Z_i, which we will often identify with the Lorentz squares of three (complex) vectors $\varkappa_i$, fulfilling energy-momentum conservation:

$$Z_i = -\varkappa_i^2, \ i=1,2,3; \quad \sum_{i=1}^{3} \varkappa_i = 0; \quad \varkappa_i = p_i + ik_i, \ i=1,2,3. \quad (35)$$

(We use the matric $\varkappa^2 = \bar{\varkappa}^2 - \varkappa_0^2$.)

We will further assume in order to introduce particle states into the discussion that the fields admit weak asymptotic limits to free fields in such a way that a reduction formalism can be developed. The free fields describe "incoming" and "outgoing" A-, B- and C-particles with masses m_1, m_2 and m_3 respectively. We will without further discussion assume that the field theory is "renormalized" in the sense that e.g. for a matrix element between a "stable" one A-particle state with momentum $\bar{p}_1$ and the vacuum the following relation is valid [9]

$$\begin{gathered}\langle 0|A(x_1)|\bar{p}_1\rangle = \langle 0|A^{\text{in}}(x_1)|\bar{p}_1\rangle = \langle 0|A^{\text{out}}(x_1)|\,\bar{p}_1\rangle = \frac{1}{N_1} e^{ip_1x_1}; \\ N_1 = \sqrt{2VZ_1}\,. \end{gathered} \quad (36)$$

The quantities N_j are normalisation constants, which depend upon the energies $E_j = \sqrt{\bar{p}_j^2 + m_j^2}$ as well as upon the quantisation volume V. The quantity V occurs because we will handle the problems connected with Haag's theorem [10] by the conventional procedure of introducing a "quantisation-box" with periodic boundary conditions.

By means of reduction formalism [7] in the way described in some detail in papers II and III (but without attempts at mathematical rigour) we may relate matrix elements, between two one-particle states for the operators, to different boundary values of the vertex function in momentum space.

We will use the same notations as in papers II and III and talk about the retarded (R) and advanced (A) boundary values (for short "causal boundary values", CBV) and about the time-ordered (T^+) and antitime-ordered (T^-) boundary values (for short TBV and $ATBV$) of the vertex function.

In that way we may write

$$\langle 0|(m_2^2 - \Box_2)\, B|\bar{p}_1, \bar{p}_3\rangle\, N_1 N_3 = \begin{Bmatrix} R_B(p_1, p_3) \\ A_B(p_1, p_3) \\ T^+(-p_1^2, -(p_1+p_3)^2, -p_3^2) \\ T^-(-p_1^2, -(p_1+p_3)^2, -p_3^2) \end{Bmatrix}. \quad (37)$$

We have in connection with Eq. (37) not specified whether the state $|\bar{p}_1, \bar{p}_3\rangle$ corresponding to a state with one A-particle and one C-particle is an "out-state" or an "in-state". In connection with R_B and T^+ we deal, however, with "in-states" ("particles at a time long before the interaction") while in connection with A_B and T^- we deal with "out-states" [8]. All the quantities R_B, A_B, T^+ and T^- have Fourier transforms in terms of vacuum expectation values (VEV) of different operator products:

$$\left\{\begin{matrix} R_B(p_1, p_3) \\ A_B(p_1, p_3) \\ T^+(-p_1^2, -(p_1+p_3)^2, -p_3^2) \\ T^-(-p_1^2, -(p_1+p_3)^2, -p_3^2) \end{matrix}\right\} = -\int dx_1\, dx_3\, e^{ip_1(x_1-x_2)+ip_3(x_3-x_2)} \left\{\begin{matrix} r_B(x_2-x_3, x_2-x_1) \\ a_B(x_3-x_2, x_1-x_2) \\ t^+(x_1, x_2, x_3) \\ t^-(x_1, x_2, x_3) \end{matrix}\right\} \tag{38}$$

$$\begin{aligned} r_B(x_2-x_3, x_2-x_1) &= \prod_{j=1}^{3} (m_j^2 - \Box_j)\, \{\langle 0|\, \Theta(21)\, \Theta(13)\, [C, [A, B]]\, |0\rangle \\ &\quad + \langle 0|\, \Theta(23)\, \Theta(31)\, [A, [C, B]]\, |0\rangle\} \\ t^+(x_1, x_2, x_3) &= \prod_{j=1}^{3} (m_j^2 - \Box_j)\, \langle 0|\, \{\Theta(12)\, \Theta(23)\, ABC + \Theta(32)\, \Theta(21)\, CBA \\ &\quad + \Theta(13)\, \Theta(32)\, ACB + \Theta(23)\, \Theta(31)\, BCA \\ &\quad + \Theta(21)\, \Theta(13)\, BAC + \Theta(31)\, \Theta(12)\, CAB\}\, |0\rangle\,. \end{aligned} \tag{39}$$

We have for practical reasons used the notation $\Theta(12) = \Theta(x_1 - x_2)$ for the step function with a vector argument, meaning that the vector belongs to the forward light-cone V^+. The field-points of the fields A, B and C are in all cases of Eq. (39) x_1, x_2 respectively x_3.

The quantity r_B is apart from the product of Klein-Gordon operators, stemming from the reduction formalism, the VEV of the combination of operator products called the retarded commutator. It is only a matter of algebra to show that r_B has support only when the field-points of the fields A and C are "retarded" (in the backward light-cone) with respect to the field-points of the field B [7, 8]. At this point local commutativity is a necessary assumption in order to get the support properties for r_B and a_B as well as to restore Lorentz covariance [7, 8].

All the quantities R_B, A_B, T^+ and T^- defined in this way correspond to different boundary values of the vertex function G. To be specific we have

$$R_B = \lim_{\substack{k_1 \in V^+ \to 0 \\ k_3 \in V^+ \to 0}} G(Z)$$

$$A_B = \lim_{\substack{k_1 \in V^- \to 0 \\ k_3 \in V^- \to 0}} G(Z) \tag{40}$$

$$T^{\pm}(\zeta_1, \zeta_2, \zeta_3) = \lim_{\varepsilon, \varepsilon', \varepsilon'' \to 0} G(\zeta_1 \pm i\varepsilon', \zeta_2 \pm i\varepsilon'', \zeta_3 \pm i\varepsilon) .$$

The quantities ε, ε', ε'' are as usual positive.

The particular boundary values occurring in Eq. (37) are notified by the restrictions

$$\begin{aligned} p_1^2 &= -m_1^2, \quad p_1 \in V^+ \\ p_3^2 &= -m_3^2, \quad p_3 \in V^+ . \end{aligned} \tag{41}$$

For other values of the vectors p_1 and p_3 the CBV R_B describe other matrix elements in a well-known way [8] and we have e.g. for

$$\begin{aligned} p_1^2 &= -m_1^2, \quad p_1 \in V^+ \\ p_3^2 &= -m_3^2, \quad p_3 \in V^- \end{aligned} \tag{42}$$

that

$$R_B(p_1, p_3) = N_1 N_3 \langle -\bar{p}_3 | (m_2^2 - \Box_2) B | \bar{p}_1 \rangle . \tag{43}$$

It should be noted that the signs of the imaginary parts of the limiting arguments of the vertex function G in Eq. (40) are $(+ + +)$ in connection with Eq. (41) and $(+ + -)$ in connection with Eq. (42). In Eq. (43) the A-particle and C-particle states are actually "instates" in the sense described above. It is, however, immediately evident that the occurring boundary value $G(m_1^2 + i\varepsilon', -(p_1 + p_3)^2 + i\varepsilon'', m_3^2 - i\varepsilon)$ also describes the matrix element

$$N_1 N_3 \langle -\bar{p}_1 \text{ out} | (m_2^2 - \Box_2) B | \bar{p}_3 \text{ out} \rangle \equiv A_B(p_1, p_3) \tag{44}$$

if the vectors p_1 and p_3 are restricted by

$$\begin{aligned} p_1^2 &= -m_1^2, \quad p_1 \in V^- \\ p_3^2 &= -m_3^2, \quad p_3 \in V^+ . \end{aligned} \tag{45}$$

This is an expression for the important field-theoretical symmetry called CPT-invariance. (We would like to add the small technical remark that we have in these connections actually assumed that $m_1^2 > m_3^2$ which implies that the vector $p_1 + p_3$ belongs to the same light-cone as the vector p_1, whenever time-like or light-like.)

Along these lines it is possible to express some of the different boundary values of the vertex function which are required for the integrands in the dispersion relations of Section 3 by means of the different CBV.

Using the notations

$$\zeta_i = -p_i^2, \qquad i=1,2,3$$
$$\sum_{i=1}^{3} p_i = 0 \tag{46}$$

it is obvious that we can write e.g.

$$G(\zeta_1 \pm i\varepsilon', \zeta_2 \pm i\varepsilon'', \zeta_3 \mp i\varepsilon) = \Theta(\pm p_1)\,\Theta(\mp p_3)\,[\Theta(\mp p_2)]\,R_B(p_1, p_3) + \Theta(\mp p_1)\,\Theta(\pm p_3)\,[\Theta(\pm p_2)]\,A_B(p_1, p_3) \tag{47}$$

in the cases

$$\zeta_i > 0 \qquad i=1,2,3$$
$$\zeta_1 > (\sqrt{\zeta_2} + \sqrt{\zeta_3})^2 \tag{48}$$

and

$$\zeta_1 > 0, \quad \zeta_3 > 0, \quad \zeta_2 < 0. \tag{49}$$

The notation with a step function inside square brackets $[\Theta(\pm p_2)]$ in Eq. (47) means that the step function is only relevant when the corresponding quantity $\zeta_2 \geqq 0$. The upper and lower signs correspond to each other in Eq. (47). Further we note that the inequality in Eq. (48) implies that the vector $(p_1 + p_3) = -p_2$ whenever time-like or light-like belongs to the same light cone as the vector p_1. Finally the occurrence of two terms in Eq. (47) is once again an expression for the *CPT*-invariance.

The same boundary value of the vertex function can for the cases

$$\zeta_i > 0 \qquad i=1,2,3$$
$$\zeta_2 > (\sqrt{\zeta_1} + \sqrt{\zeta_3})^2 \tag{50}$$

and

$$\zeta_2 > 0, \quad \zeta_3 > 0, \quad \zeta_1 < 0 \tag{51}$$

be written as

$$G(\zeta_1 \pm i\varepsilon', \zeta_2 \pm i\varepsilon'', \zeta_3 \mp i\varepsilon) = [\Theta(\mp p_1)]\,\Theta(\pm p_2)\,\Theta(\mp p_3)\,R_A(p_2, p_3) + [\Theta(\pm p_1)]\,\Theta(\mp p_2)\,\Theta(\pm p_3)\,A_A(p_2, p_3) \tag{52}$$

with the *CBV* R_A and A_A defined in the same way as R_B and A_B in Eqs. (38) to (40) except that the field A is distinguished in the same way as the field B is there.

Further for the cases

$$\zeta_1 > 0, \quad \zeta_3 > 0, \quad \zeta_2 < (\sqrt{\zeta_1} - \sqrt{\zeta_3})^2$$
$$\zeta_2 > 0, \quad \zeta_3 > (\sqrt{\zeta_1} + \sqrt{\zeta_2})^2 \tag{53}$$

and similarly for the cases

$$\zeta_2 > 0, \quad \zeta_3 > 0, \quad \zeta_1 < (\sqrt{\zeta_2} - \sqrt{\zeta_3})^2$$
$$\zeta_1 > 0, \quad \zeta_3 > (\sqrt{\zeta_1} + \sqrt{\zeta_2})^2 \tag{54}$$

we may write

$$G(\zeta_1 \pm i\varepsilon', \zeta_2 \mp i\varepsilon'', \zeta_3 \pm i\varepsilon) = [\Theta(\pm p_1)]\,[\Theta(\pm p_2)]\,\Theta(\mp p_3)\,A_A + [\Theta(\mp p_1)]\,[\Theta(\mp p_2)]\,\Theta(\pm p_3)\,R_A \tag{55}$$

with the same meaning of the symbols $[\Theta(\pm p_i)]$ as above. For the symmetrical case when the index 1 is exchanged to the index 2 and Eq. (53) or Eq. (54) is valid, we get in the same way

$$G(\zeta_1 \mp i\varepsilon', \zeta_2 \pm i\varepsilon'', \zeta_3 \pm i\varepsilon) = [\Theta(\pm p_1)]\,[\Theta(\pm p_2)]\,\Theta(\mp p_3)\,A_B + [\Theta(\mp p_1)]\,[\Theta(\mp p_2)]\,\Theta(\pm p_3)\,R_B\,. \tag{56}$$

In Eqs. (47), (52), (55), and (56) we have by different CBV actually expressed all the boundary values of the vertex function G which are required for the dispersion relations of Section 3 in all the six octants of the variable-space where the signs of the limiting imaginary parts of the arguments are "mixed", i.e., when one of them has different sign compared to the remaining two.

For the remaining two octants where the signs of the limiting imaginary parts are equal, we can in a well-known way use the TBV or $ATBV$ of the vertex function.

5. The Dispersion Relations in Terms of the On-Mass-Shell Matrix Elements of the Operators

We will in this section by means of the definitions given in Section 4 describe the occurring differences between the boundary values of the vertex function in the integrands of the dispersion relations of Section 3 in terms of the matrix elements of the operators.

We will start by discussing the dispersion relation of Eq. (17) in some detail and from these results the general procedure should be sufficiently clear to allow only a brief mentioning of the remaining cases.

In connection with the formula for the vertex function $G(Z_1, Z_2, \zeta_3 + i\varepsilon)$ we need for the integrand in Eq. (17) the contribution $\Delta_1^{(+)}$ to the Z_1-cut, i.e., to the integration range

$$\begin{aligned} &\zeta_1 > r > 0 \\ &\zeta_2 < r \\ &\zeta_3 > 0 \\ &r\zeta_3 + (r - \zeta_1)(r - \zeta_2) = 0\,, \end{aligned} \tag{57}$$

$$\Delta_1^{(+)} = G(\zeta_1 + i\varepsilon', \zeta_2 + i\varepsilon'', \zeta_3 + i\varepsilon) - G(\zeta_1 - i\varepsilon', \zeta_2 - i\varepsilon'', \zeta_3 + i\varepsilon)\,. \tag{58}$$

Eq. (57) imply the easily proved inequalities

$$\begin{aligned} \zeta_2 &< (\sqrt{\zeta_1} - \sqrt{\zeta_3})^2 \\ \zeta_1 &> (\sqrt{\zeta_2} + \sqrt{\zeta_3})^2 \quad \text{if} \quad \zeta_2 > 0 \end{aligned} \tag{59}$$

and we can then according to Eqs. (47)–(49) of Section 4 write for the quantity $\Delta_1^{(+)}$ in Eq. (58):

$$\begin{aligned} \Delta_1^{(+)} = &\,\Theta(-p_1)\,\Theta(p_3)\,[\Theta(p_2)]\,\{T^+ - R_B\} \\ &+ \Theta(p_1)\,\Theta(-p_3)\,[\Theta(-p_2)]\,\{T^+ - A_B\}\,. \end{aligned} \tag{60}$$

Using the representation formulas for the occurring CBV and TBV given in Eq. (38) of Section 4 we deduce the following Fourier representation for the quantity $\Delta_1^{(+)}$:

$$\begin{aligned} \Delta_1^{(+)} = &-\int dx_1\,dx_3\,e^{i\sum_{j=1}^{3} p_j x_j} \\ &\cdot\{\Theta(-p_1)\,\Theta(p_3)\,[\Theta(p_2)]\,\{t^+ - r_B\} + \Theta(p_1)\,\Theta(-p_3)\,[\Theta(-p_2)]\,\{t^+ - a_B\}\}. \end{aligned} \tag{61}$$

The difference $(t^+ - r_B)$ in this expression can according to Eq. (39) after some algebra be written as

$$\begin{aligned} (t^+ - r_B) = \prod_{j=1}^{3} (m_j^2 - \Box_j)\,\langle 0|\,\{&A\,\Theta(23)\,[B, C] + T(AC)\,B \\ &+ C\,\Theta(21)\,[B, A]\}\,|0\rangle\,. \end{aligned} \tag{62}$$

We note that in Eq. (62) there is only *one* step function in the integration vectors. This means that we can perform one of the integrals over the field-points in Eq. (61) by the introduction of a complete set of intermediate states with given values of the energy-momentum vectors. Due to the assumed spectral properties of the energy-momentum operators and the indicated light-cone properties of the vectors p_j in Eq. (61) we will actually only get contributions from the first two terms of the VEV in Eq. (62). A similar treatment of the difference $(t^+ - a_B)$ results in the following non-vanishing contributions (note the CPT-symmetry between the results in Eq. (62) and Eq. (63)):

$$(t^+ - a_B) = \prod_{j=1}^{3} (m_j^2 - \Box_j)\,\langle 0|\,\{\Theta(32)\,[C, B]\,A + B\,T(AC)\}\,|0\rangle\,. \tag{63}$$

If we had from the beginning been interested in the formula for $G(Z_1, Z_2, \zeta_3 - i\varepsilon)$, i.e., for the vertex function on the other side of the Z_3-cut compared to the situation discussed above, we would instead of the quantity $\Delta_1^{(+)}$ get the following contribution to the Z_1-cut in the dispersion relation of Eq. (17):

$$\Delta_1^{(-)} = G(\zeta_1 + i\varepsilon', \zeta_2 + i\varepsilon'', \zeta_3 - i\varepsilon) - G(\zeta_1 - i\varepsilon', \zeta_2 - i\varepsilon'', \zeta_3 - i\varepsilon) \tag{64}$$
$$= \Theta(-p_1)\Theta(p_3)[\Theta(p_2)]\{A_B - T^-\} + \Theta(p_1)\Theta(-p_3)[\Theta(-p_2)]\{R_B - T^-\}$$

(cf. Eq. (47)).

The *difference* between the two expressions in Eq. (60) and Eq. (64) i.e., the discontinuity stemming from the combination of the Z_1-cut and the Z_3-cut has a particularly simple expression in terms of the matrix elements and we find by the same methods employed for Eq. (62) and (63):

$$\Delta_1^{(+)} - \Delta_1^{(-)} = -\int dx_1\, dx_3\, e^{i\sum_{j=1}^{3} p_j x_j} \prod_{j=1}^{3} (m_j^2 - \Box_j)$$
$$\cdot \{\Theta(-p_1)\,\Theta(p_3)\,[\Theta(p_2)]\,\langle 0|A(x_1)\,B(x_2)\,C(x_3)|0\rangle \tag{65}$$
$$+ \Theta(p_1)\,\Theta(-p_3)\,[\Theta(-p_2)]\,\langle 0|C(x_3)\,\Theta(x_2)\,A(x_1)|0\rangle\}\,.$$

The interesting point is that the integrand of Eq. (65) contains *no* step functions in the integration vectors and we can therefore immediately perform the integrals by the introduction of complete sets $|n\rangle$ and $|m\rangle$ of intermediate states with given energy-momentum vectors p_n and p_m:

$$\Delta_1^{(+)} - \Delta_1^{(-)} = -(2\pi)^8 \prod_{j=1}^{3} (m_j^2 - \zeta_j) \sum_{|n\rangle|m\rangle} \{\Theta(-p_1)\Theta(p_3)\delta(p_1 + p_n)\delta(p_3 - p_m)$$
$$\cdot \langle 0|A|n\rangle\langle n|B|m\rangle\langle m|C|0\rangle + \Theta(p_1)\Theta(-p_3)\delta(p_1 - p_n)\delta(p_3 + p_m) \tag{66}$$
$$\cdot \langle 0|C|m\rangle\langle m|B|n\rangle\langle n|A|0\rangle\} = (i\cdot 2\pi)^2\, G_B(\zeta_1, \zeta_2, \zeta_3)\,.$$

A similar treatment of the corresponding discontinuity stemming from the combination of the Z_2-cut and the Z_3-cut results in the following formula:

$$\Delta_2^{(+)} - \Delta_2^{(-)} = \{G(\zeta_1 + i\varepsilon', \zeta_2 + i\varepsilon'', \zeta_3 + i\varepsilon) - G(\zeta_1 - i\varepsilon', \zeta_2 - i\varepsilon'', \zeta_3 + i\varepsilon)\}$$
$$- \{G(\zeta_1 + i\varepsilon', \zeta_2 + i\varepsilon'', \zeta_3 - i\varepsilon) - G(\zeta_1 - i\varepsilon', \zeta_2 - i\varepsilon'', \zeta_3 - i\varepsilon)\}$$
$$= -(2\pi)^8 \prod_{j=1}^{3} (m_j^2 - \zeta_j) \sum_{|n\rangle|m\rangle} \{\Theta(-p_2)\,\Theta(p_3)\,\delta(p_2 + p_n)\,\delta(p_3 - p_m) \tag{67}$$
$$\cdot\ \langle 0|B|n\rangle\langle n|A|m\rangle\langle m|C|0\rangle + \Theta(p_2)\Theta(-p_3)\delta(p_2 - p_n)\delta(p_3 + p_m)$$
$$\cdot\ \langle 0|C|m\rangle\langle m|A|n\rangle\langle n|B|0\rangle\} \equiv (i\cdot 2\pi)^2\, G_A(\zeta_1, \zeta_2, \zeta_3)\,.$$

In that case Eq. (52) has been used because we note that this contribution to the difference integral for $G(Z_1, Z_2, \zeta_3 + i\varepsilon) - G(Z_1, Z_2, \zeta_3 - i\varepsilon)$ in

Eq. (17) only occurs for values of the quantities ζ_i according to:

$$\begin{aligned} &\zeta_1 < r \\ &\zeta_2 > r > 0 \\ &\zeta_3 > 0 \\ &r\zeta_3 + (r-\zeta_1)(r-\zeta_2) = 0\,, \end{aligned} \tag{68}$$

i.e., when the inequalities (cf. Eqs. (50) and (51)) are fulfilled

$$\begin{aligned} &\zeta_1 < (\sqrt{\zeta_2} - \sqrt{\zeta_3})^2 \\ &\zeta_2 > (\sqrt{\zeta_1} + \sqrt{\zeta_3})^2 \quad \text{if} \quad \zeta_1 > 0\,. \end{aligned} \tag{68'}$$

Thus we may write for the contribution to the dispersion relations of Eq. (17) when one considers the discontinuity across the Z_3-cut:

$$\begin{aligned} G(Z_1,Z_2,\zeta_3+i\varepsilon) - G(Z_1,Z_2,\zeta_3-i\varepsilon) &= 2\pi i \int d\zeta_1\, d\zeta_2\, \delta(r\zeta_3+(r-\zeta_1)(r-\zeta_2)) \\ &\cdot \Big\{ G_B(\zeta_1,\zeta_2,\zeta_3)\cdot\tfrac{1}{2}\cdot\frac{\zeta_1+Z_1-2r}{\zeta_1-Z_1}\,\Theta(\zeta_1-r)\,\Theta(r-\zeta_2) \\ &+ G_A(\zeta_1,\zeta_2,\zeta_3)\cdot\tfrac{1}{2}\cdot\frac{\zeta_2+Z_2-2r}{\zeta_2-Z_2}\cdot\Theta(r-\zeta_1)\,\Theta(\zeta_2-r)\Big\} \end{aligned} \tag{69}$$

$$(Z_1-r)(Z_2-r)+r\zeta_3=0\,; \qquad r>0,\ \zeta_3>0$$

$$\operatorname{Im}(Z_1)\cdot\operatorname{Im}(Z_2)>0\,.$$

Using the same methods for the corresponding difference in the case $\operatorname{Im}(Z_1)\ \operatorname{Im}(Z_2)<0$ of Eq. (30), in which case we will have use of the formulas of Eqs. (53)–(56) of Section 4, we get the following result:

$$\begin{aligned} G(Z_1,Z_2,\zeta_3+i\varepsilon) - G(Z_1,Z_2,\zeta_3-i\varepsilon) &= 2\pi i \int d\zeta_1\, d\zeta_2\, \delta(\alpha\zeta_2+\beta\zeta_1-\alpha\beta\zeta_3) \\ &\cdot \Big\{ G_B(\zeta_1,\zeta_2,\zeta_3)\frac{\alpha}{\zeta_1-Z_1}\,\Theta(\zeta_1) + G_A(\zeta_1,\zeta_2,\zeta_3)\frac{\beta}{\zeta_2-Z_2}\,\Theta(\zeta_2)\Big\} \end{aligned} \tag{70}$$

$$\beta Z_1+\alpha Z_2=\alpha\beta\zeta_3\,; \qquad \zeta_3>0$$

$$\alpha+\beta=1, \quad \alpha>0, \quad \beta>0$$

$$\operatorname{Im}Z_1\cdot\operatorname{Im}Z_2<0\,.$$

Eqs. (69) and (70) are the main results of this paper. They express the contribution to the vertex function across one of the cut-surfaces (here the Z_3-cut) with the two remaining variables arbitrary complex (inside certain domains of the two-dimensional complex space) only in terms of the matrix elements of the locally commuting field operators, i.e., the (in general distribution-valued) weight functions G_B and G_A of Eqs. (66) and (67). The domains inside which the formula allows free variation of the variables Z_1 and Z_2, are seen to be of three real dimensions and we can describe these domains as neighbourhoods to the intersection between the Z_3-cut and the "axiomatic" boundaries of the domain of holomorphy for the three-point-function.

We further note that the formulas are valid for arbitrary real positive values of the parameters ζ_3 and r in connection with Eq. (69) as well as for ζ_3 and the actually occurring parameter α/β in connection with Eqs. (70).

We would like to stress that the assumptions behind the results are essentially only the rather general assumptions of Källén and Wightman (cf. Section 1), the particle interpretation according to reduction formalism (cf. Section 4) and some moderate integrability and boundedness properties (cf. Eqs. (9) and (27)).

6. Concluding Remarks

1. The Eqs. (69) and (70) can be immediately generalised along the lines indicated in Eqs. (18) and (19) respectively (32)–(34) if there should be need for "convergence powers" in Eqs. (9) and (27). Such modifications, which are usually called "subtractions", do not change Eqs. (69) and (70) except for the fact that the resulting expressions will become somewhat larger, and we will not give explicit formulas.

2. The Eqs. (69) and (70) have of course analogues for the cases when we, e.g., want to consider the discontinuity across the Z_1-cut with the variables Z_2 and Z_3 arbitrary complex in a similar neighbourhood of the corresponding intersection surface of the Källén-Wightman boundaries. Such formulas can be immediately written down by straight-forward permutations of indices, e.g.

$$\begin{aligned} &G(\zeta_1+i\varepsilon, Z_2, Z_3) - G(\zeta_1 - i\varepsilon, Z_2, Z_3) = (2\pi i)\int d\zeta_1\, d\zeta_3\, \delta(r\zeta_1 + (r-\zeta_2)(r-\zeta_3)) \\ &\quad \cdot \left\{ G_C(\zeta_1, \zeta_2, \zeta_3)\cdot \tfrac{1}{2} \cdot \frac{\zeta_2 + Z_2 - 2r}{\zeta_2 - Z_2}\, \Theta(\zeta_2 - r)\, \Theta(r-\zeta_3) \right. \\ &\quad \left. + G_B(\zeta_1, \zeta_2, \zeta_3)\cdot \tfrac{1}{2}\cdot \frac{\zeta_3 + Z_3 - 2r}{\zeta_3 - Z_3}\, \Theta(r-\zeta_2)\, \Theta(\zeta_3 - r) \right\} \\ &(Z_2 - r)(Z_3 - r) + r\zeta_1 = 0\,; \qquad r>0, \quad \zeta_3 > 0 \\ &\mathrm{Im}(Z_2)\cdot \mathrm{Im}(Z_3) > 0\,, \end{aligned} \tag{71}$$

$$\begin{aligned} &G(\zeta_1+i\varepsilon, Z_2, Z_3) - G(\zeta_1 - i\varepsilon, Z_2, Z_3) = (2\pi i)\int d\zeta_1\, d\zeta_3\, \delta(\alpha\zeta_2 + \beta\zeta_3 - \alpha\beta\zeta_1) \\ &\quad \cdot \left\{ G_C(\zeta_1, \zeta_2, \zeta_3)\frac{\beta}{\zeta_2 - Z_2}\, \Theta(\zeta_2) + G_B(\zeta_1, \zeta_2, \zeta_3)\frac{\alpha}{\zeta_3 - Z_3}\, \Theta(\zeta_3) \right\} \\ &\alpha Z_2 + \beta Z_3 - \alpha\beta\zeta_1\,; \qquad \zeta_1 > 0 \\ &\alpha + \beta = 1, \quad \alpha > 0, \quad \beta > 0 \\ &\mathrm{Im}(Z_2)\,\mathrm{Im}(Z_3) < 0\,. \end{aligned} \tag{72}$$

In this case the quantity G_C is a matrix element defined in a similar way as G_B and G_A of Eqs. (66) and (67):

$$G_C(\zeta_1,\zeta_2,\zeta_3)=(2\pi)^6\prod_{j=1}^{3}(m_j^2-\zeta_j)\sum_{|n\rangle|m\rangle}\{\Theta(-p_1)\Theta(p_2)\delta(p_1+p_n)\delta(p_2-p_m)$$
$$\cdot\langle 0|A|n\rangle\langle n|C|m\rangle\langle m|B|0\rangle+\Theta(p_1)\Theta(-p_2)\delta(p_1-p_n)\delta(p_2+p_m) \quad (73)$$
$$\cdot\langle 0|B|m\rangle\langle m|C|n\rangle\langle n|A|0\rangle\}\,.$$

The remaining case with index 2 and 3 permuted compared to Eqs. (69) and (70) should be obvious.

3. The weight functions G_A, G_B and G_C in the dispersion relations above also occur in other representation formulas connected to the vertex function, in the well-known representation formulas [4, 8] for the coordinate-space function, i.e., the *VEV* of non-time-ordered operator products. We have, e.g., for the operator product $[(m_1^2-\Box_1)\,A(x_1)]\cdot[(m_2^2-\Box_2)\,B(x_2)]\,[(m_3^2-\Box_3)\,C(x_3)]=P_{ABC}$

$$\langle 0|P_{ABC}|0\rangle=\frac{1}{(2\pi)^6}\int dp_1\,dp_3\,e^{ip_1(x_1-x_2)+ip_3(x_3-x_2)}\,\Theta(p_1)\,\Theta(-p_3)$$
$$\cdot\,G_B(-p_1^2,\,-(p_1+p_3)^2,\,-p_3^2)\,. \quad (74)$$

The weight function G_B further occurs in the similar representation formula for the product in the opposite order, P_{CBA} (note the *CPT*-symmetry):

$$\langle 0|P_{CBA}|0\rangle=\frac{1}{(2\pi)^6}\int dp_1\,dp_3\,e^{ip_1(x_1-x_2)+ip_3(x_3-x_2)}\,\Theta(-p_1)\,\Theta(p_3)$$
$$\cdot\,G_B(-p_1^2,\,-(p_1+p_3)^2,\,-p_3^2)\,. \quad (75)$$

4. It is interesting to note that the same situation occurs in connection with the two-point function where the well-known Källén-Lehmann representation [9, 11] can be written for the corresponding coordinate space function [8]:

$$\langle 0|(m_1^2-\Box_1)A(x_1)(m_2^2-\Box_2)B(x_2)|0\rangle=\frac{1}{(2\pi)^3}\int dp\,e^{ip(x_1-x_2)}\Theta(p)G_{AB}(-p^2)$$
$$\Theta(p)\,G_{AB}(-p^2)=(2\pi)^3(m_1+p^2)\,(m_2+p^2) \quad (76)$$
$$\cdot\sum_{|n\rangle}\delta(p-p_n)\,\langle 0|A|n\rangle\,\langle n|B|0\rangle\,.$$

The corresponding formula for the momentum-space two-point function G defined via reduction technique by

$$\begin{aligned}\langle 0|(m_1^2-\Box_1)A(0)|\bar{p}_2\rangle N_2 &= -i\int dx_2 e^{ip_2(x_2-x_1)}(m_1^2-\Box_1)(m_2^2-\Box_2)\\ &\quad\cdot\langle 0|\Theta(x_1-x_2)[B(x_2),A(x_1)]|0\rangle \qquad (77)\\ &= \lim_{k_2\in V^+} G(-(p_2+ik_2)^2)\end{aligned}$$

is then

$$G(Z)=\int_0^\infty \frac{d\zeta}{\zeta-Z}\,G_{AB}(\zeta)\,. \qquad (78)$$

In the case of the two-point function, the weight function G_{AB} can, essentially because of local commutativity [8], be proved to be equal to the weight function G_{BA} defined by

$$\Theta(p)G_{BA}(-p^2)=(2\pi)^3(p^2+m_1^2)(p^2+m_2^2)\sum_{|n\rangle}\delta(p-p_n)\langle 0|B|n\rangle\langle n|A|0\rangle\,. \qquad (79)$$

This function occurs in the representation formula corresponding to Eq. (76) for the operator product in the opposite order, i.e.

$$[(m_2^2-\Box_2)B(x_2)(m_1^2-\Box_1)A(x_1)]\,.$$

Due essentially to the representation formulas in Eqs. (69) and (70) and the boundedness properties of Eqs. (9) and (27), there are actually definite relations also between the three weight functions G_A, G_B and G_C, though not as stringent as for the two-point case. Such relations will be further investigated in a forthcoming publication.

5. Representation formulas for the coordinate-space vertex function F similar to the ones derived above for the momentum-space function G can be written down along rather similar lines. In that case different VEV will occur instead of the weight functions G_A, G_B and G_C.

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